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Non-linear rough heat equations

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Abstract This article is devoted to define and solve an evolution equation of the form $dy_t = \Delta y_t\, dt + dX_t(y_t)$, where Δ stands for the Laplace operator on a space of the form $L^p(\mathbb{R}^n)$, and X is a finite dimensional noisy nonlinearity whose typical form is given by $X_t(\varphi) = \sum_{i=1}^N x_t^i f_i(\varphi)$, where each $x = (x^{(1)}, \ldots, x^{(N)})$ is a γ-Hölder function generating a rough path and each f_i is a smooth enough function defined on $L^p(\mathbb{R}^n)$. The generalization of the usual rough path theory allowing to cope with such kind of system is carefully constructed.

Keywords Rough paths theory · Stochastic PDEs · Fractional Brownian motion

Mathematics Subject Classification (2000) 60H05 · 60H07 · 60G15

1 Introduction

The rough path theory, which was first formulated in the late 90's by Lyons [32,33] and then reworked by various authors [18,20], offers a both elegant and efficient way

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of defining integrals driven by some irregular signals. This pathwise approach enables to handle the standard (rough) differential system

$$dy_t = \sigma(y_t)\, dx_t, \quad y_0 = a, \tag{1}$$

where x is a non-differentiable process which allows the construction of a so-called rough path $\mathbf{x}$, morally represented by the iterated integrals of the process (see Definition 6.2 for a 2-rough path). The method also applies to the treatment of less classical (rough) finite-dimensional systems such that the delay equation [36] or the integral Volterra systems [12,13]. In all of those situations, the pathwise interpretation of the associated stochastic system (for a random x) reduces to the construction of a rough path $\mathbf{x}$ above x, which is now well-established for a large class of stochastic processes that for instance includes fractional Brownian motion (see [18] for many other examples).

In the last few years, several authors provided some kind of similar pathwise treatment for quasi-linear equations associated to non-bounded operators, that is to say of the rather general form

$$dy_t = Ay_t\, dt + dX_t(y_t), \qquad t \in [0, T] \tag{2}$$

where T is a strictly positive constant, A is a non-bounded operator defined on a (dense) subspace of some Banach space V and $X \in \mathcal{C}([0, T] \times V; V)$ is a noise which is irregular in time and which evolves in the space of vector fields acting on the Banach space at stake. Their results apply in particular to some specific partial differential equations perturbed by samples of (infinite-dimensional) stochastic processes.

To our knowledge, two different approaches have been used to tackle the issue of giving sense to (2):

(i) The first one essentially consists in returning to the usual formulation (1) by means of classical transformations of the initial system (2). One is then allowed to resort to the numerous results established in the standard framework of rough paths analysis. As far as this general method is concerned, let us quote the work of Caruana and Friz [5], Caruana et al. [6], Friz and Oberhauser [19] as well as the promising approach of Teichmann [45].

(ii) The second approach, contained in [25], is due to the last two authors of the present paper, and is based on a formalism which combines (analytical) semigroup theory and rough paths methods. This formulation can be seen as a "twisted" version of the classical rough path theory. The key ingredients of the standard theory of SPDEs, namely the stochastic integral and the stochastic convolution, are here replaced with a couple of operators, the so-called standard and twisted increment operators, together with a suitable notion of infinite-dimensional rough path.

Of course, one should also have in mind the huge literature concerning the case of evolution equations driven by usual Brownian motion, for which we refer to [9] for the infinite dimensional setting and to [8] for the multiparametric framework. In the particular case of the stochastic heat equation driven by an infinite dimensional Brownian motion, some sharp existence and uniqueness results have (for instance) been obtained in [39] in a Hilbert space context, and in [3,4,27,53] for Banach valued solutions (closer to the situation we shall investigate). In the Young integration context, some recent efforts have also been made in order to define solutions to parabolic [24,35] or wave type [42] equations. We would like to mention also the application of rough path ideas to the solution of dispersive equation (both deterministic and stochastic) with low-regularity initial conditions [22].

The present article goes back to the setting (ii), and proposes to fill two gaps left by [25]. More specifically, we mainly focus (for sake of clarity) on the case of the heat equation in $\mathbb{R}^n$ with a non-linear fractional perturbation, and our aim is to give a reasonable sense and solve the equation

$$dy_t = \Delta y_t \, dt + dX_t(y_t), \tag{3}$$

where Δ is the Laplacian operator considered on some $L^p(\mathbb{R}^n)$ space (with p chosen large enough and specified later on), namely

$$\Delta : D(\Delta) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

Then the first improvement we propose here consists in considering a rather general noisy nonlinearity X evolving in a Hölder space $\mathcal{C}^\gamma(L^p(\mathbb{R}^n); L^p(\mathbb{R}^n))$, with $\gamma < 1/2$, instead of the polynomial perturbations studied in [25]. A second line of generalization is that we show how to apply our results to a general 2-rough path, which goes beyond the standard Brownian case.

As usual in the stochastic evolution setting, we study Eq. (3) in its mild form, namely:

$$y_t = S_t y_0 + \int_0^t S_{t-s} dX_s(y_s), \tag{4}$$

where $S_t : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ designates the heat semigroup on $\mathbb{R}^n$. This being said, and before we state an example of the kind of result we have obtained, let us make few remarks on the methodology we have used.

(a) The main price to pay in order to deal with a general nonlinearity is that we only consider a finite dimensional noisy input. Namely, we stick here to a noise generated by a γ-Hölder path $x = (x^{(1)}, \ldots, x^{(N)})$ and evolving in a finite-dimensional subspace of $\mathcal{C}(L^p(\mathbb{R}^n); L^p(\mathbb{R}^n))$, which can be written as:

$$X_t(\varphi) = \sum_{i=1}^N x_t^i f_i(\varphi), \tag{5}$$

with some fixed elements $\{f_i\}_{i=1,\ldots,N}$ of $\mathcal{C}(L^p(\mathbb{R}^n); L^p(\mathbb{R}^n))$, chosen of the particular form

$$f_i(\varphi)(\xi) = \sigma_i(\xi, \varphi(\xi))$$

for sufficiently smooth functions $\sigma_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

Note that the hypothesis of a finite-dimensional noise is also assumed in [5] or [45]. Once again, our aim in [25] was to deal with irregular homogeneous noises in space, but we were only able to tackle the case of a linear or polynomial dependence on the unknown. As far as the form of the nonlinearity is concerned, let us mention that [5] deals with a linear case, while the assumptions in [45] can be read in our setting as: one is allowed to define an extended function $\tilde{f}_i(t, \varphi) := S_{-t} f_i(S_t \varphi)$, which is still a smooth enough function of the couple (t, φ). As we shall see, the conditions we ask in the present article for f_i are much less stringent, and we shall recover partially the results of [45] at Sect. 5.

(b) In order to interpret (4), the reasoning we will resort to is largely inspired by the analysis of the standard rough integrals. For this reason, let us recall briefly the main features of the theory, as it is presented in [20]: the interpretation of $\int y_s \, dx_s$ (with x a finite-dimensional irregular noise) stems from some kind of dissection process of the usual Riemann–Lebesgue integral $\int y \, d\tilde{x}$, when $\tilde{x}$ is a regular driving process. This work appeals to two recurrent operators acting on spaces of k-variables functions ($k \geq 1$): the so-called increment operator δ (see (26)) and its inverse, the sewing map Λ, the existence of which hinges on some specific regularity conditions. If y is a 1-variable function, then δ is simply defined as $(\delta y)_{ts} := y_t - y_s$, while if $z_{ts} = \int_s^t (y_t - y_u) \, d\tilde{x}_u$, then $(\delta z)_{tus} := z_{ts} - z_{tu} - z_{us} = (\delta y)_{tu} (\delta \tilde{x})_{us}$. With such notations, one has for instance

$$\int_s^t y_u \, d\tilde{x}_u = \left(\int_s^t d\tilde{x}_u \right) y_s + \int_s^t (y_t - y_u) \, d\tilde{x}_u = \left(\int_s^t d\tilde{x}_u \right) y_s + \left(\delta^{-1} \left((\delta y)(\delta \tilde{x}) \right) \right)_{ts}.$$

Of course, the latter equality makes only sense once the invertibility of δ has been justified, which is the main challenge of the strategy.

During the process of dissection, it early appears, and this is the basic principles of the rough path theory, that in order to give sense to $\int y_s \, dx_s$ for a large class of Hölder-processes y, it suffices to justify the existence of the iterated integrals associated to x: $\mathbf{x}^{\mathbf{1}}_{ts} = \int_s^t dx_u$, $\mathbf{x}^{\mathbf{2}}_{ts} = \int_s^t dx_u \int_s^u dx_v$, etc., up to an order which is linked to the Hölder regularity of x. If x is γ-Hölder for some $\gamma > 1/2$, then only $\mathbf{x}^{\mathbf{1}}$ is necessary, whereas if $\gamma \in (1/3, 1/2)$, then $\mathbf{x}^{\mathbf{2}}$ must come into the picture.

Once the integral has been defined, solving the system

$$(\delta y)_{ts} = \int_s^t \sigma(y_u) \, dx_u, \quad y_0 = a, \tag{6}$$

where σ is a regular function, is a matter of standard fixed-point arguments.

(c) As far as (4) is concerned, the presence of the semigroup inside the integral prevents us from writing this infinite-dimensional system under the general form (6). If y is a solution of (4) (suppose such a solution exists), its variations are actually governed by the equation (let $s < t$)

$$(\delta y)_{ts} = y_t - y_s = S_t y_0 - S_s y_0 + \int_0^s \left[S_{t-u} - S_{s-u}\right] dX_u(y_u) + \int_s^t S_{t-u}\, dX_u(y_u),$$

which, owing to the multiplicative property of the semigroup, reduces to

$$(\delta y)_{ts} = a_{ts} y_s + \int_s^t S_{t-u}\, dX_u(y_u), \tag{7}$$

where $a_{ts} = S_{t-s} - \mathrm{Id}$. Here occurs the simple idea of replacing δ with the new operator $\hat{\delta}$ defined by $(\hat{\delta} y)_{ts} := (\delta y)_{ts} - a_{ts} y_s$. Equation (7) then takes the more familiar form

$$(\hat{\delta} y)_{ts} = \int_s^t S_{t-u} dX_u(y_u), \quad y_0 = \psi. \tag{8}$$

In the second section of the article, we will see that the operator $\hat{\delta}$, properly extended to act on k-variables functions ($k \geq 1$), satisfies properties analogous to δ. In particular, the multiplicative property of S enables to retrieve the cohomology relation $\hat{\delta}\hat{\delta}$, which is at the core of the most common constructions based on δ. For sake of consistence, we shall adapt the notion of regularity of a process to this context: a 1-variable function will be said to be γ-Hölder *in the sense of* $\hat{\delta}$ if for any s, t, $|(\hat{\delta} y)_{ts}| \leq c\,|t-s|^{\gamma}$. It turns out that the properties of $\hat{\delta}$ suggest the possibility of inverting $\hat{\delta}$ through some operator $\hat{\Lambda}$, just as Λ inverts δ. This is the topic of Theorem 3.6, which was the starting point of [25] and also the cornerstone of all our present constructions.

(d) Sections 3 and 4 will then be devoted to the interpretation of the integral appearing in (8). To this end, we will proceed as with the standard system (6), which means that we will suppose at first that X is regular in time and under this hypothesis, we will look for a decomposition of the integral in terms of "iterated integrals" depending only on X. For some obvious stability reasons, it matters that the dissection mainly appeal to the operators $\hat{\delta}$ and $\hat{\Lambda}$.

However, in the course of the reasoning, some intricate interplay between twisted and non-twisted increments will force us to analyze the spatial regularity of some terms of the form $a_{ts} y_s$, where y is the candidate solution to (4). This can be achieved by letting the fractional Sobolev spaces come into play. Namely, we set $\mathcal{B}_p = L^p(\mathbb{R}^n)$ and for $\alpha \in [0, 1/2)$, we also write $\mathcal{B}_{\alpha,p}$ for the fractional Sobolev space of order α based on $\mathcal{B}_p$ (the definition will be elaborated on in Sect. 3). One can then resort to the relation

$$\text{if } \varphi \in \mathcal{B}_{\alpha,p}, \quad \|a_{ts}\varphi\|_{\mathcal{B}_p} \leq c\,|t-s|^{\alpha}\, \|\varphi\|_{\mathcal{B}_{\alpha,p}}.$$

Of course, we will have to pay attention to the fact that this time regularity gain occurs to the detriment of the spatial regularity. It is also easily conceived that we will require

$\mathcal{B}_{\alpha,p}$ to be an algebra of continuous functions, which explains why we work in some L^p spaces with p large enough.

These additional terms of the form $a_{ts} y_s$ are specific to the non-linear case, for which Taylor type expansions are required, and explain a part of the technical difficulties we have met in the current article. If the vector fields $\{f_i\}_{i=1,\dots,N}$ are linear, then we don't need any recourse to the Taylor formula and the decomposition of the candidate solution can be written thanks to $\hat{\delta}$ and $\hat{\Lambda}$ only. This particular case has been dealt with in [25], as well as the polynomial case, for which specific and individual treatments based on trees-indexed integral [21,23] are suggested. In our situation, we shall see that the landmarks of the construction, that is to say the counterparts of the usual step-2 rough path $(\int dx, \iint dx \otimes dx)$, are (morally) some operators acting on $\mathcal{B}_p$, defined as follows: for $\varphi, \psi \in \mathcal{B}_p$, set

$$X_{ts}^{x,i}(\varphi) = \int_s^t S_{tu}(\varphi)\, dx_u^i, \quad X_{ts}^{xa,i}(\varphi, \psi) = \int_s^t S_{tu}\,[a_{us}(\varphi)\cdot\psi]\, dx_u^i, \tag{9}$$

$$X_{ts}^{xx,ij}(\varphi) = \int_s^t S_{tu}(\varphi)\,\delta x_{us}^j\, dx_u^i, \tag{10}$$

for $i, j = 1, \dots, N$, where $\varphi \cdot \psi$ is the pointwise multiplication operator of φ by ψ. In some way, it is through those three operators that the (stochastic) convolution mechanisms commonly used in the treatment of SPDEs (see [9]) will appear.

In a quite natural way, the results established in Sect. 3.2 by using expansions at first order only, will be applied to a γ-Hölder process x with $\gamma > 1/2$. The considerations of Sect. 4, which involve more elaborate developments, will then enable the treatment of the case $1/3 < \gamma \le 1/2$.

It is also crucial to see how our theory applies to concrete situations. To this purpose, using an elementary integration by parts argument, we will see in Sect. 6 that in order to define the operators given by (9) and (10) properly, the additional assumptions on x reduce to the standard rough-paths hypotheses. In this way, the results of this article can be applied to a N-dimensional fractional Brownian motion x with Hurst index $H > 1/3$, thanks to the previous works of Coutin–Qian [7] or Unterberger [49] (see Remark 6.4). This also means that in the end, the solution to the rough PDE (3) is a continuous function of the initial condition and $\mathbf{x^1}$, $\mathbf{x^2}$, which suggests (as [6,45] does) that one can also solve the noisy heat equation by means of a variant of the classical rough path theory. However, we claim that our construction is really well suited for the evolution equation setting, insofar as the arguments developed here can be extended naturally to an infinite dimensional noise, at the price of some more intricate technical considerations. We plan go back to this issue in a further publication.

With all these considerations in mind, we can now give an example of the kind of result which shall be obtained in the sequel of the paper (given here in the first non trivial rough case for X, that is a Hölder continuity exponent $1/3 < \gamma \le 1/2$):

Theorem 1.1 *Let X be a noisy nonlinearity of the form* (5)*, where:*

(i) *The noisy part x is a N-dimensional Hölder-continuous signal in $\mathcal{C}^\gamma([0,T];\mathbb{R}^N)$ for a given $\gamma > 1/3$. Moreover, we assume that x allows to define a Levy area $\mathbf{x^2}$ in the sense given by Definition 6.2.*

(ii) *Each nonlinearity f_i can be written as $[f_i(\varphi)](\xi) = \sigma_i(\xi,\varphi(\xi))$, where the function $\sigma_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfies both conditions (C1) and (C2)$_k$ of Definition 2.1, for $k = 3$. Then for any couple $(\kappa, p) \in (\frac{1}{3},\gamma) \times \mathbb{N}^*$ such that $\gamma - \kappa > \frac{n}{2p}$, and any initial condition y_0 in the fractional Sobolev space $\mathcal{B}_{\kappa,p}$ (see Notation 2.3), Eq. (4) admits a unique solution $y \in \mathcal{C}^\kappa(\mathcal{B}_{\kappa,p})$ on an interval $[0,T]$, for a strictly positive time T which depends on x, $\mathbf{x^2}$ and y_0. Furthermore, the Itô map $(y_0, x, \mathbf{x^2}) \mapsto y$ is locally Lipschitz: if y (resp. $\tilde{y}$) denotes the solution of the equation on $[0,T]$ (resp. $[0,\tilde{T}]$) associated to a driving path $(x,\mathbf{x^2})$ (resp. $(\tilde{x},\tilde{\mathbf{x}}^\mathbf{2})$) and an initial condition y_0 (resp. $\tilde{y}_0$), then*

$$\mathcal{N}[y-\tilde{y};\mathcal{C}^\kappa([0,T^*];\mathcal{B}_{\kappa,p})] \le c_{x,\tilde{x},y_0,\tilde{y}_0}\left\{\|y_0-\tilde{y}_0\|_{\mathcal{B}_{\kappa,p}} + \|x-\tilde{x}\|_\gamma + \|\mathbf{x^2}-\tilde{\mathbf{x}}^\mathbf{2}\|_{2\gamma}\right\}, \tag{11}$$

where $T^* = \inf(T,\tilde{T})$ and

$$c_{x,\tilde{x},y_0,\tilde{y}_0} = C(\|y_0\|_{\mathcal{B}_{\kappa,p}}, \|\tilde{y}_0\|_{\mathcal{B}_{\kappa,p}}, \|x\|_\gamma, \|\tilde{x}\|_\gamma, \|\mathbf{x^2}\|_{2\gamma}, \|\tilde{\mathbf{x}}^\mathbf{2}\|_{2\gamma})$$

for some function $C : (\mathbb{R}^+)^6 \to \mathbb{R}^+$ growing with its arguments.

Some additional comments spring from Theorem 1.1:

(1) As the reader may have noticed, only local solutions are obtained in the general case, due to the fact that our nonlinearity cannot be considered as a bounded function on the Sobolev spaces $\mathcal{B}_{\alpha,p}$. However:
- We do obtain a global solution in the case of a Hölder continuity exponent $\gamma \in (1/2, 1]$. We shall also introduce a smoothing procedure for the nonlinearity which induces a global solution at Sect. 5.
- When x is a N-dimensional Brownian motion, the identification of our solution with the one obtained by Itô integration also yields a global solution, as detailed at Sect. 6.2. As an immediate consequence of the procedure, we retrieve an original (to the best of our knowledge) continuity statement for the Brownian solution with respect to the initial condition (Corollary 6.12), which gives an idea of other possible spins-off of the rough paths approach to (3).
- The changes of variables extensively used in [19] have a nonlinear counterpart, as recently pointed out in [14]. This additional information certainly opens the door to a global solution to Eq. (4) in the next future.

(2) As mentioned before, we have sticked to the case of the Laplace operator Δ in our presentation of Eq. (4), for sake of clarity. However, our algebraic and analytic setting only relies on a set of abstract assumptions on the semigroup S_t. As we shall see at Sect. 6.3, these conditions are still met for a fairly general second order differential operator A given in divergence form.

In conclusion, the current paper has to be understood as a general approach to a rough path analysis of SPDEs, which can be generalized and improved in several different directions.

Here is how our paper is structured: In Sect. 2, we fix the general framework of our study and put together a few basic facts about fractional Sobolev spaces and the heat semigroup. Section 3 is then intended to recall the main features of algebraic integration with respect to a semigroup of operators, taken from [25]. As a first illustration of our method, we deal with the easy case of Young integration at Sect. 3.2. The first nontrivial rough case, that is a Hölder continuity exponent $\gamma \in (1/3, 1/2]$, is handled at Sect. 4. Observe that the abstract results obtained there are expressed in terms of the operators X^x, X^{xa} and X^{xx} defined at Eqs. (9) and (10). Section 5 shows that considering a smoothed version of the nonlinearity, a global solution to Eq. (4) can be constructed. Section 6 is then devoted to the application of the abstract results to concrete (stochastic) situations. The case of a standard Brownian motion is discussed at Sect. 6.2, and a few words are finally said about possible extensions of our results to more general elliptic operators at Sect. 6.3.

2 Assumptions and setting

For sake of clarity, we shall start with labelling the assumptions evoked in the introduction as far as the linear operator and the perturbation term are concerned. We also take profit of this section to state a few preliminary results that will be at the core of our method.

2.1 Assumptions

Let us first rewrite the equation in the general abstract form:

$$y_0 = \psi, \qquad dy_t = \Delta y_t \, dt + dX_t(y_t), \qquad t \in [0, T]. \tag{12}$$

All through the paper, we will stick to the framework delimited by the two following assumptions:

Assumption A We focus on the heat equation case on the whole Euclidean space $\mathbb{R}^n$, and we try to interpret and solve the equation in $L^p(\mathbb{R}^n)$, for some integer p that will be precised during the study. In this context, remember that the Laplacian operator $\Delta = \Delta_p : \mathcal{D}(\Delta_p) \subset L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ generates an analytic semigroup of contraction S, explicitly given (independently of p) by the convolution formula

$$S_t\varphi = g_t * \varphi, \quad \text{with } g_t(\xi) = \frac{2}{(2\pi t)^{n/2}} e^{-|\xi|^2/2t}. \tag{13}$$

Assumption B The perturbation term can be decomposed as

$$X_t\varphi = \sum_{i=1}^{N} f_i(\varphi)\, x_t^i, \tag{14}$$

with, for each $i = 1, \ldots, N$, $x^i : [0, T] \to \mathbb{R}$ a scalar process and $f_i : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$. Moreover, we assume that each f_i is given as a Nemytskii operator: there exists a mapping $\sigma_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ such that, for any function φ,

$$f_i(\varphi)(\xi) = \sigma_i(\xi, \varphi(\xi)). \tag{15}$$

With the above assumptions in mind, the aim of our study is to find a reasonable interpretation of the following mild formulation of (12):

$$y_t = S_t\psi + \sum_{i=1}^{N} \int_0^t S_{t-u} f_i(y_u)\, dx_u^i, \quad t \in [0, T], \tag{16}$$

where ψ is an initial condition living in a functional space that will be specified later on, and T is a finite horizon. The additional assumptions relative to the driving process x will stem from our analysis of (12) (Hypotheses 1–3), and it would be futile and non pedagogical to remove those assumptions of their context. At this point, let us just have a mind that x should morally admit some γ-Hölder regularity, $\gamma \in (0, 1)$. Some applications to concrete (stochastic) processes will anyway be provided at Sect. 6.

Let us now anticipate a little bit the next sections by introducing in a more precise way the class of vector fields that will allow a reasonable interpretation of the equation:

Definition 2.1 For $k \geq 1$, we define $\mathcal{X}_k$ as the set of vector fields f whose components can be written as in (15), for some mappings $\sigma_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ ($i = 1, \ldots, N$) such that:

(C1) σ_i is of uniformly compact support in the first variable, ie $\sigma_i(\cdot, \eta) = 0$ outside of a ball $B_{\mathbb{R}^N}(0, M)$, independently of $\eta \in \mathbb{R}$.

$(\text{C2})_k$ the following inequality holds:

$$\sup_{\xi\in\mathbb{R}^n, \eta\in\mathbb{R}} \max_{n=0,\ldots,k} |\nabla_\eta^n \sigma_i(\xi, \eta)| + \max_{n=0,\ldots,k-1} |\nabla_\xi \nabla_\eta^n \sigma_i(\xi, \eta)| < +\infty.$$

Remark 2.2 In the above definition, condition (C1) is essentially designed to make up for the noncompacity of the space setting. In particular, if σ_i is also uniformly bounded in its second variable, then f_i is uniformly bounded as a map from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Perhaps this hypothesis could be retrieved by working with weighted L^p-spaces, as in [39]. The strategy would however require a careful adaptation of the properties exhibited in the next subsection, and for sake of conciseness, we leave this analysis to future work. As far as condition $(C2)_k$ is concerned, it is a quite standard hypothesis in the context of a rough paths type procedure (see [32]).

2.2 Preliminary results

As pointed out at point (d) of the introduction, the interplay between the linear and the non-linear part of the equation will invite us to let the fractional Sobolev spaces come into the picture:

Notation 2.3 *For any $\alpha > 0$, for any $p \in \mathbb{N}^*$, we will denote by $\mathcal{B}_{\alpha,p}$ the space $(Id - \Delta)^{-\alpha}(L^p(\mathbb{R}^n))$, endowed with the norm*

$$\|\varphi\|_{\mathcal{B}_{\alpha,p}} = \|\varphi\|_{L^p(\mathbb{R}^n)} + \|(-\Delta)^{\alpha}\varphi\|_{L^p(\mathbb{R}^n)}. \tag{17}$$

Set also $\mathcal{B}_p = \mathcal{B}_{0,p} = L^p(\mathbb{R}^n)$ for any $p \in \mathbb{N}^* \cup \{\infty\}$.

The space $\mathcal{B}_{\alpha,p}$ is also referred to as the *Bessel potential* of order $(2\alpha, p)$. Adams [1] or Stein [43] gave a thorough description of those fractional Sobolev spaces. Let us indicate here the two classical properties that we will resort to in the sequel:

- *Sobolev inclusions*: If $0 \leq \mu \leq 2\alpha - \frac{n}{p}$, then the following continuous embedding holds

$$\mathcal{B}_{\alpha,p} \subset \mathcal{C}^{0,\mu}(\mathbb{R}^n), \tag{18}$$

  where $\mathcal{C}^{0,\mu}(\mathbb{R}^n)$ stands for the space of bounded, μ-Hölder functions.
- *Algebra*: If $2\alpha p > n$, then $\mathcal{B}_{\alpha,p}$ is a Banach algebra with respect to pointwise multiplication, or in other words

$$\|\varphi \cdot \psi\|_{\mathcal{B}_{\alpha,p}} \leq \|\varphi\|_{\mathcal{B}_{\alpha,p}} \|\psi\|_{\mathcal{B}_{\alpha,p}}. \tag{19}$$

The general theory of fractional powers of operators then provides us with sharp estimates for the semigroup S_t (see for instance [40] or [15]):

Proposition 2.4 *Fix a time $T > 0$. S_t satisfies the following properties:*

- *Contraction: For all $t \geq 0$, $\alpha \geq 0$, S_t is a contraction operator on $\mathcal{B}_{\alpha,p}$.*
- *Regularization: For all $t \in (0, T]$, $\alpha \geq 0$, S_t sends $\mathcal{B}_p$ on $\mathcal{B}_{\alpha,p}$ and*

$$\|S_t\varphi\|_{\mathcal{B}_{\alpha,p}} \leq c_{\alpha,T}\, t^{-\alpha} \|\varphi\|_{\mathcal{B}_p}. \tag{20}$$

- *Hölder regularity. For all $t \in (0, T]$, $\varphi \in \mathcal{B}_{\alpha,p}$,*

$$\|S_t\varphi - \varphi\|_{\mathcal{B}_p} \leq c_{\alpha,T}\, t^{\alpha} \|\varphi\|_{\mathcal{B}_{\alpha,p}}. \tag{21}$$

$$\|\Delta S_t\varphi\|_{\mathcal{B}_p} \leq c_{\alpha,T}\, t^{-1+\alpha} \|\varphi\|_{\mathcal{B}_{\alpha,p}}. \tag{22}$$

At some point of our study, the interpretation of the integral $\int_s^t S_{tu}\, dx_u^i\, f_i(y_u)$ will require a Taylor expansion of the (regular) function f_i. As a result, pointwise multiplications of elements of $\mathcal{B}_p$ appear, giving birth to elements of $\mathcal{B}_{p/k}$, $k \in \{1, \ldots, p\}$. In order to go back to the base space $\mathcal{B}_p$, we shall resort to the following additional properties of S_t, which accounts for our use of the spaces $\mathcal{B}_p$ $(p \geq 2)$ instead of the classical Hilbert space $\mathcal{B}_2$:

Proposition 2.5 *For all* $t > 0$, $k \in \{1, \ldots, p\}$, $\varphi \in \mathcal{B}_{p/k}$, *one has*

$$\|S_t\varphi\|_{\mathcal{B}_p} \leq c_{k,n} t^{-\frac{n(k-1)}{2p}} \|\varphi\|_{\mathcal{B}_{p/k}}, \tag{23}$$

$$\|\Delta S_t\varphi\|_{\mathcal{B}_p} \leq c_{k,n} t^{-1-\frac{n(k-1)}{2p}} \|\varphi\|_{\mathcal{B}_{p/k}}. \tag{24}$$

Proof Those are direct consequences of the Riesz–Thorin theorem. Indeed, for any $\varphi \in \mathcal{B}_{p/k}$,

$$\|S_t\varphi\|_{\mathcal{B}_p} \leq \|g_t * \varphi\|_{\mathcal{B}_p} \leq \|g_t\|_{\mathcal{B}_{p/(p-k+1)}} \|\varphi\|_{\mathcal{B}_{p/k}} \leq c_{k,n} t^{-\frac{n(k-1)}{2p}} \|\varphi\|_{\mathcal{B}_{p/k}}.$$

The second inequality can be proved in the same way, since $\Delta S_t\varphi = \left(\frac{dS_t}{dt}\right)\varphi = \partial_t g_t * \varphi$. □

Let us finally point out the following result of Strichartz [44], which will be one of the cornerstones of our fixed-point arguments through its immediate corollary:

Proposition 2.6 *For all* $\alpha \in (0, 1/2)$, *for all* $p > 1$, *set*

$$T_\alpha f(\xi) = \left(\int_0^1 r^{-1-4\alpha} \left[\int_{|\eta| \leq 1} |f(\xi + r\eta) - f(\xi)|\, d\eta \right]^2 dr \right)^{1/2}.$$

Then $f \in \mathcal{B}_{\alpha,p}$ *if and only if* $f \in \mathcal{B}_p$ *and* $T_\alpha f \in \mathcal{B}_p$, *and*

$$\|f\|_{\mathcal{B}_{\alpha,p}} \sim \|f\|_{\mathcal{B}_p} + \|T_\alpha f\|_{\mathcal{B}_p}.$$

Corollary 2.7 *If* $f \in \mathcal{X}_1$, *then for any* $\varphi \in \mathcal{B}_{\alpha,p}$, $f(\varphi) \in \mathcal{B}_{\alpha,p}$ *and*

$$\mathcal{N}[f(\varphi); \mathcal{B}_{\alpha,p}] \leq c_f \left\{1 + \mathcal{N}[\varphi; \mathcal{B}_{\alpha,p}]\right\}. \tag{25}$$

3 Algebraic integration associated to the heat semigroup

As in [25], our interpretation of the equation will be based on a preliminary dissection procedure that appeals to a particular coboundary operator $\hat{\delta}$, as well as its inverse $\hat{\Lambda}$. This section is meant to remind the reader with the definition and main properties of those two tools. As an illustrative example, the treatment of the Young case is also provided here.

3.1 The twisted coboundary $\hat{\delta}$

Notice that we shall work on n^{th} dimensional simplexes of $[0, T]$, which will be denoted by

$$\mathcal{S}_T^n = \big\{(s_1, \dots, s_n) \in [0, T]^n;\ s_1 \le s_2 \le \cdots s_n\big\}.$$

We will also set $\mathcal{C}_n = \mathcal{C}_n(\mathcal{S}_T^n, V)$ for the continuous n-variables functions from $\mathcal{S}_T^n$ to V, for a given Banach space V. Observe that we work on those simplexes just because the operator S_{t-u} is defined for $t \ge u$ (i.e. on $\mathcal{S}_T^2$) only.

Let us recall now two basic notations of usual algebraic integration, as explained in [20] and also recalled in [25]: we define first an coboundary operator, denoted by δ, which acts on the set $\mathcal{C}_n = \mathcal{C}_n(\mathcal{S}_T^n, V)$ of the continuous n-variables functions according to the formula:

$$\delta : \mathcal{C}_n \to \mathcal{C}_{n+1}, \quad (\delta g)_{t_1 \dots t_{n+1}} = \sum_{i=1}^{n+1} (-1)^i g_{t_1 \dots \hat{t_i} \dots t_n} \tag{26}$$

where the notation $\hat{t_i}$ means that this particular index is omitted. In this definition, V stands for any vector space. Next, a convention for products of elements of $\mathcal{C}_n$ is needed, and it is recalled in the following notation:

Notation 3.1 *If $g \in \mathcal{C}_n(\mathcal{L}(V, W))$ and $h \in \mathcal{C}_m(W)$, then the product $gh \in \mathcal{C}_{m+n-1}(W)$ is defined by the formula*

$$(gh)_{t_1 \dots t_{m+n-1}} = g_{t_1 \dots t_n} h_{t_n \dots t_{n+m-1}}.$$

In point (b) of the introduction, we (briefly) explains why the standard increment δ was not really well-suited to the study of (4). We will rather use a twisted version of δ, denoted by $\hat{\delta}$, and defined by:

Definition 3.2 For any $n \in \mathbb{N}^*$, $y \in \mathcal{C}_n(\mathcal{B}_{\alpha,p})$, for all $t_1 \le \cdots \le t_{n+1}$,

$$(\hat{\delta} y)_{t_{n+1} \dots t_1} = (\delta y)_{t_{n+1} \dots t_1} - a_{t_{n+1} t_n} y_{t_n \dots t_1}, \quad \text{with } a_{ts} = S_{t-s} - \text{Id si } s \le t. \tag{27}$$

The operator $a : (t, s) \mapsto a_{ts}$ is only defined on the simplex $\{t \ge s\}$. As a consequence, we will have to pay attention to the decreasing order of the time variables throughout our calculations below. Note that we will often resort to the notation S_{ts} for S_{t-s}, so as to get a consistent notational convention for the indexes.

The rest of this subsection is devoted to the inventory of some of those results. The associated proofs can be found in [25].

Let us start with the fundamental property:

Proposition 3.3 *The operator $\hat\delta$ satisfies the cohomological relation $\hat\delta\hat\delta = 0$. Besides, $Ker\,\hat\delta_{|\mathcal{C}_{n+1}(\mathcal{B}_{\alpha,p})} = Im\,\hat\delta_{|\mathcal{C}_n(\mathcal{B}_{\alpha,p})}$.*

Now, let us turn to a more trivial result, which will be exploited in the sequel. Remember that we use the notational convention 3.1 for time variables.

Proposition 3.4 *If $L \in \mathcal{C}_{n-1}(V)$ and $M \in \mathcal{C}_2(\mathcal{L}(V))$, then*

$$\hat\delta(ML) = (\hat\delta M)L - M(\delta L). \tag{28}$$

The following result is the equivalent of Chasles relation in the $\hat\delta$ setting. It is an obvious consequence of the multiplicative property of S.

Proposition 3.5 *Let x a differentiable process. If $y_{ts} = \int_s^t S_{tu}\, dx_u\, f_u$, then $(\hat\delta y)_{tus} = 0$ for all $s \le u \le t$.*

From an analytical point of view, the notion of Hölder-regularity of a process should be adapted to this context, and thus, we define, for any $\alpha \in [0, 1/2)$, $p \in \mathbb{N}^*$, $\kappa \in (0, 1)$,

$$\hat{\mathcal{C}}_1^\kappa(\mathcal{B}_{\alpha,p}) := \left\{ y \in \mathcal{C}_1(\mathcal{B}_{\alpha,p}) :\ \sup_{s<t} \frac{\|(\hat\delta y)_{ts}\|_{\mathcal{B}_{\alpha,p}}}{|t-s|^\kappa} < \infty \right\}. \tag{29}$$

Let us take profit of this subsection to introduce the Hölder spaces commonly used in the k-increment theory. They are the subspaces of $\mathcal{C}_1(V)$, $\mathcal{C}_2(V)$ and $\mathcal{C}_3(V)$ respectively induced by the norms (V stands for any Banach space):

$$\mathcal{N}[y; \mathcal{C}_1^\kappa(V)] := \sup_{s<t} \frac{\|(\delta y)_{ts}\|_V}{|t-s|^\kappa}, \quad \mathcal{N}[y; \mathcal{C}_2^\kappa(V)] := \sup_{s<t} \frac{\|y_{ts}\|_V}{|t-s|^\kappa},$$
$$\mathcal{N}[y; \mathcal{C}_3^{\kappa,\rho}(V)] := \sup_{s<u<t} \frac{\|y_{tus}\|_V}{|t-u|^\kappa\, |u-s|^\rho},$$
$$\mathcal{N}[y; \mathcal{C}_3^\mu(V)] := \inf \left\{ \sum_i \mathcal{N}[y^i; \mathcal{C}_3^{\kappa,\mu-\kappa}(V)] :\ y = \sum_i y^i \right\}.$$

Now let us state the main result of this subsection which allows to invert the twisted coboundary operator $\hat\delta$ by means of a map $\hat\Lambda$. This inversion operator is the convolutional analog of the sewing map Λ at the core of the standard rough paths constructions contained in [20].

Theorem 3.6 *Fix a time $T > 0$, a parameter $\kappa \ge 0$ and let $\mu > 1$. For any $h \in \mathcal{C}_3^\mu([0,T]; \mathcal{B}_{\kappa,p}) \cap Ker\,\hat\delta_{|\mathcal{C}_3(\mathcal{B}_{\kappa,p})}$, there exists a unique element*

$$\hat\Lambda h \in \cap_{\alpha\in[0,\mu)} \mathcal{C}_2^{\mu-\alpha}([0,T]; \mathcal{B}_{\kappa+\alpha,p})$$

such that $\hat\delta(\hat\Lambda h) = h$. Moreover, $\hat\Lambda h$ satisfies the following contraction property: for all $\alpha \in [0, \mu)$,

$$\mathcal{N}[\hat\Lambda h; \mathcal{C}_2^{\mu-\alpha}([0,T]; \mathcal{B}_{\kappa+\alpha,p})] \le c_{\alpha,\mu,T}\, \mathcal{N}[h; \mathcal{C}_3^\mu([0,T]; \mathcal{B}_{\kappa,p})]. \tag{30}$$

The analogy between Λ and $\hat{\Lambda}$ is made even clearer by the following result (compare with [20, Proposition 1]), which will also enable to make the link with a more classical formulation of the rough integration theory by means of Riemann sums.

Proposition 3.7 *Let $g \in \mathcal{C}_2(\mathcal{B}_{\kappa,p})$ such that $\hat{\delta} g \in \mathcal{C}_3^{\mu}(\mathcal{B}_{\kappa,p})$ with $\mu > 1$. Then the increment $\hat{\delta} f = (Id - \hat{\Lambda}\hat{\delta})g \in \mathcal{C}_2(\mathcal{B}_{\kappa,p})$ satisfies*

$$(\hat{\delta} f)_{ts} = \lim_{|\Pi_{ts}|\to 0} \sum_{(t_k)\in\Pi_{ts}} S_{tt_{k+1}} g_{t_{k+1}t_k} \quad \text{in } \mathcal{B}_{\kappa,p},$$

for all $s \le t$.

3.2 Example: the Young case

In order to illustrate in a simple setting the adaptation of the *dissection* method to the convolutional context, let us have a look in this subsection at the so-called 'Young case', which refers to the fact that only expansions at first order will be involved here. Observe that this kind of considerations has already been explored in [24] under more general hypotheses concerning the spatial regularity of the noise. We will see in Sect. 6 that the general result of Theorem 3.10 can be applied to a noise generated by a (finite-dimensional) γ-Hölder process x, with $\gamma > 1/2$. This is an improvement with respect to [25], where the unnatural condition $\gamma > 5/6$ had to be assumed. Throughout this subsection, we fix a parameter $\gamma \in (1/2, 1)$, which (morally) represents the Hölder regularity of x.

The aim here is to give an interpretation of the twisted Young integral $\int_s^t S_{tu}\, dx_u\, z_u$ in terms of δ, $\hat{\delta}$ and $\hat{\Lambda}$. To this purpose, we follow the same reasoning as in [20,25]: we assume first that x and z are smooth processes, and obtain a decomposition of the integral $\int_s^t S_{tu}\, dx_u\, z_u := \int_s^t S_{tu}\, x'_u z_u\, du$ in terms of δ and $\hat{\Lambda}$ in this particular case. This allows then to extend the notion of twisted integral to Hölder continuous signals with Hölder continuity coefficient greater than 1/2.

Thus, assume, at first, that x is real valued and regular (for instance Lipschitz, or even differentiable) in time, as well as the integrand z, and look at the decomposition

$$\int_s^t S_{tu}\, dx_u\, z_u = \left(\int_s^t S_{tu}\, dx_u \right) z_s + \int_s^t S_{tu}\, dx_u\, (\delta z)_{us}. \tag{31}$$

If we set $r_{ts} = \int_s^t S_{tv}\, dx_v\, (\delta z)_{vs}$, it is easily seen that

$$(\hat{\delta} r)_{tus} = \int_s^t S_{tv}\, dx_v\, (\delta z)_{vs} - \int_u^t S_{tv}\, dx_v\, (\delta z)_{vu} - S_{tu} \int_s^u S_{uv}\, dx_v\, (\delta z)_{vs},$$

which, using the fact that $S_{tu}S_{uv} = S_{tv}$, reduces to

$$(\hat{\delta}r)_{tus} = \left(\int_u^t S_{tv}\, dx_v\right)(\delta z)_{us}. \tag{32}$$

This first elementary step lets already emerge the object which plays the role of the a priori first order increment associated to the heat equation, namely

$$X^{x,i}_{ts} = \int_s^t S_{tv}\, dx^i_v.$$

We are then in position to invert $\hat{\delta}$ in (32) thanks to Theorem 3.6. Indeed, one easily deduces, owing to the regularity of x and z,

$$X^x(\delta z) \in \mathcal{C}_3^2(\mathcal{B}_{\alpha,p}) \quad \text{for some } \alpha \in [0, 1/2).$$

Consequently, we get

$$\int_s^t S_{tu}\, dx_u\, z_u = X^{x,i}_{ts} z^i_s + \hat{\Lambda}_{ts}\left(X^{x,i}\, \delta z^i\right). \tag{33}$$

As in the standard case algebraic integration setting in the Young setting, we now wonder if the right-hand-side of (33) remains well-defined in a less regular context:

- *From an analytical point of view*. The regularity assumption of Theorem 3.6 imposes the condition: for all $i \in \{1, \ldots, N\}$,

$$X^{x,i}\delta z^i \in \mathcal{C}_3^{\mu}(\mathcal{B}_{\alpha,p}) \quad \text{with } \alpha \in [0, 1/2) \text{ and } \mu > 1.$$

  Therefore, we shall be led to suppose that z^i is κ-Hölder (in the classical sense), with values in $\mathcal{B}_{\alpha,p}$, or in other words $z^i \in \mathcal{C}_1^{\kappa}(\mathcal{B}_{\alpha,p})$, and we will also assume that $X^{x,i} \in \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p}))$, with $\kappa + \gamma > 1$.
- *From an algebraic point of view*. We know that $\hat{\Lambda}$ is defined on the spaces $\mathcal{C}_3^{\mu}(\mathcal{B}_{\alpha,p}) \cap \operatorname{Ker} \hat{\delta}$. This constrains us to assume that $\hat{\delta}(X^{x,i}\, \delta z^i) = 0$, which, by (28), is satisfied once we admit that $\hat{\delta}X^{x,i} = 0$.

Let us record those two conditions under the abstract hypothesis:

Hypothesis 1 From x, one can build processes $X^{x,i}$ ($i \in \{1, \ldots, N\}$) of two variables such that, for all i:

- For any $\alpha \in [0, 1/2)$ such that $2\alpha p > 1$, $X^{x,i} \in \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p}))$
- The algebraic relation $\hat{\delta}X^{x,i} = 0$ is satisfied.

Remark 3.8 Actually, the additional condition $2\alpha p > 1$ could have been skipped in the latter hypothesis. We have notified it so that Hypothesis 1 meets the more general Hypothesis 2 of Sect. 4.

We are then allowed to use expression (33) for irregular integrands:

Proposition 3.9 *Under Hypothesis 1, we define, for all processes z such that $z^i \in \mathcal{C}_1^0(\mathcal{B}_{\kappa,p}) \cap \mathcal{C}_1^\kappa(\mathcal{B}_p)$, $i = 1, \dots, N$, with $\kappa < \gamma$ and $\kappa + \gamma > 1$, the integral*

$$\mathcal{J}_{ts}(\hat{d}x\, z) = X_{ts}^{x,i} z_s^i + \hat{\Lambda}_{ts}\big(X^{x,i}\, \delta z^i\big). \tag{34}$$

In that case:

- *$\mathcal{J}(\hat{d}x\, z)$ is well-defined and there exists an element $\hat{z} \in \hat{\mathcal{C}}_1^\gamma(\mathcal{B}_{\kappa,p})$ such that $\hat{\delta}\hat{z}$ is equal to $\mathcal{J}(\hat{d}x\, z)$.*
- *It holds that*

$$\mathcal{N}[\hat{z}; \hat{\mathcal{C}}_1^\gamma(\mathcal{B}_{\kappa,p})] \le c_x \left\{ \mathcal{N}[z; \mathcal{C}_1^0(\mathcal{B}_{\kappa,p})] + \mathcal{N}[z; \mathcal{C}_1^\kappa(\mathcal{B}_p)] \right\}, \tag{35}$$

with

$$c_x \le c \left\{ \mathcal{N}[X^x; \mathcal{C}_2^\gamma(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p))] + \mathcal{N}[X^x; \mathcal{C}_2^\gamma(\mathcal{L}(\mathcal{B}_{\kappa,p}, \mathcal{B}_{\kappa,p})] \right\} \tag{36}$$

- *The integral can be written as*

$$\mathcal{J}_{ts}(\hat{\delta}x\, z) = \lim_{|\Delta| \to 0} \sum_{(t_k) \in \Delta} S_{tt_{k+1}} X_{t_{k+1}t_k}^{x,i} z_{t_k}^i, \tag{37}$$

where the limit is taken over partitions $\Delta_{[s,t]}$ of the interval $[s,t]$, as their mesh tends to 0. Hence it coincides with the Young type integral $\int_s^t S_{tu}\, dx_u\, z_u$.

Proof The fact that $\mathcal{J}_{ts}(\hat{d}x\, z)$ is well defined is a direct consequence of Hypothesis 1, and the Chasles relation $\hat{\delta}\mathcal{J}(\hat{d}x\, z)$, which accounts for the existence of $\hat{z}$, can be shown by straightforward computations using (28).

For the second point, notice that, thanks to Hypothesis 1, one has

$$\begin{aligned} &\mathcal{N}[\mathcal{J}(\hat{d}x\, z); \mathcal{C}_2^\gamma(\mathcal{B}_{\kappa,p})] \\ &\quad \le \mathcal{N}[X^{x,i}; \mathcal{C}_2^\gamma(\mathcal{L}(\mathcal{B}_{\kappa,p}, \mathcal{B}_{\kappa,p}))]\, \mathcal{N}[z^i; \mathcal{C}_1^0(\mathcal{B}_{\kappa,p})] + \mathcal{N}[\hat{\Lambda}(X^{x,i}\, \delta z^i); \mathcal{C}_2^\gamma(\mathcal{B}_{\kappa,p})], \end{aligned}$$

since $X^{x,i}\, \delta z^i \in \mathcal{C}_3^{\gamma+\kappa}(\mathcal{B}_p)$. By the contraction property (30) of $\hat{\Lambda}$, we then deduce

$$\mathcal{N}[\hat{\Lambda}(X^{x,i}\, \delta z^i); \mathcal{C}_2^\gamma(\mathcal{B}_{\kappa,p})] \le c\, \mathcal{N}[X^{x,i}; \mathcal{C}_2^\gamma(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p))]\, \mathcal{N}[z^i; \mathcal{C}_1^\kappa(\mathcal{B}_p)],$$

which completes the proof of (35). According to Proposition 3.7, (37) is a consequence of the reformulation

$$\mathcal{J}(\hat{d}x\, z) = (\mathrm{Id} - \hat{\Lambda}\hat{\delta})(X^{x,i} z^i). \tag{38}$$

□

Using the above formalism, we can show the existence and uniqueness of a global solution to the equation. In the following statement, remember that the spaces $\mathcal{X}_k$ have been introduced in Definition 2.1.

Theorem 3.10 *Assume Hypothesis 1 with $\gamma > 1/2$, and assume also that $f = (f_1, \ldots, f_N)$ with $f_i \in \mathcal{X}_2$ for $i = 1, \ldots, N$. For any $\kappa < \gamma$ such that $\gamma + \kappa > 1$ and $2\kappa p > n$, consider the space $\hat{\mathcal{C}}_1^{0,\kappa}([0,T], \mathcal{B}_{\kappa,p}) = \mathcal{C}_1^0([0,T], \mathcal{B}_{\kappa,p}) \cap \hat{\mathcal{C}}_1^{\kappa}([0,T], \mathcal{B}_{\kappa,p})$, provided with the norm*

$$\mathcal{N}[.; \hat{\mathcal{C}}_1^{0,\kappa}([0,T], \mathcal{B}_{\kappa,p})] = \mathcal{N}[.; \mathcal{C}_1^0([0,T], \mathcal{B}_{\kappa,p})] + \mathcal{N}[.; \hat{\mathcal{C}}_1^{\kappa}([0,T], \mathcal{B}_{\kappa,p})].$$

Then the infinite-dimensional system

$$(\hat{\delta} y)_{ts} = \mathcal{J}_{ts}(\hat{d}x\, f(y)), \quad y_0 = \psi \in \mathcal{B}_{\kappa,p}, \tag{39}$$

interpreted with Proposition 3.9, admits a unique global solution in $\hat{\mathcal{C}}_1^{0,\kappa}([0,T], \mathcal{B}_{\kappa,p})$. Besides, the Itô map $(\psi, X^{x,i}) \mapsto y$, where y is the unique solution of (39), is locally Lipschitz.

Remark 3.11 In the last statement, we consider the operators $X^{x,i}$ as elements of the incremental space $\mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)) \cap \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\kappa,p}, \mathcal{B}_{\kappa,p}))$. The regularity of the Itô map with respect to $X^{x,i}$ is then relative to the norm

$$\mathcal{N}[.; \mathcal{CL}^{\kappa,\gamma,p}] = \mathcal{N}[.; \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p))] + \mathcal{N}[.; \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\kappa,p}, \mathcal{B}_{\kappa,p}))].$$

The following notation will be used in the proof, and also in the sequel of the paper.

Notation 3.12 *Let A, B be two positive quantities, and a a parameter lying in a certain vector space V. We say that $A \lesssim_a B$ if there exists a positive constant c_a depending on a such that $A \le c_a B$. When we don't want to specify the dependence on a, we just write $A \lesssim B$. Notice also that the value of the constants c or c_a in our computations can change from line to line, throughout the paper.*

Proof It is a classical fixed-point argument. We will only prove the existence and uniqueness of a local solution. The reasoning which enables to extend the local solution into a solution on the whole interval $[0, T]$ is standard; some details about the general procedure can be found in [20] (in a slightly different context).

We consider an interval $I = [0, T_*]$ with T_* a time that may change during the proof, and the application $\Gamma : \hat{\mathcal{C}}_{1,\psi}^{0,\kappa}(I, \mathcal{B}_{\kappa,p}) \to \hat{\mathcal{C}}_{1,\psi}^{0,\kappa}(I, \mathcal{B}_{\kappa,p})$ defined by $\Gamma(y)_0 = \psi$ and $(\hat{\delta}\Gamma(y))_{ts} = \mathcal{J}_{ts}(\hat{d}x\, f(y))$.

Invariance of a ball. Let $y \in \hat{\mathcal{C}}^{0,\kappa}_{1,\psi}(I, \mathcal{B}_{\kappa,p})$ and $z = \Gamma(y)$. By (35), we know that

$$\mathcal{N}[z; \hat{\mathcal{C}}^{\kappa}_1(I, \mathcal{B}_{\kappa,p})] \le c_x\, |I|^{\gamma-\kappa} \left\{ \mathcal{N}[f_i(y); \mathcal{C}^{\kappa}_1(I, \mathcal{B}_p)] + \mathcal{N}[f_i(y); \mathcal{C}^{0}_1(\mathcal{B}_{\kappa,p})] \right\}. \tag{40}$$

Recalling our convention in Notation 3.12, the assumption $f_i \in \mathcal{X}_1$ is enough to guarantee that the following bounds holds for f_i: $\mathcal{N}[f_i(\varphi) - f_i(\psi); \mathcal{B}_p] \lesssim_f \mathcal{N}[\varphi - \psi; \mathcal{B}_p]$ and $\mathcal{N}[f_i(\varphi); \mathcal{B}_{\kappa,p}] \lesssim_f 1 + \mathcal{N}[\varphi; \mathcal{B}_{\kappa,p}]$ (see Corollary 2.7) for arbitrary test functions φ, ψ. So we have

$$\begin{aligned} \mathcal{N}[f_i(y); \mathcal{C}^{\kappa}_1(I, \mathcal{B}_p)] &\lesssim_f \mathcal{N}[y; \mathcal{C}^{\kappa}_1(I, \mathcal{B}_p)] \\ &\lesssim_f \mathcal{N}[y; \mathcal{C}^{0}_1(I, \mathcal{B}_{\kappa,p})] + \mathcal{N}[y; \hat{\mathcal{C}}^{\kappa}_1(I, \mathcal{B}_{\kappa,p})] \\ &\lesssim_f \mathcal{N}[y; \hat{\mathcal{C}}^{0,\kappa}_1(I, \mathcal{B}_{\kappa,p})], \end{aligned}$$

where, to get the second inequality, we have used the property (21) of the semigroup. We get also $\mathcal{N}[f_i(y); \mathcal{C}^{0}_1(\mathcal{B}_{\kappa,p})] \lesssim_f 1 + \mathcal{N}[y; \mathcal{C}^{0}_1(\mathcal{B}_{\kappa,p})]$, which, going back to (40), leads to

$$\mathcal{N}[z; \hat{\mathcal{C}}^{\kappa}_1(I, \mathcal{B}_{\kappa,p})] \lesssim_{x,f} |I|^{\gamma-\kappa} \left\{ 1 + \mathcal{N}[y; \hat{\mathcal{C}}^{0,\kappa}_1(I, \mathcal{B}_{\kappa,p})] \right\}.$$

Besides, $z_s = (\hat{\delta} z)_{s0} + S_s \psi$, hence, since S_s is a contraction operator on $\mathcal{B}_{\kappa,p}$,

$$\mathcal{N}[z; \mathcal{C}^{0}_1(I, \mathcal{B}_{\kappa,p})] \le |I|^{\kappa}\, \mathcal{N}[z; \hat{\mathcal{C}}^{\kappa}_1(I, \mathcal{B}_{\kappa,p})] + \|\psi\|_{\mathcal{B}_{\kappa,p}}.$$

Finally,

$$\mathcal{N}[z; \hat{\mathcal{C}}^{0,\kappa}_1(I, \mathcal{B}_{\kappa,p})] \le \|\psi\|_{\mathcal{B}_{\kappa,p}} + c_x\, |I|^{\gamma-\kappa} \left\{ 1 + \mathcal{N}[y; \hat{\mathcal{C}}^{0,\kappa}_1(I, \mathcal{B}_{\kappa,p})] \right\}.$$

Then we choose $I = [0, T_1]$ such that $c_x T_1^{\gamma-\kappa} \le \frac{1}{2}$ to get the invariance by Γ of the balls

$$B^R_{T_0,\psi} = \{ y \in \hat{\mathcal{C}}^{0,\kappa}_1([0, T_0], \mathcal{B}_{\kappa,p}) : \ y_0 = \psi, \quad \mathcal{N}[y; \hat{\mathcal{C}}^{0,\kappa}_1([0, T_0], \mathcal{B}_{\kappa,p})] \le R \},$$

for any $T_0 \le T_1$, with (for instance) $R = 1 + 2\|\psi\|_{\mathcal{B}_{\kappa,p}}$.
Contraction property. Let $y, \tilde{y} \in \hat{\mathcal{C}}^{0,\kappa}_{1,\psi}(I, \mathcal{B}_{\kappa,p})$ and $z = \Gamma(y)$, $\tilde{z} = \Gamma(\tilde{y})$. By (35),

$$\begin{aligned} &\mathcal{N}[z - \tilde{z}; \hat{\mathcal{C}}^{\kappa}_1(\mathcal{B}_{\kappa,p})] \\ &\quad \le c_x\, |I|^{\gamma-\kappa} \left\{ \mathcal{N}[f_i(y) - f_i(\tilde{y}); \mathcal{C}^{0}_1(\mathcal{B}_{\kappa,p})] + \mathcal{N}[f_i(y) - f_i(\tilde{y}); \mathcal{C}^{\kappa}_1(\mathcal{B}_p)] \right\}. \end{aligned} \tag{41}$$

In order to estimate the Hölder norm $\mathcal{N}[f_i(y)-f_i(\tilde{y});\mathcal{C}_1^{\kappa}(\mathcal{B}_p)]$, we rely on the decomposition

$$\begin{aligned}
&\sigma_i(\xi, y_t(\xi)) - \sigma_i(\xi, \tilde{y}_t(\xi)) - \sigma_i(\xi, y_s(\xi)) + \sigma_i(\xi, \tilde{y}_s(\xi)) \\
&\quad = \delta(y-\tilde{y})_{ts}(\xi) \int_0^1 dr\, \sigma_i'(\xi, y_s(\xi) + r(\delta y)_{ts}(\xi)) \\
&\qquad + (\delta\tilde{y})_{ts}(\xi) \int_0^1 dr \left\{\sigma_i'(\xi, y_s(\xi) + r(\delta y)_{ts}(\xi)) - \sigma_i'(\xi, \tilde{y}_s(\xi) + r(\delta\tilde{y})_{ts}(\xi))\right\}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathcal{N}[f_i(y) - f_i(\tilde{y}); \mathcal{C}_1^{\kappa}(\mathcal{B}_p)] \\
&\quad \le c_f \left\{\mathcal{N}[y-\tilde{y}; \hat{\mathcal{C}}_1^{0,\kappa}(\mathcal{B}_{\kappa,p})] + \mathcal{N}[\tilde{y}; \hat{\mathcal{C}}_1^{0,\kappa}(\mathcal{B}_{\kappa,p})]\,\mathcal{N}[y-\tilde{y}; \mathcal{C}_1^{0}(\mathcal{B}_{\infty})]\right\}.
\end{aligned}$$

Remember that we have assumed that $2\kappa p > n$, so that, by the Sobolev continuous inclusion $\mathcal{B}_{\kappa,p} \subset \mathcal{B}_\infty$, $\mathcal{N}[y-\tilde{y};\mathcal{C}_1^0(\mathcal{B}_\infty)] \le \mathcal{N}[y-\tilde{y};\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]$ and as a result

$$\mathcal{N}[f_i(y) - f_i(\tilde{y}); \mathcal{C}_1^{\kappa}(\mathcal{B}_p)] \le c\,\mathcal{N}[y-\tilde{y}; \hat{\mathcal{C}}_1^{0,\kappa}(\mathcal{B}_{\kappa,p})] \left\{1 + \mathcal{N}[\tilde{y}; \hat{\mathcal{C}}_1^{0,\kappa}(\mathcal{B}_{\kappa,p})]\right\}.$$

The same kind of argument easily leads to

$$\begin{aligned}
&\mathcal{N}[f_i(y) - f_i(\tilde{y}); \mathcal{C}_1^{0}(\mathcal{B}_{\kappa,p})] \\
&\quad \le c\,\mathcal{N}[y-\tilde{y}; \hat{\mathcal{C}}_1^{0,\kappa}(\mathcal{B}_{\kappa,p})] \left\{1 + \mathcal{N}[y; \hat{\mathcal{C}}_1^{0,\kappa}(\mathcal{B}_{\kappa,p})] + \mathcal{N}[\tilde{y}; \hat{\mathcal{C}}_1^{0,\kappa}(\mathcal{B}_{\kappa,p})]\right\},
\end{aligned}$$

The last two estimations, together with (41), provide a control of $\mathcal{N}[z-\tilde{z};\hat{\mathcal{C}}_1^{\kappa}(\mathcal{B}_{\kappa,p})]$ in terms of $y, \tilde{y}$. Moreover, as $z_0 = \tilde{z}_0 = \psi$,

$$\mathcal{N}[z-\tilde{z}; \mathcal{C}_1^{0}(\mathcal{B}_{\kappa,p})] \le |I|^{\kappa}\,\mathcal{N}[z-\tilde{z}; \hat{\mathcal{C}}_1^{\kappa}(\mathcal{B}_{\kappa,p})].$$

Now, if $y, \tilde{y}$ both belong to one of the invariant balls $B^R_{T_0,\psi}$, with $T_0 \le T_1$, the above results give

$$\mathcal{N}[z-\tilde{z}; \hat{\mathcal{C}}_1^{0,\kappa}([0,T_0], \mathcal{B}_{\kappa,p})] \le c_x T_0^{\gamma-\kappa}\{1+2R\}\,\mathcal{N}[y-\tilde{y}; \hat{\mathcal{C}}_1^{0,\kappa}([0,T_0], \mathcal{B}_{\kappa,p})].$$

It only remains to pick $T_0 \le T_1$ such that $c_x T_0^{\gamma-\kappa}\{1+2R\} \le \frac{1}{2}$, and we get the contraction property of the application $\Gamma : B^R_{T_0,\psi} \to B^R_{T_0,\psi}$. This statement obviously completes the proof of the existence and uniqueness of a solution to (39) defined on $[0, T_0]$. □

Remark 3.13 When the driving process x is smooth, X^x is implicitly defined by the (Lebesgue) integral $\int_s^t S_{tu}\, dx_u = \int_s^t S_{tu}\, x'_u\, du$ and the above construction has been done in such a way that the solution y given by Theorem 3.10 coincides with the (usual) solution of the mild form of (12):

$$y_0 = \psi \in \mathcal{B}_{\kappa,p}, \quad y_t = S_t\psi + \int_0^t S_{tu}\, f(y_u)\, x'_u\, du.$$

In this case, it may be worth noticing that if in addition the initial condition ψ belongs to the domain $\mathcal{B}_{1,p}$ of Δ_p, then y is also the strong solution of (12). This is a consequence of [40, Theorem 6.1.6], owing to the Lipschitz continuity (in both variables) of the mapping $(u, \varphi) \mapsto f(\varphi)\, x'_u$ from $[0, T] \times \mathcal{B}_p$ to $\mathcal{B}_p$.

4 Rough case

The aim now is to go one step further than the Young case. We would like to conceive more sophisticated developments of the integral so as to cope with a γ-Hölder driving process, with $\gamma \in (1/3, 1/2)$.

4.1 Heuristic considerations

The strategy to give a (reasonable) sense to the integral $\int_s^t S_{tu}\, dx^i_u\, f_i(y_u)$ will be largely inspired by the reasoning followed for the standard integral $\int_s^t y_u\, dx_u$, explained in [20,25]. Thus, let us suppose at first that the process x is differentiable, and divide the procedure to reach a suitable decomposition of the integral into two steps:

- Identify the space $\mathcal{Q}$ of controlled processes which will accommodate the solution of the system.
- Decompose $\int_s^t S_{tu}\, dx^i_u\, f_i(y_u)$ as an element of $\mathcal{Q}$ when y belongs itself to $\mathcal{Q}$, until we get an expression likely to remain meaningful if x is less regular.

This heuristic reasoning essentially aims at identifying the algebraic structures which will come into play. The details concerning the analytical conditions will be checked a posteriori. Remember that Assumption B still prevails, which means that the noisy nonlinearity is given by

$$X_t(\varphi) = \sum_{i=1}^{N} x^i_t\, f_i(\varphi), \quad \text{with} \quad f_i(\varphi)(\xi) = \sigma_i(\xi, \varphi(\xi)),$$

and we shall see that σ_i has to be considered as an element of $\mathcal{X}_2$ (according to Definition 2.1).

Step 1: Identification of the controlled processes. The first elementary decomposition still consists in:

$$\int_s^t S_{tu}\, dx^i_u\, f_i(y_u) = \left(\int_s^t S_{tu}\, dx^i_u\right) f_i(y_s) + \int_s^t S_{tu}\, dx^i_u\, \delta(f_i(y))_{us}. \tag{42}$$

It is then natural to think that the potential solution of the system is to belong to a space structured by the relation

$$(\hat{\delta} y)_{ts} = \left(\int_s^t S_{tu}\, dx_u^i \right) y_s^{x,i} + y_{ts}^{\sharp},$$

with $y^{\sharp}$ admitting a Hölder regularity twice higher than y. For the solution itself, we would have $y_s^{x,i} = f_i(y_s)$, $y_{ts}^{\sharp} = \int_s^t S_{tu}\, dx_u^i\, \delta(f_i(y))_{us}$. Hence the potential algebraic structure of the controlled processes

$$\mathcal{Q} = \{y : \hat{\delta} y = X_{ts}^{x,i} y_s^{x,i} + y_{ts}^{\sharp}\}, \quad \text{with } X_{ts}^{x,i} = \int_s^t S_{tu}\, dx_u^i.$$

Remember that the latter operator satisfies the algebraic relation

$$\hat{\delta} X^{x,i} = 0. \tag{43}$$

Besides, it will turn out useful in the sequel to write $X^{x,i}$ as

$$X^{x,i} = X^{ax,i} + \delta x^i, \quad \text{with } X_{ts}^{ax,i} = \int_s^t a_{tv}\, dx_v^i. \tag{44}$$

Morally, $X^{ax,i}$ admits a higher Hölder regularity than x owing to the property (21) of the semigroup. We will go back over the usefulness of this trivial decomposition in Remark 4.3. In the following we will sometimes omit the vector indexes $i, j, \ldots$ whenever the contractions are obvious.

Step 2: Decomposition of $\int_s^t S_{tu}\, dx_u\, f_i(y_u)$ when $y \in \mathcal{Q}$. Going back to expression (42), we see that it is more exactly the integral $\int_s^t S_{tu}\, dx_u\, \delta(f_i(y))_{us}$ that remains to be dissected when $y \in \mathcal{Q}$, that is to say when the $\hat{\delta}$-increment of y can be written as $(\hat{\delta} y)_{ts} = X_{ts}^{x,i} y_s^{x,i} + y_{ts}^{\sharp}$. To this purpose, let us introduce a new notation which will appear in many of our future computations:

Notation 4.1 *For any $f \in \mathcal{X}_2$ (see Definition 2.1), we set*

$$[f'(\varphi)](\xi) = \nabla_2 \sigma(\xi, \varphi(\xi)),$$

where ∇_2 stands for the derivative with respect to the second variable. The function f' is understood as a mapping from $\mathcal{B}_p$ to $\mathcal{B}_p$ for any $p \geq 1$.

Using this notational convention, notice that

$$\begin{aligned}\delta(f_i(y))_{ts} &= (\delta y)_{ts} \cdot f_i'(y_s) + \int_0^1 dr\, \left[f_i'(y_s + r(\delta y)_{ts}) - f_i'(y_s)\right] \cdot (\delta y)_{ts} \\ &= (a_{ts} y_s) \cdot f_i'(y_s) + (\hat{\delta} y)_{ts} \cdot f_i'(y_s) + f_i(y)^{\sharp,1}_{ts} \\ &= (a_{ts} y_s) \cdot f_i'(y_s) + (X^{x,j}_{ts} y^{x,j}_s) \cdot f_i'(y_s) + f_i(y)^{\sharp,1}_{ts} + f_i(y)^{\sharp,2}_{ts} \\ &= (a_{ts} y_s) \cdot f_i'(y_s) + (\delta x^j)_{ts} \cdot y^{x,j}_s \cdot f_i'(y_s) + f_i(y)^{\sharp,1}_{ts} + f_i(y)^{\sharp,2}_{ts} + f_i(y)^{\sharp,3}_{ts},\end{aligned} \tag{45}$$

where we have successively introduced the notations

$$f_i(y)^{\sharp,1}_{ts} = \int_0^1 dr\, \left[f_i'(y_s + r(\delta y)_{ts}) - f_i'(y_s)\right] \cdot (\delta y)_{ts}, \quad f_i(y)^{\sharp,2}_{ts} = y^{\sharp}_{ts} \cdot f_i'(y_s), \tag{46}$$

$$f_i(y)^{\sharp,3}_{ts} = (X^{ax,j}_{ts} y^{x,j}_s) \cdot f_i'(y_s). \tag{47}$$

Observe that, in the course of those computations, we have used some additional conventions that we make explicit for further use:

Notation 4.2 *Let φ, ψ be two elements of $\mathcal{B}_p$. Then $\varphi \cdot \psi$ is the element of $\mathcal{B}_{p/2}$ defined by the pointwise multiplication $[\varphi \cdot \psi](\xi) = \varphi(\xi)\, \psi(\xi)$. If we assume furthermore that M, N are two elements of $\mathcal{L}(\mathcal{B}_p; \mathcal{B}_p)$, then the bilinear form $B(M, N)$ is defined as:*

$$B(M, N) : \mathcal{B}_p \times \mathcal{B}_p \to \mathcal{B}_{p/2}, \quad (\varphi, \psi) \mapsto [B(M, N)](\varphi, \psi) = M(\varphi) \cdot N(\psi).$$

We will also make use of the standard product notation

$$M \times N : \mathcal{B}_p \times \mathcal{B}_p \to \mathcal{B}_p \times \mathcal{B}_p, \quad (\varphi, \psi) \mapsto [M \times N](\varphi, \psi) = (M(\varphi), N(\psi)).$$

With this convention in mind, the algebraic decomposition (45) of $f_i(y)$ can now be read as:

$$\delta(f_i(y))_{ts} = B(a_{ts}, \mathrm{Id})(y, f_i'(y))_s + (\delta x^j)_{ts} \cdot y^{x,j}_s \cdot f_i'(y_s) + f_i(y)^{\sharp}_{ts}. \tag{48}$$

If we analyze the regularity of the terms of this expression, it seems reasonable to consider the first two terms as elements of order one and $f_i(y)^{\sharp}$ as an element of order two. Let us make two comments about this intuition:

(a) To assert that $B(a_{ts}, \mathrm{Id})(y, f_i'(y))_s$ admits a strictly positive Hölder regularity, otherwise stated to retrieve increments $|t-s|^{\alpha}$ from the operator a_{ts}, we must use the property (21) of the semigroup. It means in particular that a change of space will occur: if $y_s \in \mathcal{B}_{\alpha,p}$, then $B(a_{ts}, \mathrm{Id})(y, f_i'(y))_s$ will be estimated as an element of $\mathcal{B}_p$. This remark also holds for $f_i(y)^{\sharp,3}_{ts} = (X^{ax,j}_{ts} y^{x,j}_s) \cdot f_i'(y_s)$.

(b) The term $f_i(y)^{\sharp,1}$ is considered as a second order element insofar as it is easily (pointwise) estimated by (a constant times) $|(\delta y)_{ts}|^2$. However, as far as the spatial regularity is concerned, this supposes that $f_i(y)^{\sharp,1}$ has to be seen as an element of $\mathcal{B}_{p/2}$, if $y \in \mathcal{B}_p$. To go back to the base space $\mathcal{B}_p$, we shall use the regularization property (23) of the semigroup, through the operator X^x (see (60) in Hypothesis 2).

Now, inject decomposition (48) into (42) to obtain

$$\int_s^t S_{tu}\, dx_u^i\, f_i(y_u) = X^{x,i}_{ts}\, f_i(y)_s + X^{xa,i}_{ts}(y, f_i'(y))_s + X^{xx,ij}_{ts}(y^{x,j} \cdot f_i'(y))_s + \int_s^t S_{tu}\, dx_u^i\, f_i(y)^{\sharp}_{us}, \tag{49}$$

where we have introduced the following operators of order two (which act on some spaces that will be detailed later on):

$$X^{xa,i}_{ts} = \int_s^t S_{tu}\, dx_u^i\, B(a_{us}, \mathrm{Id}) \quad \text{and} \quad X^{xx,ij}_{ts} = \int_s^t S_{tu}\, dx_u^i\, (\delta x^j)_{us}. \tag{50}$$

A little more specifically, those operators act on couples (φ, ψ) in some Sobolev type spaces, and

$$X^{xa,i}_{ts}(\varphi, \psi) = \int_s^t S_{tu}\, dx_u^i\, [a_{us}(\varphi) \cdot \psi] \quad \text{and} \quad X^{xx,ij}_{ts}(\varphi) = \int_s^t S_{tu}\, dx_u^i\, (\delta x^j)_{us}\, [\varphi].$$

Then, since we have assumed that $f_i(y)^\sharp$ admitted a "double" regularity, we can see the residual term $r_{ts} = \int_s^t S_{tu}\, dx_u^i\, f_i(y_u)^\sharp$ as a third order element, whose regularity is expected to be greater than 1 as soon as the Hölder regularity of x is greater than $1/3$. Thus, we are in the same position as in (31), and just as in the latter situation, r will be interpreted thanks to $\hat{\Lambda}$.

In order to compute $\hat{\delta} r$, rewrite r using (49):

$$r_{ts} = \int_s^t S_{tu}\, dx_u^i\, f_i(y_u) - X^{x,i}_{ts}\, f_i(y_s) - X^{xa,i}_{ts}(y, f_i'(y))_s - X^{xx,ij}_{ts}(y^{x,j} \cdot f_i'(y))_s.$$

Therefore, with the help of the algebraic formula (28), we get

$$(\hat{\delta} r)_{tus} = X^{x,i}_{tu}\delta(f_i(y))_{us} - (\hat{\delta} X^{xa,i})_{tus}(y, f_i'(y))_s + X^{xa,i}_{tu}\delta(y, f_i'(y))_{us}$$
$$-(\hat{\delta} X^{xx,ij})_{tus}(y^{x,j}\cdot f_i'(y))_s + X^{xx,ij}_{tu}\delta(y^{x,j}\cdot f_i'(y))_{us}.$$

Going back to the very definition of $X^{xa,i}$ and $X^{xx,ij}$, it is quite easy to show that the following relations are satisfied whenever x is a smooth function:

$$(\hat{\delta} X^{xa,i})_{tus} = X^{xa,i}_{tu}(a_{us}\times \mathrm{Id}) + X^{x,i}_{tu}B(a_{us}, \mathrm{Id}), \tag{51}$$

$$(\hat{\delta} X^{xx,ij})_{tus} = X^{x,i}_{tu}(\delta x^j)_{us}. \tag{52}$$

By combining these two relations together with (48), we deduce

$$(\hat{\delta} r)_{tus} = X^{x,i}_{tu}(f_i(y)^{\sharp}_{us}) + X^{xa,i}_{tu}((\hat{\delta} y)_{us}, f_i'(y_s)) + X^{xa,i}_{tu}(y_u, \delta(f_i'(y))_{us})$$
$$+X^{xx,ij}_{ts}\delta(y^{x,j}\cdot f_i'(y))_{us} := J_{tus}. \tag{53}$$

All the terms of this decomposition are (morally) of order three. Now, remember that we wish to tackle the case $3\gamma > 1$, so that it seems actually wise to invert $\hat{\delta}$ at this point, and we get

$$\int_s^t S_{tu}\, dx^i_u\, f_i(y_u) = X^{x,i}_{ts}\, f_i(y_s)$$
$$+X^{xa,i}_{ts}(y, f_i'(y))_s + X^{xx,ij}_{ts}(y^{x,j}\cdot f_i'(y))_s + \hat{\Lambda}_{ts}(J), \tag{54}$$

where J_{tus} is given by (53). Notice once again that we have obtained a decomposition valid for some smooth functions x and y, but this decomposition can now be extended to an irregular situation up to $\gamma > 1/3$.

In a natural way, we will use (54) as the definition of the integral in the prescribed context of a γ-Hölder process with $\gamma > 1/3$. To conclude this heuristic reasoning, let us summarize the different hypotheses we have (roughly) raised during the procedure:

- The process x generates four operators X^x, X^{ax}, X^{xa} and X^{xx}, which satisfy the algebraic relations (43), (51) and (52). As for the Hölder regularity of those operators, X^x admits the same regularity as x, X^{xx} twice the regularity of x, just as X^{ax} and X^{xa} (even if one must change the space one works with, according to the above point (a)).
- The increments $(\hat{\delta} y)_{ts}$ can be decomposed as $(\hat{\delta} y)_{ts} = X^x_{ts}y^x_s + y^{\sharp}_{ts}$, where $y^{\sharp}$ is twice more regular than y. Besides, according to (a) again, the process y must evolve in a space $\mathcal{B}_{\alpha,p}$, with $\alpha > 0$. These remarks will give birth to the spaces $\mathcal{Q}^{\kappa}_{\alpha,p}$.
- The functions f_i belong to $\mathcal{X}_2$.

Remark 4.3 If one has a look at the constructions established in [25], it seems more natural, at first sight, to search for a decomposition of the integral based on the (twisted) iterated integral

$$\tilde{X}_{ts}^{xx,ij} = \int_s^t S_{tu}\, dx_u^i\, B(X_{us}^{x,j}, \mathrm{Id}) = \int_s^t S_{tu}\, dx_u^i\, B\left(\int_s^u S_{uv}\, dx_v^j, \mathrm{Id}\right), \tag{55}$$

rather than on the area X_{ts}^{xx} we have introduced in (50). In a way, the definition of $\tilde{X}_{ts}^{xx}$ is actually more consistent with the general iteration scheme of the rough path procedure. Nevertheless, when it comes to applying the results to a fBm x (with Hurst index $H \in (1/3, 1/2)$) for instance, it seems difficult to justify the existence of the iterated integral (55). According to our computations, this difficulty is due to a lack of regularity for the term S_{uv} in (55). Indeed, if one refers to [2], the definition of the integral would require a condition like

$$\mathcal{N}[S_{uv} - S_{uu}; \mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p})] \lesssim |u - v|^{\nu},$$

for some $\nu > 0$, but this kind of inequality cannot be satisfied in this general form, since the Hölder property (21) of the semigroup requires a change of space. This is why we have turned to a formulation with X_{ts}^{xx}, which is made possible by the introduction of the operator X_{ts}^{ax} (defined by (44)) in decomposition (45). As we shall see in Sect. 6, the definition and the estimation of the regularity of X^{xx} are much simpler, since this can be done by means of an integration by parts argument.

4.2 Definition of the integral

In this subsection, we will only make the previous assumptions and constructions more formal. From now on, we fix a coefficient $\gamma > 1/3$, which (morally) represents the Hölder regularity of the driving process x. The definition of the rough path above x associated to the heat equation is then the following:

Hypothesis 2 We assume that the process x allows to define operators $X^{x,i}$, $X^{ax,i}$, $X^{xa,i}$, $X^{xx,ij}$ ($i, j \in \{1, \ldots, N\}$), such that, recalling our Notation 4.2:

(H1) From an algebraic point of view:

$$\hat{\delta} X^{x,i} = 0 \tag{56}$$

$$X^{x,i} = X^{ax,i} + \delta x^i \tag{57}$$

$$\hat{\delta} X^{xa,i} = X^{xa,i}(a \times \mathrm{Id}) + X^{x,i} B(a, \mathrm{Id}) \tag{58}$$

$$\hat{\delta} X^{xx,ij} = X^{x,i}(\delta x^j). \tag{59}$$

(H2) From an analytical point of view: if $2\alpha p > n$, then

$$X^{x,i} \in \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)) \cap \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p})) \cap \mathcal{C}_2^{\gamma - n/(2p)}(\mathcal{L}(\mathcal{B}_{p/2}, \mathcal{B}_p)) \tag{60}$$

$$X^{ax,i} \in \mathcal{C}_2^{\gamma+\alpha}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_p)) \tag{61}$$

$$X^{xa,i} \in \mathcal{C}_2^{\gamma+\alpha-n/(2p)}(\mathcal{L}(\mathcal{B}_{\alpha,p} \times \mathcal{B}_p, \mathcal{B}_p)) \cap \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p} \times \mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p})) \tag{62}$$

$$X^{xx,ij} \in \mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)) \cap \mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_{\alpha,p})) \cap \mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p}, \mathcal{B}_p)). \tag{63}$$

We will denote by $\mathbf{X} = (X^x, X^{ax}, X^{xa}, X^{xx})$ the path so defined. $\mathbf{X}$ belongs to a product of operators spaces, denoted by $\mathcal{CL}^{\gamma,\kappa,p}$, and furnished with a natural norm build with the norms of each space.

The formal definition of controlled process takes the following form:

Definition 4.4 For all $\alpha \in (0, 1/2)$, $\kappa \in (0, 1)$, we define

$$\hat{\mathcal{Q}}^{\kappa}_{\alpha,p} = \hat{\mathcal{Q}}^{\kappa}_{\alpha,p}([0,T]) = \Big\{ y \in \hat{\mathcal{C}}_1^{\kappa}([0,T], \mathcal{B}_{\alpha,p}) : (\hat{\delta} y)_{ts} = X^{x,i}_{ts} y^{x,i}_s + y^{\sharp}_{ts},$$
$$y^{x,i} \in \mathcal{C}_1^0([0,T], \mathcal{B}_{\alpha,p}) \cap \mathcal{C}_1^{\kappa}([0,T], \mathcal{B}_p),\ y^{\sharp} \in \mathcal{C}_2^{\gamma}([0,T], \mathcal{B}_{\alpha,p}) \cap \mathcal{C}_2^{2\kappa}([0,T], \mathcal{B}_p) \Big\}.$$

We will call $\hat{\mathcal{Q}}^{\kappa}_{\alpha,p}$ the space of κ-controlled processes of $\mathcal{B}_{\alpha,p}$, together with the norm

$$\mathcal{N}[y; \hat{\mathcal{Q}}^{\kappa}_{\alpha,p}] = \mathcal{N}[y; \hat{\mathcal{C}}_1^{\kappa}(\mathcal{B}_{\alpha,p})] + \sum_{i=1}^{N} \Big\{ \mathcal{N}[y^{x,i}; \mathcal{C}_1^0(\mathcal{B}_{\alpha,p})] + \mathcal{N}[y^{x,i}; \mathcal{C}_1^{\kappa}(\mathcal{B}_p)] \Big\}$$
$$+ \mathcal{N}[y^{\sharp}; \mathcal{C}_2^{\gamma}(\mathcal{B}_{\alpha,p})] + \mathcal{N}[y^{\sharp}; \mathcal{C}_2^{2\kappa}(\mathcal{B}_p)],$$

where the time interval $[0, T]$ is omitted for sake of clarity.

Observe that, in what follows, we will only consider the spaces $\hat{\mathcal{Q}}^{\kappa}_{\kappa,p}$, with $2\kappa p > 1$.

We can now show how nonlinearities of the form given in Assumption B act on a controlled process.

Lemma 4.5 *Assume that $f_i \in \mathcal{X}_2$ for $i = 1, \dots, N$ and let $\kappa \in (1/3, \gamma)$. If $y \in \hat{\mathcal{Q}}^{\kappa}_{\kappa,p}$ admits the decomposition $\hat{\delta} y = X^{x,i} y^{x,i} + y^{\sharp}$, then the increment $\delta f_i(y)$ can be written as*

$$\delta(f_i(y))_{ts} = B(a_{ts}, Id)(y, f_i'(y))_s + (\delta x^j)_{ts} \cdot (y^{x,j} \cdot f_i'(y))_s + f_i(y)^{\sharp}_{ts}, \tag{64}$$

with $f_i(y)^{\sharp} = f_i(y)^{\sharp,1} + f_i(y)^{\sharp,2} + f_i(y)^{\sharp,3}$, where the elements $f_i(y)^{\sharp,k}$ are given by (46) and (47). Moreover, one has

$$\mathcal{N}[f_i(y)^{\sharp,1}; \mathcal{C}_2^{2\kappa}(\mathcal{B}_{p/2})] \le c_{f,\mathbf{X}} \Big\{ \mathcal{N}[y; \mathcal{C}_1^0(\mathcal{B}_{\alpha,p})]^2 + \mathcal{N}[y; \hat{\mathcal{Q}}^{\kappa}_{\kappa,p}]^2 \Big\} \tag{65}$$

$$\mathcal{N}[f_i(y)^{\sharp,2}; \mathcal{C}_2^{2\kappa}(\mathcal{B}_p)] \le c_{f,\mathbf{X}} \mathcal{N}[y; \hat{\mathcal{Q}}^{\kappa}_{\kappa,p}], \quad \mathcal{N}[f_i(y)^{\sharp,3}; \mathcal{C}_2^{2\kappa}(\mathcal{B}_p)] \le c_{f,\mathbf{X}} \mathcal{N}[y; \hat{\mathcal{Q}}^{\kappa}_{\kappa,p}]. \tag{66}$$

Proof This refers to decomposition (45). The estimate of $f_i(y)^{\sharp,2}$ is obvious, while the estimate of $f_i(y)^{\sharp,3}$ stems from the Hypothesis (61). As for $f_i(y)^{\sharp,1}$, notice that

$$\|f_i(y)^{\sharp,1}_{ts}\|_{\mathcal{B}_{p/2}} \lesssim c_f \|(\delta y)^2_{ts}\|_{\mathcal{B}_{p/2}} \lesssim \|(\delta y)_{ts}\|^2_{\mathcal{B}_p} \lesssim \|(\hat{\delta} y)_{ts}\|^2_{\mathcal{B}_p} + \|a_{ts} y_s\|^2_{\mathcal{B}_p},$$

and the result then comes from property (21). □

We are now in position to justify the use of (54) as a definition for the integral:

Proposition 4.6 *Let $y \in \hat{\mathcal{Q}}^{\kappa}_{\kappa,p}([0,T])$ admitting the decomposition $\hat{\delta} y = X^{x,i} y^{x,i} + y^{\sharp}$, with $\kappa \in (1/3, \gamma)$ and $p \in \mathbb{N}^*$ such that $\gamma - \kappa > n/(2p)$. Assume that $f = (f_1, \ldots, f_N)$ with $f_i \in \mathcal{X}_2$ for $i = 1, \ldots, N$. We set, for all $s < t$,*

$$\mathcal{J}_{ts}(\hat{d}x\, f(y)) = X^{x,i}_{ts} f_i(y_s) + X^{xa,i}_{ts}(y, f_i'(y))_s + X^{xx,ij}_{ts}(y^{x,j} \cdot f_i'(y))_s + \hat{\Lambda}_{ts}(J), \tag{67}$$

where we recall our Notation 4.1 for f_i', and with

$$J_{tus} = X^{x,i}_{tu}(f_i(y)^{\sharp}_{us}) + X^{xa,i}_{tu}((\hat{\delta} y)_{us}, f_i'(y_s)) + X^{xa,i}_{tu}(y_u, \delta(f_i'(y))_{us}) + X^{xx,ij}_{ts} \delta(y^{x,j} \cdot f_i'(y))_{us}, \tag{68}$$

the term $f(y)^{\sharp}$ being defined by decomposition (64). Then one has:

(1) *$\mathcal{J}(\hat{d}x\, f(y))$ is well-defined and there exists $z \in \mathcal{Q}^{\kappa}_{\kappa,p}([0,T])$ such that $\hat{\delta} z$ is equal to the increment $\mathcal{J}(\hat{d}x\, f(y))$. Furthermore, for any $0 \le s < t \le T$, the integral $\mathcal{J}_{ts}(\hat{d}x\, f(y))$ coincides with a Riemann type integral for two regular functions x and y.*

(2) *The following estimation holds true*

$$\mathcal{N}[z; \mathcal{Q}^{\kappa}_{\kappa,p}([0,T])] \le c_{f,\mathbf{X}} \left\{1 + \mathcal{N}[y; \mathcal{C}^0_1(\mathcal{B}_{\kappa,p})]^2 + T^{\alpha} \mathcal{N}[y; \mathcal{Q}^{\kappa}_{\kappa,p}]^2\right\}, \tag{69}$$

for some $\alpha > 0$.

(3) *For all $s < t$,*

$$\mathcal{J}_{ts}(\hat{d}x\, f(y)) = \lim_{|\Delta_{[s,t]}| \to 0} \sum_{(t_k) \in \Delta_{[s,t]}} S_{tt_{k+1}} \big\{ X^{x,i}_{t_{k+1}t_k} f_i(y_{t_k}) + X^{xa,i}_{t_{k+1}t_k}(y, f_i'(y))_{t_k} + X^{xx,ij}_{t_{k+1}t_k}(y^{x,j}, f_i'(y))_{t_k} \big\}, \tag{70}$$

where the limit is taken over partitions $\Delta_{[s,t]}$ of the interval $[s,t]$, as their mesh tends to 0.

Proof The fact that $\mathcal{J}_{ts}(\hat{d}x\, f(y))$ coincides with a Riemann type integral for two regular functions x and y is just what has been derived at Eq. (54). As far as the second claim of our proposition is concerned, it is a direct consequence of Hypothesis 2, together with the estimations of Lemma 4.5. Let us check for instance the regularity of J:

- for $X^{x,i}(f_i(y)^\sharp)$, we get, by (60) and (66),

$$\mathcal{N}[X^{x,i}(f_i(y)^{\sharp,2}+f_i(y)^{\sharp,3});\mathcal{C}_3^{\gamma+2\kappa}(\mathcal{B}_p)]\le c_{f,\mathbf{X}}\,\mathcal{N}[y;\hat{\mathcal{Q}}^\kappa_{\kappa,p}],$$

while, owing to (60) and (65),

$$\mathcal{N}[X^{x,i}f_i(y)^{\sharp,1};\mathcal{C}_3^{\gamma+2\kappa-n/(2p)}(\mathcal{B}_p)]\le c_{f,\mathbf{X}}\left\{\mathcal{N}[y;\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]^2+\mathcal{N}[y;\hat{\mathcal{Q}}^\kappa_{\kappa,p}]\right\}.$$

- for $X^{xa,i}((\hat\delta y), f_i'(y))$, we use (62) to obtain

$$\mathcal{N}[X^{xa,i}((\hat\delta y), f_i'(y));\mathcal{C}_3^{\gamma+2\kappa-n/(2p)}(\mathcal{B}_p)]\le c_{f,\mathbf{X}}\,\mathcal{N}[y;\hat{\mathcal{C}}_1^\kappa(\mathcal{B}_{\kappa,p})]\le c_{f,\mathbf{X}}\,\mathcal{N}[y;\hat{\mathcal{Q}}^\kappa_{\kappa,p}].$$

- for $X^{xa,i}(y,\delta(f_i'(y)))$, one has, by (62) again,

$$\begin{aligned}&\mathcal{N}[X^{xa,i}(y,\delta(f_i'(y)));\mathcal{C}_3^{\gamma+2\kappa-n/(2p)}(\mathcal{B}_p)]\\&\le c_{f,\mathbf{X}}\,\mathcal{N}[y;\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]\mathcal{N}[y;\mathcal{C}_1^\kappa(\mathcal{B}_p)]\\&\le c_{f,\mathbf{X}}\,\mathcal{N}[y;\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]\left\{\mathcal{N}[y;\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]+\mathcal{N}[y;\hat{\mathcal{Q}}^\kappa_{\kappa,p}]\right\}.\end{aligned}$$

- for $X^{xx,ij}\delta(y^{x,j}\cdot f_i'(y))$, we deduce from (63) that

$$\begin{aligned}&\mathcal{N}[X^{xx,ij}\delta(y^{x,j}\cdot f_i'(y));\mathcal{C}_3^{2\gamma+\kappa}(\mathcal{B}_p)]\\&\le c_{f,\mathbf{X}}\left\{\mathcal{N}[y^{x,j};\mathcal{C}_1^\kappa(\mathcal{B}_p)]+\mathcal{N}[y^{x,j};\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]\mathcal{N}[y;\mathcal{C}_1^\kappa(\mathcal{B}_p)]\right\}\\&\le c_{f,\mathbf{X}}\left\{1+\mathcal{N}[y;\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]^2+\mathcal{N}[y;\hat{\mathcal{Q}}^\kappa_{\kappa,p}]^2\right\}.\end{aligned}$$

Moreover, thanks to the algebraic relations stated in Hypothesis 2 and decomposition (64), it is easy to show that

$$J=-\hat\delta\left(X^{x,i}(f_i(y))+X^{xa,i}(y,f_i'(y))+X^{xx,ij}(y^{x,j}\cdot f_i'(y))\right).$$

Therefore, $J\in\operatorname{Ker}\hat\delta\cap\mathcal{C}_3^\mu(\mathcal{B}_p)$, with $\mu=\gamma+2\kappa-n/(2p)>1$, and we are allowed to apply $\hat\Lambda$. Besides, using the contraction property (30), we get

$$\mathcal{N}[\hat\Lambda(J);\mathcal{C}_2^{\gamma+2\kappa-n/(2p)}(\mathcal{B}_p)]\le c_{f,\mathbf{X}}\left\{1+\mathcal{N}[y;\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]^2+\mathcal{N}[y;\hat{\mathcal{Q}}^\kappa_{\kappa,p}]^2\right\},$$

and also

$$\mathcal{N}[\hat\Lambda(J);\mathcal{C}_2^{\gamma+\kappa-n/(2p)}(\mathcal{B}_{\kappa,p})]\le c_{f,\mathbf{X}}\left\{1+\mathcal{N}[y;\mathcal{C}_1^0(\mathcal{B}_{\kappa,p})]^2+\mathcal{N}[y;\hat{\mathcal{Q}}^\kappa_{\kappa,p}]^2\right\}.$$

The regularity of the other terms of (67) can be proved with similar arguments. As for the expression (70), it is a consequence of Proposition 3.7, since one can write

$$\mathcal{J}(\hat{d}x\, f(y)) = \left(\mathrm{Id} - \hat{\Lambda}\hat{\delta}\right)\left(X^{x,i}(f_i(y)) + X^{xa,i}(y, f_i'(y)) + X^{xx,ij}(y^{x,j} \cdot f_i'(y))\right).$$

□

Once our integral for controlled processes is defined, the existence and uniqueness of a local solution for our equation is easily proved:

Theorem 4.7 *Assume that $f = (f_1, \ldots, f_N)$ with $f_i \in \mathcal{X}_3$ for $i = 1, \ldots, N$. For any pair $(\kappa, p) \in (1/3, \gamma) \times \mathbb{N}$ such that $\gamma - \kappa > n/(2p)$, there exists a time $T > 0$ for which the system*

$$(\hat{\delta} y)_{ts} = \mathcal{J}_{ts}(\hat{d}x\, f(y)), \quad y_0 = \psi \in \mathcal{B}_{\kappa,p}, \tag{71}$$

interpreted with Proposition 4.6, admits a unique solution y in $\mathcal{Q}^{\kappa}_{\kappa,p}([0,T])$.

Proof As in the proof of Theorem 3.10, this local solution is obtained via a fixed-point argument. The invariance of a well-chosen ball of $\mathcal{Q}^{\kappa}_{\kappa,p}([0,T])$ is easy to establish from (69), once one has noticed that the latter estimate entails (with the notations of Proposition 4.6)

$$\mathcal{N}[z; \mathcal{Q}^{\kappa}_{\kappa,p}([0,T])] \leq c_{f,\mathbf{X}}\left\{1 + \|\psi\|^2_{\mathcal{B}_{\kappa,p}} + T^{\alpha}\mathcal{N}[y; \mathcal{Q}^{\kappa}_{\kappa,p}(I)]\right\}, \tag{72}$$

for some parameter $\alpha > 0$.
As for the contraction property, it stems from long but elementary computations, essentially similar to the estimates of the proof of Theorem 3.10. Write for instance, if $y, \tilde{y} \in \hat{\mathcal{Q}}^{\kappa}_{\kappa,p}(I)$ $(I = [0,T])$ are such that $y_0 = \tilde{y}_0 = \psi$,

$$X^{xa,i}_{ts}(y, f_i'(y))_s - X^{xa,i}_{ts}(\tilde{y}, f_i'(\tilde{y}))_s = X^{xa,i}_{ts}(y - \tilde{y}, f_i'(y))_s + X^{xa,i}_{ts}(\tilde{y}, f_i'(y) - f_i'(\tilde{y}))_s.$$

Then by (62),

$$\begin{aligned}
&\mathcal{N}[X^{xa,i}(y - \tilde{y}, f_i'(y)); \mathcal{C}_2^{\gamma+\kappa-n/(2p)}(I; \mathcal{B}_p)] \\
&\quad \leq c_{\mathbf{X}}\mathcal{N}[y - \tilde{y}; \mathcal{C}_1^0(I; \mathcal{B}_{\kappa,p})]\mathcal{N}[f_i'(y); \mathcal{C}_1^0(\mathcal{B}_p)] \leq c_{\mathbf{X},f}T^{\kappa}\mathcal{N}[y - \tilde{y}; \mathcal{Q}^{\kappa}_{\kappa,p}(I)],
\end{aligned}$$

while

$$\begin{aligned}
&\mathcal{N}[X^{xa,i}(\tilde{y}, f_i'(y) - f_i'(\tilde{y})); \mathcal{C}_2^{\gamma+\kappa-n/(2p)}(I; \mathcal{B}_p)] \\
&\quad \leq c_{\mathbf{X},f}\mathcal{N}[\tilde{y}; \mathcal{C}_1^0(I; \mathcal{B}_{\kappa,p})]\mathcal{N}[y - \tilde{y}; \mathcal{C}_1^0(I; \mathcal{B}_{\kappa,p})] \\
&\quad \leq c_{\mathbf{X},f,\psi}T^{\kappa}\left\{1 + \mathcal{N}[\tilde{y}; \mathcal{Q}^{\kappa}_{\kappa,p}(I)]\right\}\mathcal{N}[y - \tilde{y}; \mathcal{Q}^{\kappa}_{\kappa,p}(I)].
\end{aligned}$$

By following the same lines with the other terms of the decomposition of

$$\hat{\delta}(z - \tilde{z}) = \mathcal{J}\left(\hat{d}x\,\left[f(y) - f(\tilde{y})\right]\right),$$

we deduce

$$\mathcal{N}[z-\tilde{z};\mathcal{Q}^{\kappa}_{\kappa,p}(I)] \le c_{\mathbf{X},f,\psi} T^{\lambda} \mathcal{N}[y-\tilde{y};\mathcal{Q}^{\kappa}_{\kappa,p}(I)] \times \left\{1+\mathcal{N}[y;\mathcal{Q}^{\kappa}_{\kappa,p}(I)]^2+\mathcal{N}[\tilde{y};\mathcal{Q}^{\kappa}_{\kappa,p}(I)]^2\right\}$$

for some $\lambda > 0$, which allows to settle the expected fixed-point argument on a small enough interval $[0, T]$. □

As an almost immediate consequence of this rough paths approach, we get the following continuity statement for the solution:

Proposition 4.8 *Assume that the three parameters (γ, κ, p) are chosen as in Theorem 4.7. Then the Itô map associated to Eq. (71) is locally Lipschitz in the following sense: if y (resp. $\tilde{y}$) denotes the solution on $[0, T]$ (resp. $[0, \tilde{T}]$) of (71) given by Theorem 4.7, for a driving process x (resp. $\tilde{x}$) satisfying Hypothesis 2, and with initial condition ψ (resp. $\tilde{\psi}$), then*

$$\mathcal{N}[y-\tilde{y};\mathcal{C}^{\kappa}_1([0,T^*];\mathcal{B}_{\kappa,p})] \le c_{\mathbf{X},\tilde{\mathbf{X}},\psi,\tilde{\psi}} \left\{\|\psi-\tilde{\psi}\|_{\mathcal{B}_{\kappa,p}}+\mathcal{N}[\mathbf{X}-\tilde{\mathbf{X}};\mathcal{CL}^{\gamma,\kappa,p}]\right\}, \tag{73}$$

where $T^* = \inf(T, \tilde{T})$ and $\mathbf{X}$ (resp. $\tilde{\mathbf{X}}$) stands for the path above x (resp. $\tilde{x}$) described by Hypothesis 2. As for the constant in (73), it can be written as

$$c_{\mathbf{X},\tilde{\mathbf{X}},\psi,\tilde{\psi}} = C(\mathcal{N}[\mathbf{X};\mathcal{CL}^{\gamma,\kappa,p}],\mathcal{N}[\tilde{\mathbf{X}};\mathcal{CL}^{\gamma,\kappa,p}],\|\psi\|_{\mathcal{B}_{\kappa,p}},\|\tilde{\psi}\|_{\mathcal{B}_{\kappa,p}}),$$

for some function $C : (\mathbb{R}^+)^4 \to \mathbb{R}$ growing with its four arguments.

Proof The strategy is exactly the same as in the standard diffusion case: for any interval I contained in $[0, T^*]$, we introduce the quantity

$$\begin{aligned}\mathcal{N}[y-\tilde{y};\mathcal{Q}^{x,\tilde{x}}_{\kappa}(I)] = {} & \mathcal{N}[y-\tilde{y};\hat{\mathcal{C}}^{\gamma}_1(I;\mathcal{B}_{\kappa,p})] \\ & + \mathcal{N}[y^{x,i}-\tilde{y}^{x,i};\mathcal{C}^{\kappa}_1(I;\mathcal{B}_p)\cap\mathcal{C}^0_1(I;\mathcal{B}_{\kappa,p})] \\ & + \mathcal{N}[y^{\sharp}-\tilde{y}^{\sharp};\mathcal{C}^{2\kappa}_2(I;\mathcal{B}_p)\cap\mathcal{C}^{\gamma}_2(I;\mathcal{B}_{\kappa,p})],\end{aligned}$$

and then we show from the decomposition of $\hat{\delta}(y-\tilde{y}) = \mathcal{J}(\hat{d}x^i\, f_i(y)) - \mathcal{J}(\hat{d}\tilde{x}^i\, f_i(\tilde{y}))$ that if $I = I^{\varepsilon}_k = [k\varepsilon, (k+1)\varepsilon]$,

$$\begin{aligned}&\mathcal{N}[y-\tilde{y};\mathcal{Q}^{x,\tilde{x}}_{\kappa}(I^{\varepsilon}_k)] \\ &\quad \le c_{\mathbf{X},\tilde{\mathbf{X}},\psi,\tilde{\psi}} \left\{\varepsilon^{\kappa}\mathcal{N}[y-\tilde{y};\mathcal{Q}^{x,\tilde{x}}_{\kappa}(I^{\varepsilon}_k)]+\mathcal{N}[\mathbf{X}-\tilde{\mathbf{X}};\mathcal{CL}^{\gamma,\kappa,p}]+\|y_{k\varepsilon}-\tilde{y}_{k\varepsilon}\|_{\mathcal{B}_{\kappa,p}}\right\}.\end{aligned}$$

By taking ε small enough, we deduce, for any k,

$$\begin{aligned}
&\mathcal{N}[y-\tilde{y};\mathcal{Q}_{\kappa}^{x,\tilde{x}}(I_k^{\varepsilon})]\\
&\quad\le c_{\mathbf{X},\tilde{\mathbf{X}},\psi,\tilde{\psi}}\left\{\mathcal{N}[\mathbf{X}-\tilde{\mathbf{X}};\mathcal{CL}^{\gamma,\kappa,p}]+\|y_{k\varepsilon}-\tilde{y}_{k\varepsilon}\|_{\mathcal{B}_{\kappa,p}}\right\}\\
&\quad\le c_{\mathbf{X},\tilde{\mathbf{X}},\psi,\tilde{\psi}}\left\{\mathcal{N}[\mathbf{X}-\tilde{\mathbf{X}};\mathcal{CL}^{\gamma,\kappa,p}]+\|\psi-\tilde{\psi}\|_{\mathcal{B}_{\kappa,p}}+\varepsilon^{\gamma}\sum_{l=0}^{k-1}\mathcal{N}[y-\tilde{y};\mathcal{Q}_{\kappa}^{x,\tilde{x}}(I_l^{\varepsilon})]\right\}.
\end{aligned}$$

Inequality (73) is then a consequence of Gronwall Lemma, together with the obvious control $\mathcal{N}[y-\tilde{y};\hat{\mathcal{C}}_1^{\gamma}([0,T^*];\mathcal{B}_{\kappa,p}]\le\sum_{k=0}^{N_\varepsilon}\mathcal{N}[y-\tilde{y};\mathcal{Q}_{\kappa}^{x,\tilde{x}}(I_k^{\varepsilon})]$, where N_ε is the smallest integer such that $N_\varepsilon\cdot\varepsilon\ge T^*$. □

5 Global solution under stronger regularity assumptions

The aim of this section is to show that a regularization in the nonlinearity involved in our heat equation can yield a global solution. Specifically, this section is devoted to the proof of the existence and uniqueness of a global solution to the (slightly) modified system

$$(\hat{\delta}y)_{ts}=\int_s^t S_{tu}\,dx_u^{(i)}\,S_\varepsilon f_i(y_u),\quad y_0=\psi, \tag{74}$$

where $f_i\in\mathcal{X}_3$, $\psi\in\mathcal{B}_{\alpha,p}$ for some $\alpha\ge 0$ to be precised, and ε is a strictly positive fixed parameter. Owing to the regularizing effect of S_ε, we will see that such a system is much easier to handle than the original formulation (71).

Note that we have chosen a regularization by S_ε in (74) in order to be close to Teichmann's framework [45]. However, it will be clear from the considerations below that an extension to a convolutional nonlinearity of the form

$$[\tilde{f}_i(y)](\xi)=\int_{\mathbb{R}^n}K(\xi,\eta)\,f_i(y)(\eta)\,d\eta,\quad \xi\in\mathbb{R}^n,$$

with a smooth enough kernel K and $f_i\in\mathcal{X}_3$, is possible. The technical argument which enables to extend the local solution into a global one are taken from a previous work of two of the authors [13].

5.1 Heuristic considerations

The regularizing property (20) of the semigroup S_ε allows us to turn to a decomposition of $\int_s^t S_{tu}\,dx_u^{(i)}\,S_\varepsilon f_i(y_u)$ similar to the finite-dimensional case, or otherwise stated written without the help of the mixed operator X^{xa}. Indeed, let us go back to decomposition (45):

$$\delta(f_i(y))_{ts} = (\delta x)_{ts} y_s^x \cdot f_i'(y_s) + \Big[a_{ts} y_s \cdot f_i'(y_s) + y_{ts}^{\sharp} \cdot f_i'(y_s) + (X_{ts}^{ax,i} y_s^{x,i}) \cdot f_i'(y_s) + \int_0^1 dr \, \big[f_i'(y_s + r(\delta y)_{ts}) - f_i'(y_s) \big] \cdot (\delta y)_{ts} \Big], \quad (75)$$

but this time, let us consider the whole term into brackets as a remainder term evolving in $\mathcal{B}_p$ (or maybe $\mathcal{B}_{p/2}$), and denote it by $f_i(y)^{\sharp}$. This point of view is for instance justified if we let the process y evolve in $\mathcal{B}_{1,p}$, insofar as, for any $s, t \in I$,

$$\|a_{ts} y_s \cdot f_i'(y_s)\|_{\mathcal{B}_p} \lesssim |t-s| \, \|f_i'\|_\infty \|y_s\|_{\mathcal{B}_{1,p}} \lesssim |t-s|^{2\kappa} \, |I|^{1-2\kappa} \, \|f_i'\|_\infty \|y_s\|_{\mathcal{B}_{1,p}}.$$

For obvious stability reasons, the strong assumption $y_s \in \mathcal{B}_{1,p}$ then implies that the residual term stemming from the decomposition of $\int_s^t S_{tu} \, dx_u^{(i)} \, S_\varepsilon f_i(y_u)$ should also be seen as an element of $\mathcal{B}_{1,p}$. This is made possible through the action of S_ε. Indeed, owing to (20), one has

$$\|S_\varepsilon(f(y)^{\sharp})\|_{\mathcal{B}_{1,p}} \le c\,\varepsilon^{-1} \, \|f(y)^{\sharp}\|_{\mathcal{B}_p}, \quad \text{for some constant } c > 0.$$

5.2 Definition of the integral

According to the above considerations, only the processes $X^{x,i}$, $X^{ax,i}$ and $X^{xx,i}$ will come into play. Therefore, let us focus on the following simplified version of Hypothesis 2:

Hypothesis 3 We assume that the process x allows to define operators $X^{x,i}$, $X^{ax,i}$, $X^{xx,ij}$ ($i, j \in \{1, \ldots, N\}$), such that, recalling our Notation 4.2:

(H1) From an algebraic point of view:

$$\hat{\delta} X^{x,i} = 0 \quad (76)$$

$$X^{x,i} = X^{ax,i} + \delta x^i \quad (77)$$

$$\hat{\delta} X^{xx,ij} = X^{x,i}(\delta x^j). \quad (78)$$

$$\text{The operators } X^{x,i} \text{ and } X^{xx,ij} \text{ commute with } S_\varepsilon. \quad (79)$$

(H2) From an analytical point of view:

$$X^{x,i} \in \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)) \cap \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{1,p}, \mathcal{B}_{1,p})) \cap \mathcal{C}_2^{\gamma - n/(2p)}(\mathcal{L}(\mathcal{B}_{p/2}, \mathcal{B}_p)) \quad (80)$$

$$X^{ax,i} \in \mathcal{C}_2^{1+\gamma}(\mathcal{L}(\mathcal{B}_{1,p}, \mathcal{B}_p)) \quad (81)$$

$$X^{xx,ij} \in \mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p)) \cap \mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_{1,p}, \mathcal{B}_{1,p})). \quad (82)$$

Remark 5.1 The assumption (79) is trivially met when x is a differentiable process and $X^{x,i}$ is defined by $X^{x,i}_{ts} = \int_s^t S_{tu}\, dx^{(i)}_u$. It will remain true in rough cases, following the constructions of Sect. 6. This commutativity property will be resorted to in the proofs of Propositions 5.3 and 5.4.

The notion of controlled processes which has been introduced in Definition 4.4 can also be simplified in this context:

Definition 5.2 For any $\kappa < \gamma$, let us define the space

$$\tilde{\mathcal{Q}}_{\kappa,p} = \Big\{ y \in \mathcal{C}_1^{\gamma}(\mathcal{B}_{1,p}) : \ (\hat{\delta} y)_{ts} = X^{x,i}_{ts}\, y^{x,i}_s + y^{\sharp}_{ts},\ y^{x,i} \in \mathcal{C}_1^{\kappa}(\mathcal{B}_{1,p}) \cap \mathcal{C}_1^{0}(\mathcal{B}_{1,p}),\ y^{\sharp} \in \mathcal{C}_2^{2\kappa}(\mathcal{B}_{1,p}) \Big\},$$

together with the seminorm

$$\mathcal{N}[y; \tilde{\mathcal{Q}}_{\kappa,p}] = \mathcal{N}[y^{x,i}; \mathcal{C}_1^0(\mathcal{B}_{1,p})] + \mathcal{N}[y^{x,i}; \mathcal{C}_1^{\kappa}(\mathcal{B}_{1,p})] + \mathcal{N}[y^{\sharp}; \mathcal{C}_2^{2\kappa}(\mathcal{B}_{1,p})].$$

With this notation, one has $\mathcal{N}[y; \mathcal{C}_1^{\gamma}(\mathcal{B}_{1,p})] \le c_x\, \mathcal{N}[y; \tilde{\mathcal{Q}}_{\kappa,p}]$.

In the following two propositions, let us fix an interval $I = [a, b]$ and denote $|I| = b - a$.

Proposition 5.3 *Let $y \in \tilde{\mathcal{Q}}_{\kappa,p}(I)$ with decomposition $\hat{\delta} y = X^{x,i} y^{x,i} + y^{\sharp}$, for some $(\kappa, p) \in (1/3, \gamma) \times \mathbb{N}^*$ such that $\gamma - \kappa > n/(2p)$ and initial value $h = y_a \in \mathcal{B}_{1,p}$. For any $\psi \in \mathcal{B}_{1,p}$, define a process z by the two relations: $z_a = \psi$ and for any $s < t \in I$,*

$$(\hat{\delta} z)_{ts} = \mathcal{J}_{ts}(\hat{d} x^{(i)}\, S_\varepsilon f_i(y_s)) = X^{x,i}_{ts}\, S_\varepsilon f_i(y_s) + X^{xx,ij}_{ts}\, S_\varepsilon(y^{x,j}_s \cdot f_i'(y_s)) + \hat{\Lambda}_{ts}\left(X^{x,i}\, S_\varepsilon f_i(y)^{\sharp} + X^{xx,ij}\, S_\varepsilon \delta(y^{x,j} \cdot f_i'(y)) \right),$$

where $f_i(y)^{\sharp}$ stands for the term into brackets in (75). Then:

- *z is well-defined as an element of $\tilde{\mathcal{Q}}_{\kappa,p}(I)$.*
- *The following estimation holds:*

$$\mathcal{N}[z; \tilde{\mathcal{Q}}_{\kappa,p}(I)] \le c\, \varepsilon^{-1} \left\{ 1 + |I|^{2(\gamma-\kappa)}\, \mathcal{N}[y; \tilde{\mathcal{Q}}_{\kappa,p}(I)]^2 + |I|^{2(1-\kappa)}\, \|h\|^2_{\mathcal{B}_{1,p}} \right\}, \tag{83}$$

  for some constant $c > 0$.
- *For any $s < t \in I$, $(\hat{\delta} z)_{ts}$ can also be written as*

$$(\hat{\delta} z)_{ts} = \lim_{|\mathcal{P}_{[s,t]}| \to 0} \sum_{t_k \in \mathcal{P}_{[s,t]}} \left\{ X^{x,i}_{t_{k+1} t_k}\, S_\varepsilon f_i(y_{t_k}) + X^{xx,ij}_{t_{k+1} t_k}\, S_\varepsilon \left(y^{x,j}_{t_k} \cdot f_i'(y_{t_k}) \right) \right\} \quad \text{in } \mathcal{B}_{1,p}. \tag{84}$$

Proof Let us focus on the estimation of the residual term

$$z^\sharp_{ts} = X^{xx,ij}_{ts} S_\varepsilon(y^{x,j}_s \cdot f_i'(y_s)) + \hat{\Lambda}_{ts}\left(X^{x,i} S_\varepsilon f_i(y)^\sharp + X^{xx,ij} S_\varepsilon \delta(y^{x,j} \cdot f_i'(y))\right).$$

First, using (82) and (20), we get

$$\begin{aligned}\|X^{xx,ij}_{ts} S_\varepsilon(y^{x,j}_s \cdot f_i'(y_s))\|_{\mathcal{B}_{1,p}} &\leq c_x\, |t-s|^{2\gamma}\, \varepsilon^{-1} \|y^{x,j}_s \cdot f_i'(y_s)\|_{\mathcal{B}_p} \\ &\leq c_x\, |t-s|^{2\gamma}\, \varepsilon^{-1} \|y^{x,j}_s\|_{\mathcal{B}_{1,p}} \\ &\leq c_x\, |t-s|^{2\gamma}\, \varepsilon^{-1}\, \mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)].\end{aligned}$$

Secondly, write $f_i(y)^\sharp = f_i(y)^{\sharp,1} + f_i(y)^{\sharp,2}$, with $f_i(y)^{\sharp,1}_{ts} = a_{ts} y_s \cdot f_i'(y_s) + y^\sharp_{ts} \cdot f_i'(y_s) + (X^{ax,i}_{ts} y^{x,i}_s) \cdot f_i'(y_s)$, $f_i(y)^{\sharp,2}_{ts} = \int_0^1 dr\, [f_i'(y_s + r(\delta y)_{ts}) - f_i'(y_s)] \cdot (\delta y)_{ts}$, and notice that

$$\begin{aligned}&\|X^{x,i}_{tu} S_\varepsilon f_i(y)^{\sharp,1}_{us}\|_{\mathcal{B}_{1,p}} \\ &\lesssim |t-u|^\gamma\, \varepsilon^{-1} \|f_i(y)^{\sharp,1}_{us}\|_{\mathcal{B}_p} \\ &\lesssim |t-u|^\gamma\, \varepsilon^{-1} \left\{\|(a_{us} y_s) \cdot f_i'(y_s)\|_{\mathcal{B}_p} + \|(X^{ax,i}_{us} y^{x,i}_s) \cdot f_i'(y_s)\|_{\mathcal{B}_p} + \|y^\sharp_{us} \cdot f_i'(y_s)\|_{\mathcal{B}_p}\right\} \\ &\lesssim |t-u|^\gamma\, \varepsilon^{-1} \left\{|u-s|\, \|y_s\|_{\mathcal{B}_{1,p}} + |u-s|^{1+\gamma}\, \|y^{x,i}_s\|_{\mathcal{B}_{1,p}} + \|y^\sharp_{us}\|_{\mathcal{B}_{1,p}}\right\} \\ &\lesssim |t-u|^\gamma\, \varepsilon^{-1} \left\{|u-s|^{2\kappa}\, \mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)] + |u-s| \left\{\mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)] + \|h\|_{\mathcal{B}_{1,p}}\right\}\right\} \\ &\lesssim |t-s|^{\gamma+2\kappa}\, \varepsilon^{-1} \left\{\mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)] + |I|^{1-2\kappa}\, \|h\|_{\mathcal{B}_{1,p}}\right\},\end{aligned}$$

while, owing to (79),

$$\begin{aligned}&\|X^{x,i}_{tu} S_\varepsilon f_i(y)^{\sharp,2}_{us}\|_{\mathcal{B}_{1,p}} = \|S_\varepsilon X^{x,i}_{tu} f_i(y)^{\sharp,2}_{us}\|_{\mathcal{B}_{1,p}} \\ &\lesssim \varepsilon^{-1}\, |t-u|^{\gamma-n/(2p)}\, \|f_i(y)^{\sharp,2}_{us}\|_{\mathcal{B}_{p/2}} \\ &\lesssim \varepsilon^{-1}\, |t-u|^{\gamma-n/(2p)}\, \|(\delta y)_{us}\|^2_{\mathcal{B}_p} \\ &\lesssim \varepsilon^{-1}\, |t-u|^{\gamma-n/(2p)} \left\{\|(\hat{\delta} y)_{us}\|^2_{\mathcal{B}_p} + \|a_{us} y_s\|^2_{\mathcal{B}_p}\right\} \\ &\lesssim \varepsilon^{-1}\, |t-u|^{\gamma-n/(2p)} \left\{|u-s|^{2\gamma}\, \mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)]^2 + |u-s|^2 \left\{\mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)]^2 + \|h\|^2_{\mathcal{B}_{1,p}}\right\}\right\} \\ &\lesssim \varepsilon^{-1}\, |t-s|^{3\gamma-n/(2p)} \left\{\mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)]^2 + |I|^{2(1-\gamma)}\, \|h\|^2_{\mathcal{B}_{1,p}}\right\}.\end{aligned}$$

Even more simple estimations based on (82) give

$$\begin{aligned}&\|X^{xx,ij}_{tu} S_\varepsilon \delta(y^{x,j} \cdot f_i'(y))_{us}\|_{\mathcal{B}_{1,p}} \\ &\quad\lesssim \varepsilon^{-1}\, |t-s|^{2\gamma+\kappa} \left\{1 + \mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)]^2 + |I|^{1-\kappa}\, \mathcal{N}[y;\, \tilde{Q}_{\kappa,p}(I)] \cdot \|h\|_{\mathcal{B}_{1,p}}\right\}.\end{aligned}$$

Thanks to the contraction property (30), we now easily deduce

$$\mathcal{N}[z^{\sharp}; \mathcal{C}_2^{2\kappa}(I)] \leq c\, \varepsilon^{-1} \left\{ 1 + |I|^{2(\gamma-\kappa)}\, \mathcal{N}[y; \tilde{\mathcal{Q}}_{\kappa,p}(I)]^2 + |I|^{2(1-\kappa)}\, \|h\|^2_{\mathcal{B}_{1,p}} \right\}.$$

The estimation of $\mathcal{N}[z^{x,i}; \mathcal{C}_1^{0,\kappa}(I; \mathcal{B}_{1,p})]$ can be established along the same lines. As for (84), it is a consequence of (3.7), together with the reformulation

$$\hat{\delta} z = (\mathrm{Id} - \hat{\Lambda}\hat{\delta})(X^{x,i} S_\varepsilon f_i(y) + X^{xx,ij} S_\varepsilon (y^{x,j} \cdot f_i'(y))).$$

□

In order to settle an efficient fixed-point argument in this context, the following Lipschitz relation is required:

Proposition 5.4 *If $y, \tilde{y} \in \tilde{\mathcal{Q}}_{\kappa,p}(I)$ with $y_a = \tilde{y}_a$, and if we denote by $z, \tilde{z}$ the two processes in $\tilde{\mathcal{Q}}_{\kappa,p}(I)$ such that*

$$z_0 = \tilde{z}_0 = y_0 \quad \textit{and} \quad \hat{\delta} z = \mathcal{J}(\hat{d}x^{(i)}\, S_\varepsilon f_i(y)),\ \hat{\delta}\tilde{z} = \mathcal{J}(\hat{d}x^{(i)}\, S_\varepsilon f_i(\tilde{y})),$$

then

$$\begin{aligned} \mathcal{N}[z - \tilde{z}; \tilde{\mathcal{Q}}_{\kappa,p}(I)] \leq c_x\, \varepsilon^{-1}\, |I|^{\gamma-\kappa}\, \mathcal{N}[y - \tilde{y}; \tilde{\mathcal{Q}}_{\kappa,p}(I)] \\ \Big\{ 1 + |I|^{2(\gamma-\kappa)}\, \{\mathcal{N}[y; \tilde{\mathcal{Q}}_{\kappa,p}(I)]^2 \\ + \mathcal{N}[y; \tilde{\mathcal{Q}}_{\kappa,p}(I)]^2\} + |I|^{2(1-\kappa)}\, \|h\|^2_{\mathcal{B}_{1,p}} \Big\}. \end{aligned} \tag{85}$$

Proof One has, for any $s, t \in I$,

$$\begin{aligned} \hat{\delta}(z - \tilde{z})_{ts} &= X^{x,i}_{ts} S_\varepsilon (f_i(y_s) - f_i(\tilde{y}_s)) + X^{xx,ij}_{ts} S_\varepsilon (y^{x,j}_s \cdot f_i'(y_s) - \tilde{y}^{x,j}_s \cdot f_i'(\tilde{y}_s)) \\ &\quad + \hat{\Lambda}_{ts} \left(X^{x,i} S_\varepsilon (f_i(y)^{\sharp} - f_i(\tilde{y})^{\sharp}) + X^{xx,ij} \delta (y^{x,j} \cdot f_i'(y) - \tilde{y}^{x,j} \cdot f_i'(\tilde{y})) \right). \end{aligned}$$

Let us only focus on the more intricate term, that is to say $X^{x,i} S_\varepsilon (f_i(y)^{\sharp,2} - f_i(\tilde{y})^{\sharp,2})$, where, according to the notations of the proof of Proposition 5.3,

$$f_i(y)^{\sharp,2}_{ts} = \int_0^1 dr\, \left[f_i'(y_s + r(\delta y)_{ts}) - f_i'(y_s) \right] \cdot (\delta y)_{ts}.$$

Write

$$f_i(y)^{\sharp,2}_{ts} - f_i(\tilde{y})^{\sharp,2}_{ts} = \int_0^1 dr\, \left[f_i'(y_s + r(\delta y)_{ts}) - f_i'(y_s)\right] \cdot \delta(y-\tilde{y})_{ts}$$
$$+(\delta\tilde{y})_{ts} \cdot \delta(y-\tilde{y})_{ts} \cdot \int_0^1 dr\, r \int_0^1 dr'\, f_i''(y_s + rr'(\delta y)_{ts})$$
$$+(\delta\tilde{y})^2_{ts} \cdot \int_0^1 dr\, r \int_0^1 dr'\, \left[f_i''(y_s + rr'(\delta y)_{ts}) - f_i''(\tilde{y}_s + rr'(\delta\tilde{y})_{ts})\right].$$

In this way,

$$\|f_i(y)^{\sharp,2}_{ts} - f_i(\tilde{y})^{\sharp,2}_{ts}\|_{\mathcal{B}_{p/2}} \lesssim \|\delta(y-\tilde{y})_{ts}\|_{\mathcal{B}_p} \left\{\|(\delta y)_{ts}\|_{\mathcal{B}_p} + \|(\delta\tilde{y})_{ts}\|_{\mathcal{B}_p}\right\}$$
$$+\|(\delta\tilde{y})_{ts}\|^2_{\mathcal{B}_p} \left\{\|y_s - \tilde{y}_s\|_{\mathcal{B}_\infty} + \|y_t - \tilde{y}_t\|_{\mathcal{B}_\infty}\right\}.$$

Now

$$\|\delta(y-\tilde{y})_{ts}\|_{\mathcal{B}_p} \lesssim \|\hat{\delta}(y-\tilde{y})_{ts}\|_{\mathcal{B}_{1,p}} + |t-s|\, \|(y_s - \tilde{y}_s) - S_{sa}(y_a - \tilde{y}_a)\|_{\mathcal{B}_{1,p}}$$
$$\lesssim |t-s|^\gamma\, \mathcal{N}[y-\tilde{y};\, \tilde{\mathcal{Q}}_{\kappa,p}(I)],$$

while

$$\|(\delta y)_{ts}\|_{\mathcal{B}_p} \leq \|(\hat{\delta} y)_{ts}\|_{\mathcal{B}_{1,p}} + \|a_{ts}(\hat{\delta} y)_{sa}\|_{\mathcal{B}_p} + \|a_{ts} S_{sa} h\|_{\mathcal{B}_p}$$
$$\lesssim |t-s|^\kappa \left\{|I|^{\gamma-\kappa}\, \mathcal{N}[y;\, \tilde{\mathcal{Q}}_{\kappa,p}(I)] + |I|^{1-\kappa}\, \|h\|_{\mathcal{B}_{1,p}}\right\}$$

and finally

$$\|y_s - \tilde{y}_s\|_{\mathcal{B}_\infty} \lesssim \|y_s - \tilde{y}_s\|_{\mathcal{B}_{1,p}} \lesssim \|y_s - \tilde{y}_s - S_{sa}(y_a - \tilde{y}_a)\|_{\mathcal{B}_{1,p}}$$
$$\lesssim |I|^{\gamma-\kappa}\, \mathcal{N}[y-\tilde{y};\, \tilde{\mathcal{Q}}_{\kappa,p}(I)].$$

This easily leads to

$$\mathcal{N}[f_i(y)^{\sharp,2} - f_i(\tilde{y})^{\sharp,2}; \mathcal{C}_2^{2\kappa}(\mathcal{B}_{p/2})] \lesssim |I|^{\gamma-\kappa}\, \mathcal{N}[y-\tilde{y};\, \tilde{\mathcal{Q}}_{\kappa,p}(I)]$$
$$\left\{1 + |I|^{2(\gamma-\kappa)} \left\{\mathcal{N}[y;\, \tilde{\mathcal{Q}}_{\kappa,p}(I)]^2 + \mathcal{N}[\tilde{y};\, \tilde{\mathcal{Q}}_{\kappa,p}(I)]^2\right\} + |I|^{2(1-\kappa)}\, \|h\|^2_{\mathcal{B}_{1,p}}\right\}.$$

Inequality (85) now follows from standard computations based on Hypothesis 3. □

We are now in position to prove the expected global result:

Theorem 5.5 *Let $f_i \in \mathcal{X}_3$, for $i \in \{1, \ldots, N\}$. Under Hypothesis 3, let $(\kappa, p) \in (1/3, \gamma) \times \mathbb{N}^*$ such that $\gamma - \kappa > n/(2p)$. For any $T > 0$, for any $\psi \in \mathcal{B}_{1,p}$, the differential system*

$$(\hat{\delta} y)_{ts} = \mathcal{J}_{ts}(\hat{d}x^{(i)} \, S_\varepsilon f_i(y)), \quad y_0 = \psi,$$

interpreted with Proposition 5.3, admits a unique global solution in $\tilde{Q}_{\kappa,p}([0, T])$.

Proof The strategy of the proof has been extensively developed in [13] in a similar background suitable to Volterra systems. The key point is to control both the norm of the local solution $y^{(k)}$ and the norm of the initial condition $y^{(k)}_{l_k}$ on each successive intervals $I_k^M = [l_k^M, l_{k+1}^M]$, where the sequence l_k^M is such that $l_{k+1}^M - l_k^M = \frac{1}{M+k}$, and $M \geq 1$ is a well-chosen fixed parameter. More precisely, we consider the sets

$$B_k^{\psi_k} = \left\{ y \in \mathcal{Q}^\kappa_{\kappa,p}(I_k^M) : \; y_{l_k^M} = \psi_k, \; y^{x,i}_{l_k^M} = S_\varepsilon f_i(\psi_k), \; \mathcal{N}[y; \mathcal{Q}^\kappa_{\kappa,p}(I_k^M)] \leq (M+k)^{\alpha_2} \right\},$$

where ψ_k is such that $\|\psi_k\|_{\mathcal{B}_{1,p}} \leq (M+k)^{\alpha_1}$, and prove the statement: there exist two parameters $\alpha_1, \alpha_2 > 0$ and an integer M such that for any $k \geq 0$, the (usual) map Γ is a strict contraction on the invariant set $B_k^{\psi_k}$ and the following property holds

$$\text{If } y \in B_k^{\psi_k}, \quad \text{then } \mathcal{N}[y_{l_{k+1}^M}; \mathcal{B}_{1,p}] \leq (M+k+1)^{\alpha_1}. \tag{H}$$

As in the proof of Theorem 3.10, if $y \in B_k^{\psi_k}$, $z = \Gamma(y)$ is defined as the unique element in $\mathcal{Q}^\kappa_{\kappa,p}(I_k^M)$ such that $z_{l_k^M} = \psi_k$ and for any $s, t \in I_k^M$, $(\hat{\delta} z)_{ts} = \mathcal{J}_{ts}(\hat{d}x^{(i)} \, S_\varepsilon f_i(y))$. The patching argument that leads to a global solution is then easily settled thanks to Property (H).

With the view of proving the above assertion, observe that if $y^1, y^2 \in B_k^{\psi_k}$, then by (83) (with obvious notations)

$$\begin{aligned}
&\mathcal{N}[z^1; \mathcal{Q}^\kappa_{\kappa,p}(I_k^M)] \\
&\quad \leq c_{x,f,\varepsilon} \left\{ 1 + (M+k)^{-2(\gamma-\kappa)} \mathcal{N}[y; \mathcal{Q}^\kappa_{\kappa,p}(I_k^M)]^2 + (M+k)^{-2(1-\kappa)} \|\psi_k\|^2_{\mathcal{B}_{1,p}} \right\}, \\
&\quad \leq c_{x,f,\varepsilon} \left\{ 1 + (M+k)^{-2(\gamma-\kappa)+2\alpha_2} + (M+k)^{-2(1-\kappa)+2\alpha_1} \right\},
\end{aligned} \tag{86}$$

while, owing to (85),

$$\mathcal{N}[z^1 - z^2; \mathcal{Q}^\kappa_{\kappa,p}(I_k^M)] \leq c_{x,f,\varepsilon} J_{M+k} \mathcal{N}[y^1 - y^2; \mathcal{Q}^\kappa_{\kappa,p}(I_k^M)], \tag{87}$$

where $J_k = k^{-(\gamma-\kappa)} + k^{-3(\gamma-\kappa)+2\alpha_2} + k^{-(\gamma-\kappa+2(1-\kappa))+2\alpha_1}$. As far as Property (H) is concerned, write, if $y \in B_k^{\psi_k}$,

$$y_{l_{k+1}^M} = S_{l_{k+1}^M l_k^M} \psi_k + (\hat{\delta} y)_{l_{k+1}^M l_k^M} = S_{l_{k+1}^M l_k^M} \psi_k + X^{x,i}_{l_{k+1}^M l_k^M} S_\varepsilon f_i(\psi_k) + y^\sharp_{l_{k+1}^M l_k^M},$$

which entails

$$\|y_{l^M_{k+1}}\|_{\mathcal{B}_{1,p}} \leq (M+k)^{\alpha_1} + c_{x,f,\varepsilon}(M+k)^{-\gamma} + (M+k)^{\alpha_2-2\kappa}. \tag{88}$$

From (86)–(88), the three expected properties (stability of $B_k^{\psi_k}$, contraction, (H)) are then readily translated as a system for α_1, α_2. One can finally check that if those two parameters are picked such that

$$3\kappa - 2\gamma < \alpha_2 < \gamma, \quad 1-\gamma < \alpha_1 < 1+\alpha_2+\gamma-3\kappa,$$

then there exists a sufficiently large M for which the assertion is verified. □

6 Applications

We now intend to apply the previous abstract results to concrete N-dimensional processes x. To this end, we know that it suffices to be able to construct, from x, a path $\mathbf{X} = (X^x, X^{ax}, X^{xa}, X^{xx})$ which satisfies Hypothesis 2. Indeed, the latter assumption clearly covers Hypothesis 1. We first study the general case of a 2-rough path, and then focus on the Brownian case, for which a comparison with the existing result of Itô theory is established.

6.1 The case of a 2-rough path

As usual in this paper, we shall proceed in two steps: we first work at a heuristic level, that is with smooth processes, and try to obtain an expression which can be extended to irregular situations. We then check directly Hypothesis 2 on the expression obtained in the heuristic step.

Assume for the moment that x is a smooth $\mathbb{R}^N$-valued function. Then the operators X^x, X^{ax}, X^{xa} and X^{xx} are defined by the formulae

$$X^{x,i}_{ts}(\varphi)(\xi) = \int_s^t S_{tu}(\varphi)(\xi)\, dx^i_u, \quad X^{ax,i}_{ts}(\varphi)(\xi) = \int_s^t a_{tu}(\varphi)(\xi)\, dx^i_u, \tag{89}$$

$$X^{xa,i}_{ts}(\varphi,\psi)(\xi) = \int_s^t S_{tu}((a_{us}\varphi)\cdot\psi)(\xi)\, dx^i_u \tag{90}$$

$$X^{xx,ij}_{ts}(\varphi)(\xi) = \int_s^t S_{tu}(\varphi)(\xi)\, dx^i_u\, (\delta x^j)_{us}. \tag{91}$$

Set now $\mathbf{x}^{\mathbf{2}}_{ts} = \int_s^t dx_u \otimes (\delta x)_{us}$. Then a straightforward integration by parts argument yields the following expression for the increments introduced above:

$$X^{x,i}_{ts} = (\delta x^i)_{ts} + \int_s^t \Delta S_{tu} (\delta x^i)_{us}\, du \tag{92}$$

$$X^{ax,i}_{ts} = \int_s^t \Delta S_{tu} (\delta x^i)_{us}\, du \tag{93}$$

$$X^{xa,i}_{ts} = \int_s^t X^{x,i}_{tu}\, B(\Delta S_{us}, \mathrm{Id})\, du \tag{94}$$

$$X^{xx,ij}_{ts} = \mathbf{x}^{\mathbf{2},ij}_{ts} + \int_s^t \Delta S_{tu} \mathbf{x}^{\mathbf{2},ij}_{us}\, du. \tag{95}$$

These are the expressions that we are ready to extend to irregular processes. Let us only elaborate on how to get (94). Actually, it suffices to notice that

$$\int_s^t S_{tu}((a_{us}\varphi)\cdot\psi)\, dx^i_u = -\int_s^t \partial_u (X^{x,i}_{tu})((a_{us}\varphi)\cdot\psi),$$

where, in the last integral, the partial derivative ∂_u only applies to the operator $X^{x,i}_{tu}$. Then

$$-\int_s^t \partial_u (X^{x,i}_{tu})((a_{us}\varphi)\cdot\psi) = \Big[-X^{x,i}_{tu}((a_{us}\varphi)\cdot\psi)\Big]_s^t + \int_s^t du\, X^{x,i}_{tu}(\partial_u(a_{us}\varphi)\cdot\psi)$$
$$= \int_s^t du\, X^{x,i}_{tu}((\Delta S_{us}\varphi)\cdot\psi).$$

Remark 6.1 At this point, it is not clear that the integral expressions $\int_s^t AS_{tu}(\delta x^i)_{us}\, du$,... give rise to operators defined on $\mathcal{B}_{\alpha,p}$. For the moment, we only consider those expressions as operators acting on $\mathcal{C}_c^\infty$. The extension to any space $\mathcal{B}_{\alpha,p}$ will stem from a continuity argument (see the proof of Proposition 6.3).

According to the above considerations, in order to extend expressions (92)–(95) to a Hölder path x, one is led to suppose that this process generates a standard 2-rough path, in the following sense (see [32] for further details on k-rough paths, $k \geq 2$):

Definition 6.2 For any $x \in \mathcal{C}_1^\gamma(\mathbb{R}^N)$ ($\gamma > 1/3$), we call Lévy area above x any process $\mathbf{x^2} \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^N \otimes \mathbb{R}^N)$ such that $\delta\mathbf{x^2} = \delta x \otimes \delta x$, or in other words

$$(\delta\mathbf{x}^{\mathbf{2},ij})_{tus} = (\delta x^i)_{tu}(\delta x^j)_{us}, \quad i, j = 1, \ldots, N. \tag{96}$$

The couple $(x, \mathbf{x^2})$ is referred to as a 2-rough path above x.

Once endowed with a Lévy area above x, we are in position to extend the three expressions (92)–(95). Together with Theorem 4.7, the following statement completes the proof of Theorem 1.1.

Proposition 6.3 *Let $\gamma > 1/3$. If $x \in \mathcal{C}_1^\gamma(\mathbb{R}^N)$ allows the construction of a Lévy area $\mathbf{x^2}$, the operators $X^{x,i}$, $X^{ax,i}$, $X^{xa,i}$, $X^{xx,ij}$ defined by (92)–(95), can be extended to a path $\mathbf{X}$ which satisfies Hypothesis 2. Moreover, if $(\tilde{x}, \tilde{\mathbf{x}}^{\mathbf{2}})$ is another 2-rough path, to which we associate a path $\tilde{\mathbf{X}}$, the following control holds:*

$$\mathcal{N}[\mathbf{X}-\tilde{\mathbf{X}}; \mathcal{CL}^{\gamma,\kappa,p}] \leq c_{x,\tilde{x}} \left\{\mathcal{N}[x-\tilde{x}; \mathcal{C}_1^\gamma(\mathbb{R}^N)] + \mathcal{N}[\mathbf{x^2}-\tilde{\mathbf{x}}^{\mathbf{2}}; \mathcal{C}_2^{2\gamma}(\mathbb{R}^N \otimes \mathbb{R}^N)]\right\}, \tag{97}$$

with $c_{x,\tilde{x}} = C\left(\mathcal{N}[x; \mathcal{C}_1^\gamma(\mathbb{R}^N)], \mathcal{N}[\tilde{x}; \mathcal{C}_1^\gamma(\mathbb{R}^N)]\right)$, for some growing function C. Remember that the normed space $\mathcal{CL}^{\gamma,\kappa,p}$ has been introduced in Hypothesis 2 and plays a part in the continuity statement (73).

Proof We have to check both the algebraic and analytic assumptions.
Algebraic conditions. The verification of (56)–(59) is a matter of elementary calculations. For instance, let us have a look at relation (59). For all $s < u < t$, one has

$$(\hat{\delta}X^{xx,ij})_{tus} = \mathbf{x}_{ts}^{\mathbf{2},ij} - \mathbf{x}_{tu}^{\mathbf{2},ij} - S_{tu}\mathbf{x}_{us}^{\mathbf{2},ij} + \int_u^t \Delta S_{tv}(\mathbf{x}_{vs}^{\mathbf{2},ij} - \mathbf{x}_{vu}^{\mathbf{2},ij})\, dv.$$

Then, by (96), this expression reduces to

$$\begin{aligned}
&(\hat{\delta}X^{xx,ij})_{tus} \\
&\quad = (\mathrm{Id} - S_{tu})\mathbf{x}_{us}^{\mathbf{2},ij} + (\delta x^i)_{tu}(\delta x^j)_{us} + \int_u^t \Delta S_{tv}(\mathbf{x}_{us}^{\mathbf{2},ij} + (\delta x^i)_{vu}(\delta x^j)_{us})\, dv \\
&\quad = \left[(\delta x^i)_{tu} + \int_u^t \Delta S_{tv}(\delta x^i)_{vu}\, dv\right](\delta x^j)_{us} = X_{tu}^{x,i}(\delta x^j)_{us}.
\end{aligned}$$

Analytical conditions. Let us examine the regularity of each operator individually.

Case of $X^{x,i}$. The norms at stake here are

$$\mathcal{N}[X^{x,i}; \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_p, \mathcal{B}_p))] \tag{98}$$

$$\mathcal{N}[X^{x,i}; \mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\kappa,p}, \mathcal{B}_{\kappa,p}))] \tag{99}$$

$$\mathcal{N}[X^{x,i}; \mathcal{C}_2^{\gamma-n/2p}(\mathcal{L}(\mathcal{B}_{p/2}, \mathcal{B}_p))]. \tag{100}$$

In order to establish those regularity results, let us first rewrite (92) as

$$X_{ts}^{x,i} = S_{ts}(\delta x^i)_{ts} - \int_s^t \Delta S_{tu}(\delta x^i)_{tu}\, du.$$

Then one has, for any $\kappa \in [0, 1)$,

$$\|X_{ts}^{x,i}(\varphi)\|_{\mathcal{B}_{\kappa,p}} \leq \|S_{ts}(\varphi)\|_{\mathcal{B}_{\kappa,p}} |(\delta x^i)_{ts}| + \int_s^t \|\Delta S_{tu}(\varphi)\|_{\mathcal{B}_{\kappa,p}} |(\delta x^i)_{tu}| du$$
$$\lesssim \|\varphi\|_{\mathcal{B}_{\kappa,p}} \|x^i\|_{\gamma} \left(|t-s|^{\gamma} + \int_s^t |t-u|^{-1+\gamma} du \right) \lesssim \|\varphi\|_{\mathcal{B}_{\kappa,p}} \|x^i\|_{\gamma} |t-s|^{\gamma},$$

which gives both (98) and (99). Along the same lines, in order to prove (100), we use the fact that $\|S_{ts}(\varphi)\|_{\mathcal{B}_p} \lesssim \|\varphi\|_{\mathcal{B}_{p/2}} |t-s|^{-n/2p}$ and that $\|\Delta S_{ts}(\varphi)\|_{\mathcal{B}_p} \lesssim \|\varphi\|_{\mathcal{B}_{p/2}} |t-s|^{-1-n/2p}$. Then we obtain

$$\|X_{ts}^{x,i}(\varphi)\|_{\mathcal{B}_p} \lesssim \|\varphi\|_{\mathcal{B}_{p/2}} \|x^i\|_{\gamma} |t-s|^{\gamma-n/2p}$$

for all p such that $\gamma - n/2p > 0$. Those estimations give the required bound (100).
Case of $X^{ax,i}$. We should now check that (61) is verified in our setting. To this aim, write $X^{ax,i}$ as

$$X_{ts}^{ax,i} = a_{ts}(\delta x^i)_{ts} - \int_s^t \Delta S_{tu}(\delta x^i)_{tu}\, du.$$

Then

$$\|X_{ts}^{ax,i}(\varphi)\|_{\mathcal{B}_p} = \|a_{ts}(\varphi)\|_{\mathcal{B}_p} |(\delta x^i)_{ts}| + \int_s^t \|\Delta S_{tu}(\varphi)\|_{\mathcal{B}_p} |(\delta x^i)_{tu}| du$$

and using the semigroup estimates

$$\|a_{ts}(\varphi)\|_{\mathcal{B}_p} \lesssim \|\varphi\|_{\mathcal{B}_{\kappa,p}} |t-s|^{\kappa} \qquad \|\Delta S_{tu}(\varphi)\|_{\mathcal{B}_p} \lesssim \|\varphi\|_{\mathcal{B}_{\kappa,p}} |t-u|^{-1+\kappa}$$

we easily conclude that

$$\mathcal{N}[X^{ax,i};\mathcal{L}(\mathcal{B}_{\kappa,p},\mathcal{B}_p)] \leq c_x\,|t-s|^{\gamma+\kappa}\,, \tag{101}$$

which is the expected regularity result.
Case of $X^{xa,i}$. Going back to (62), one must prove that the following norms are finite:

$$\begin{aligned}&\mathcal{N}[X^{xa,i};\mathcal{C}_2^{\gamma+\kappa-n/(2p)}(\mathcal{L}(\mathcal{B}_{\kappa,p}\times\mathcal{B}_p,\mathcal{B}_p))], \quad \text{and}\\ &\mathcal{N}[X^{xa,i};\mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\kappa,p}\times\mathcal{B}_{\kappa,p},\mathcal{B}_{\kappa,p}))].\end{aligned} \tag{102}$$

To do so, write $X^{xa,i}_{ts}$ as

$$X^{xa,i}_{ts} = X^{x,i}_{ts}B(a_{ts},\mathrm{Id}) - \int_s^t S_{tu}X^{x,i}_{us}B(\Delta S_{us},\mathrm{Id})\,du.$$

We deduce

$$\begin{aligned}\mathcal{N}[X^{xa,i}_{ts}(\varphi,\psi);\mathcal{B}_p] &\lesssim \mathcal{N}[X^{x,i};\mathcal{C}_2^{\gamma-n/(2p)}(\mathcal{L}(\mathcal{B}_{p/2},\mathcal{B}_p))]\mathcal{N}[(a_{ts}\varphi)\cdot\psi;\mathcal{B}_{p/2}]\\ &+\mathcal{N}[X^{x,i};\mathcal{C}_2^{\gamma-n/(2p)}(\mathcal{L}(\mathcal{B}_{p/2},\mathcal{B}_p))]\int_s^t |u-s|^{\gamma}\mathcal{N}[(\Delta S_{us}\varphi)\cdot\psi;\mathcal{B}_{p/2}]du\end{aligned}$$

where

$$\mathcal{N}[(a_{ts}\varphi)\cdot\psi;\mathcal{B}_{p/2}] \lesssim \mathcal{N}[a_{ts}\varphi;\mathcal{B}_p]\mathcal{N}[\psi;\mathcal{B}_p] \lesssim |t-s|^{\kappa}\mathcal{N}[\varphi;\mathcal{B}_{\kappa,p}]\mathcal{N}[\psi;\mathcal{B}_p]$$

and

$$\begin{aligned}\mathcal{N}[(\Delta S_{us}\varphi)\cdot\psi;\mathcal{B}_{p/2}] &\lesssim \mathcal{N}[\Delta S_{us}\varphi;\mathcal{B}_p]\mathcal{N}[\psi;\mathcal{B}_p]\\ &\lesssim |u-s|^{-1+\kappa}\mathcal{N}[\varphi;\mathcal{B}_{\kappa,p}]\mathcal{N}[\psi;\mathcal{B}_p].\end{aligned}$$

This allows to conclude that

$$\begin{aligned}&\mathcal{N}[X^{xa,i}_{ts}(\varphi,\psi);\mathcal{B}_p]\\ &\quad\lesssim \mathcal{N}[X^{x,i};\mathcal{C}_2^{\gamma-n/(2p)}(\mathcal{L}(\mathcal{B}_{p/2},\mathcal{B}_p))]\mathcal{N}[\varphi;\mathcal{B}_{\kappa,p}]\mathcal{N}[\psi;\mathcal{B}_p]|t-s|^{\gamma+\kappa-n/(2p)},\end{aligned}$$

and the first of the required bounds in (102) follows. For the second one, we have

$$\begin{aligned}\mathcal{N}[X^{xa,i}_{ts}(\varphi,\psi);\mathcal{B}_{\kappa,p}] &\lesssim \mathcal{N}[X^{x,i};\mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\kappa,p},\mathcal{B}_{\kappa,p}))]\mathcal{N}[(a_{ts}\varphi)\cdot\psi;\mathcal{B}_{\kappa,p}]\\ &+\mathcal{N}[X^{x,i};\mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\kappa,p},\mathcal{B}_{\kappa,p}))]\int_s^t |u-s|^{\gamma}\mathcal{N}[(\Delta S_{us}\varphi)\cdot\psi;\mathcal{B}_{\kappa,p}]du,\end{aligned}$$

and using the algebra property of $\mathcal{B}_{\kappa,p}$, we get

$$\mathcal{N}[(a_{ts}\varphi)\cdot\psi;\mathcal{B}_{\kappa,p}]\lesssim\mathcal{N}[\varphi;\mathcal{B}_{\kappa,p}]\mathcal{N}[\psi;\mathcal{B}_{\kappa,p}]$$

and

$$\begin{aligned}\mathcal{N}[(\Delta S_{us}\varphi)\cdot\psi;\mathcal{B}_{\kappa,p}]&\lesssim\mathcal{N}[\Delta S_{us}\varphi;\mathcal{B}_{\kappa,p}]\mathcal{N}[\psi;\mathcal{B}_{\kappa,p}]\\&\lesssim|u-s|^{-1}\mathcal{N}[\varphi;\mathcal{B}_{\kappa,p}]\mathcal{N}[\psi;\mathcal{B}_{\kappa,p}]\end{aligned}$$

so that

$$\begin{aligned}&\mathcal{N}[X^{xa,i}_{ts}(\varphi,\psi);\mathcal{B}_p]\\&\quad\lesssim\mathcal{N}[X^{x,i};\mathcal{C}_2^{\gamma}(\mathcal{L}(\mathcal{B}_{\kappa,p},\mathcal{B}_{\kappa,p}))]\mathcal{N}[\varphi;\mathcal{B}_{\kappa,p}]\mathcal{N}[\psi;\mathcal{B}_{\kappa,p}]\\&\quad\times\left(|t-s|^{\gamma}+\int_s^t|u-s|^{\gamma-1}du\right).\end{aligned}$$

The second estimate follows.
Case of $X^{xx,ij}$. We must estimate the norm

$$\mathcal{N}[X^{xx,ij};\mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_p,\mathcal{B}_p))],\tag{103}$$

and also $\mathcal{N}[X^{xx,ij};\mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p},\mathcal{B}_{\alpha,p}))]$ and $\mathcal{N}[X^{xx,ij};\mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_{\alpha,p},\mathcal{B}_p))]$. We focus on (103), the others terms having similar behavior using the algebra property of $\mathcal{B}_{\alpha,p}$ and the Sobolev embedding $\mathcal{B}_{\alpha,p}\subset\mathcal{B}_\infty$.

First, write $X^{xx,ij}_{ts}$ as

$$X^{xx,ij}_{ts}=S_{ts}\mathbf{x}^{\mathbf{2},ij}_{ts}-\int_s^t\Delta S_{tu}\left[\mathbf{x}^{\mathbf{2},ij}_{tu}+(\delta x^i)_{tu}(\delta x^j)_{us}\right]du.$$

From this expression, we immediately get

$$\begin{aligned}&\mathcal{N}[X^{xx,ij}_{ts}(\varphi);\mathcal{B}_p]\\&\quad\le c_x\left\{\mathcal{N}[S_{ts}(\varphi);\mathcal{B}_p]||t-s|^{2\gamma}+\int_s^t\mathcal{N}[\Delta S_{tu}(\varphi);\mathcal{B}_p][|t-u|^{2\gamma}+|t-u|^{\gamma}|u-s|^{\gamma}|]du\right\}\\&\quad\le c_x\left\{\mathcal{N}[\varphi;\mathcal{B}_p]||t-s|^{2\gamma}+\mathcal{N}[\varphi;\mathcal{B}_p]\int_s^t|t-u|^{-1}[|t-u|^{2\gamma}+|t-u|^{\gamma}|u-s|^{\gamma}|]du\right\}\\&\quad\le c_x\,\mathcal{N}[\varphi;\mathcal{B}_p]\,|t-s|^{2\gamma}.\end{aligned}$$

This gives the expected conclusion $\mathcal{N}[X^{xx,ij};\mathcal{C}_2^{2\gamma}(\mathcal{L}(\mathcal{B}_p,\mathcal{B}_p))]<\infty$.

The continuity statement (97) is easily proved with the same arguments. □

Remark 6.4 As recalled in the introduction, it is a well-known fact that one can construct a 2-rough path (in the sense of Definition 6.2) above a N-dimensional fractional Brownian motion B with Hurst parameter $H > 1/3$ (see e.g. [7,18,36,49]). Theorem 1.1 can thus be applied in order to handle the heat equation (71) driven by such a process. To the best of the authors' knowledge, this is presently the only method that provides an interpretation and a solution to the equation when $H \in (1/3, 1/2)$. We are actually able to extend the strategy to the case $H > 1/4$ by injecting third-order developments of the vector fields in the procedure described at Sect. 4. For sake of conciseness, we have preferred not to include all the technical details behind this slight improvement. What is really lacking now is a more general formulation that would allow to cope with rough paths of any order.

6.2 The Brownian case

When $x = B$ is a standard N-dimensional Brownian motion, the mild equation (4) can also be understood in the Itô sense, and the existence and uniqueness of a (global) solution is in this situation already well-established, even for small p (the main reference we have in mind here is [4], but similar results can be found in [3,27,53]). In what follows, we mean to show that under the hypotheses of Theorem 4.7, the two notions of solution (rough paths and Itô sense) actually coincide. To this end, we shall lean on the two following lemmas, borrowed respectively from [25] and [4].

Lemma 6.5 *Fix a time $T > 0$. For every $\alpha, \beta \geq 0$, $p, q \geq 1$, there exists a constant c such that for any $R \in \mathcal{C}_2([0,T]; \mathcal{B}_{\alpha,p})$,*

$$\mathcal{N}[R; \mathcal{C}_2^{\beta}([0,T]; \mathcal{B}_{\alpha,p})] \leq c \left\{ U_{\beta+\frac{2}{q},q,\alpha,p}(R) + \mathcal{N}[\hat{\delta} R; \mathcal{C}_3^{\beta}([0,T]; \mathcal{B}_{\alpha,p}] \right\},$$

where

$$U_{\beta,q,\alpha,p}(R) = \left[\int_{0 \leq u < v \leq T} \left(\frac{\| R_{vu} \|_{\mathcal{B}_{\alpha,p}}}{|v-u|^{\beta}} \right)^q du dv \right]^{1/q}.$$

Lemma 6.6 *For every $p \geq 2$, the Burkholder–Davies–Gundy inequality holds in $\mathcal{B}_p$. In other words, for any $T > 0$, if B is a one-dimensional Brownian motion and H is an adapted process with values in $L^2([0,T]; \mathcal{B}_p)$, then for any $q \geq 2$, there exists a constant c independent of H such that*

$$E\left[\sup_{0 \leq t \leq T} \left\| \int_0^t H_u \, dB_u \right\|_{\mathcal{B}_p}^q \right] \leq c\, E\left[\left(\int_0^T \| H_u \|_{\mathcal{B}_p}^2 \, du \right)^{q/2} \right]. \tag{104}$$

Remark 6.7 It is readily checked from the very Definition (17) of $\|.\|_{\mathcal{B}_{\kappa,p}}$ that $\mathcal{B}_p$ can be replaced with any $\mathcal{B}_{\kappa,p}$ in (104).

From now on, we fix three parameters (γ, κ, p) that satisfy the assumptions of Theorem 4.7, namely

$$1/3 < \kappa < \gamma < 1/2, \quad \gamma - \kappa > \frac{n}{2p}.$$

We also fix $f_i \in \mathcal{X}_3$ $(i = 1, \ldots, N)$ and we denote by Y the (continuous, $\mathcal{B}_{\kappa,p}$-valued) solution to the equation

$$Y_0 = \psi \in \mathcal{B}_{\kappa,p}, \quad Y_t = S_t\psi + \sum_{i=1}^{N} \int_0^t S_{t-u}\, dB_u^i\, f_i(Y_u), \quad t \in [0, T], \tag{105}$$

where the integral is understood in the Itô sense and the initial condition ψ is assumed to be deterministic, for more simplicity. As announced earlier, the existence and uniqueness of Y is for instance proven in [4].

Proposition 6.8 *With the notations of Sect. 4, Y almost surely belongs to $\mathcal{Q}^{\kappa}_{\kappa,p}([0, T])$, where, in the definition of the latter space, the operator-valued process $X^{B,i}_{ts} = \int_s^t S_{t-u}\, dB^i_u$ is understood in the Itô sense.*

The proof of this proposition relies on two preliminary results.

Lemma 6.9 *For every $q \geq 2$, $s < t \in [0, T]$,*

$$E\left[\|(\hat{\delta} Y)_{ts}\|^q_{\mathcal{B}_{\kappa,p}}\right] \leq c_{\psi,q,f,T}\, |t-s|^{q/2}. \tag{106}$$

Proof From the Eq. (105) itself, we first deduce

$$\begin{aligned} E\left[\|Y_t\|^q_{\mathcal{B}_{\kappa,p}}\right] &\leq c_q \left\{ \|\psi\|^q_{\mathcal{B}_{\kappa,p}} + E\left[\left(\int_0^t \|S_{t-u} f_i(Y_u)\|^2_{\mathcal{B}_{\kappa,p}}\, du\right)^{q/2}\right]\right\} \\ &\leq c_{q,f,\psi} \left\{ 1 + \left(\int_0^t |t-u|^{-2\kappa}\, du\right)^{q/2}\right\} \leq c_{q,f,\psi,T}. \end{aligned}$$

Then, since $(\hat{\delta}Y)_{ts} = \int_s^t S_{t-u} f_i(Y_u)\, dW_u^i$, one has

$$E\left[\|(\hat{\delta}Y)_{ts}\|^q_{\mathcal{B}_{\kappa,p}}\right] \le c\, E\left[\left(\int_s^t \|f_i(Y_u)\|^2_{\mathcal{B}_{\kappa,p}}\, du\right)^{q/2}\right]$$
$$\le c\left\{|t-s|^{q/2} + E\left[\left(\int_s^t \|Y_u\|^2_{\mathcal{B}_{\kappa,p}}\, du\right)^{q/2}\right]\right\}$$
$$\le c\left\{|t-s|^{q/2} + |t-s|^{q/2-1}\int_s^t E\left[\|Y_u\|^q_{\mathcal{B}_{\kappa,p}}\right] du\right\} \le c\,|t-s|^{q/2},$$

where, to get the second inequality, we have used the estimate given by Corollary 2.7. □

Lemma 6.10 *The operators X^B, X^{aB}, X^{Ba}, X^{BB} defined in the Itô sense by formulas (89)–(91), satisfy the conditions of Hypothesis 2.*

Proof Observe first that formulas (92)–(95) remain true for those operators, thanks to Itô's formula. (95) is for instance obtained by applying Itô's formula to the product $S_{t.}\mathbf{B}^{\mathbf{2},ij}_{.s}$, where $\mathbf{B}^{\mathbf{2},ij}_{.s}$ stands for the semimartingale $\mathbf{B}^{\mathbf{2},ij}_{us} = \int_s^u dB_u^i\, (\delta B^j)_{us}$, which gives

$$S_{tt}\mathbf{B}^{\mathbf{2},ij}_{ts} - S_{ts}\mathbf{B}^{\mathbf{2},ij}_{ss} = \int_s^t S_{tu}\, d\mathbf{B}^{\mathbf{2},ij}_{us} + \int_s^t \frac{d}{du}(S_{tu})\, \mathbf{B}^{\mathbf{2},ij}_{us}\, du,$$

or otherwise stated $X^{BB,ij}_{ts} = \mathbf{B}^{\mathbf{2},ij}_{ts} + \int_s^t \Delta S_{tu}\, \mathbf{B}^{\mathbf{2},ij}_{us}\, du$. Once endowed with those expressions, it suffices to follow the lines of the proof of Proposition 6.3. □

Proof of Proposition 6.8 One can of course write $(\hat{\delta}Y)_{ts} = X^{B,i}_{ts} Y^{B,i}_s + Y^\sharp_{ts}$, with $Y^{B,i}_s = f_i(Y_s)$ and $Y^\sharp_{ts} = \int_s^t S_{tu}\, dB_u^i\, \delta(f_i(Y))_{us}$. By applying Lemma 6.5 to the process $\hat{\delta}Y$, we easily deduce from (106) $Y \in \hat{\mathcal{C}}_1^\gamma(\mathcal{B}_{\kappa,p})$ a.s., and accordingly $Y^{B,i} \in \mathcal{C}_1^\kappa(\mathcal{B}_p) \cap \mathcal{C}_1^0(\mathcal{B}_{\kappa,p})$ a.s. As for $Y^\sharp$, one has $(\hat{\delta}Y^\sharp)_{tus} = X^{B,i}_{tu}\delta(f_i(Y))_{us}$, which a.s. entails $\mathcal{N}[\hat{\delta}Y^\sharp; \mathcal{C}_3^{2\kappa}(\mathcal{B}_p)] < \infty$ and also $\mathcal{N}[\hat{\delta}Y^\sharp; \mathcal{C}_3^\gamma(\mathcal{B}_{\kappa,p})] < \infty$. Besides, some estimates similar to those of the proof of Lemma 6.9 show that for any $q \ge 2$,

$$E\left[\|Y^\sharp_{ts}\|^q_{\mathcal{B}_{\kappa,p}}\right] \le c\,|t-s|^{q/2} \quad \text{and} \quad E\left[\|Y^\sharp_{ts}\|^q_{\mathcal{B}_p}\right] \le c\,|t-s|^{q(1/2+\kappa)}.$$

We are thus in position to apply Lemma 6.5 to $Y^\sharp$, which yields $Y^\sharp \in \mathcal{C}_2^{2\kappa}(\mathcal{B}_p) \cap \mathcal{C}_2^\gamma(\mathcal{B}_{\kappa,p})$. This completes the proof of the proposition. □

Proposition 6.11 *The Itô integral $\int_s^t S_{tu}\, dB_u^i\, f_i(Y_u)$ coincides with the rough path integral $\mathcal{J}_{ts}(\hat{d}B^i\, f_i(Y))$ built via Proposition 4.6 from the processes X^B, X^{aB}, X^{Ba}, X^{BB}. Consequently, Y is also solution to the equation in the rough path sense.*

Proof Decomposition (49) remains clearly true for $\int_s^t S_{tu}\, dB_u^i\, f_i(Y_u)$, that is to say (with the notations of Sect. 4) $\int_s^t S_{tu}\, dB_u^i\, f_i(Y_u) = M_{ts} + R_{ts}^1$, where

$$M_{ts} = X_{ts}^{B,i} f_i(Y_s) + X_{ts}^{Ba,i}(y, f_i'(Y))_s + X_{ts}^{BB,ij}(Y^{x,j} \cdot f_i'(Y))_s,$$
$$R_{ts}^1 = \int_s^t S_{t-u}\, dB_u^i\, f_i(Y)_{us}^\sharp.$$

It is also easily seen, with the help of Lemma 4.5, that for any $q \geq 2$, $E\left[\|R_{ts}^1\|_{\mathcal{B}_p}^q\right] \leq c\,|t-s|^{\mu q}$ with $\mu > 3\kappa$. This allows to apply Lemma 6.5 to R^1 and assert that $R^1 \in \mathcal{C}_2^{3\kappa}(\mathcal{B}_p)$ a.s., the control of $\mathcal{N}[\hat\delta R^1; \mathcal{C}_3^{3\kappa}(\mathcal{B}_p)] = \mathcal{N}[\hat\delta M; \mathcal{C}_3^{3\kappa}(\mathcal{B}_p)]$ being established in Proposition 4.6.

On the other hand, we know (see Proposition 4.6 again) that $\mathcal{J}_{ts}(\hat d B^i\, f_i(Y)) = M_{ts} + R_{ts}^2$, with $R_{ts}^2 = \hat\Lambda_{ts}(J) \in \mathcal{C}_2^{3\kappa}(\mathcal{B}_p)$. As a consequence, $R^1 - R^2 \in \operatorname{Ker} \hat\delta_{|\mathcal{C}_2(\mathcal{B}_p)} \cap \mathcal{C}_2^{3\kappa}(\mathcal{B}_p)$ and since $3\kappa > 1$, this readily entails $R^1 = R^2$ a.s. □

As a spin-off of this identification procedure, we can apply Proposition 4.8 to Eq. (105) and retrieve the following continuity statement:

Corollary 6.12 *Assume that $(\kappa, p) \in (1/3, 1/2) \times \mathbb{N}^*$ are such that $\frac{1}{2} - \kappa > \frac{n}{2p}$. Then the Itô map $\psi \mapsto Y$ associated to Eq. (105) is locally Lipschitz: if Y (resp. $\tilde Y$) stands for the solution to the equation with initial condition ψ (resp. $\tilde\psi$), then*

$$\mathcal{N}[Y - \tilde Y; \mathcal{C}_1^\kappa([0,T]; \mathcal{B}_{\kappa,p})] \leq c_{\psi,\tilde\psi}\, \|\psi - \tilde\psi\|_{\mathcal{B}_{\kappa,p}},$$

where $c_{\psi,\tilde\psi} = C(\|\psi\|_{\mathcal{B}_{\kappa,p}}, \|\tilde\psi\|_{\mathcal{B}_{\kappa,p}})$, for some growing function $C(\mathbb{R}^+)^2 \to \mathbb{R}^+$.

6.3 Extension to more general elliptic operators

The strategy we have developed all through the paper for the heat equation can actually be applied to a more general class of operators on $\mathbb{R}^n$ for which the properties exhibited in Sect. 2.2 remain (almost) true. More precisely, those properties hold for any operator of the form

$$A = \sum_{i,j=1}^n \partial_{\xi_i}(a_{ij}(\xi)\, \partial_{\xi_j}) - \mathrm{Id},$$

where the coefficients a_{ij} satisfy the following conditions:

(H1) For any $\xi \in \mathbb{R}^n$, $a_{ij}(\xi)$ is a real symmetric matrix,
(H2) For all $i, j = 1, \ldots, n$, a_{ij} is smooth, bounded, with bounded derivatives,
(H3) There exists $a_0 > 0$ such that for any $\xi \in \mathbb{R}^n$ and any $|v| = 1$, $a(\xi)v \cdot v \geq a_0$,
(H4) For all $i, j = 1, \ldots, n$, there exists a_{ij}^∞ and $\alpha \in (0,1)$ such that $\lim_{|\xi|\to\infty} a_{ij}(\xi) = a_{ij}^\infty$ and $\left|a_{ij}(\xi) - a_{ij}^\infty\right| \leq c\,|\xi|^{-\alpha}$ for any $|\xi| \geq 1$.

Let us only sketch out the arguments that indeed lead to statements similar to those of Propositions 2.4–2.6 and (18)–(19):

- According to [10, Theorem 1.8.1], A is the generator of a symmetric Markov semigroup and consequently [10, Theorems 1.4.1, 1.4.2] the generator of an analytic semigroup of contraction S, which allows to adapt Proposition 2.4 to this context.
- The domains of the fractional powers of A_p coincide with the spaces $L^{2\alpha,p} = [L^p, W^{2,p}]_\alpha$ obtained by complex interpolation of L^p with the usual (integer) Sobolev space $W^{2,p}$. This can be easily deduced from the association of [34, Theorem 11.6.1] and [41, Theorem C], both theorems holding true under (H1)–(H4). The precise definition and properties of $L^{2\alpha,p}$ can be found in [1, Chap. 7]. It is in particular proven that (18) and (19) remain valid for those spaces.
- The regularizing properties (23) and (24) of the semigroup from $L^{p/k}$ to L^p can be shown with the same arguments as in the proof of Proposition 2.5, thanks to the Gaussian estimates for the fundamental solution associated to A (see [17, Chap. 9,Theorem 8]).
- Finally, the control (25) is immediate once we have noticed the identification (see [1, Sections 7.63]) of $L^{2\alpha,p}$ with the space $W^{2\alpha,p}$ defined by the norm

$$\|\varphi\|_{W^{2\alpha,p}}^p = \|\varphi\|_{\mathcal{B}_p}^p + \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} d\eta \, \frac{|\varphi(\xi) - \varphi(\eta)|^p}{|\xi - \eta|^{n+2\alpha p}}.$$

Now, observe that if one wishes to study the following extension of (3)

$$y_0 = \psi, \quad dy_t = \tilde{A} y_t \, dt + \sum_{i=1}^N dx_t^i \, f_i(y_t), \quad \tilde{A} = \sum_{i,j=1}^n \partial_{\xi_i}(a_{ij}(\xi)\, \partial_{\xi_j}), \tag{107}$$

one must first write the system as

$$y_0 = \psi, \quad dy_t = A y_t \, dt + \left[y_t \, dt + \sum_{i=1}^N dx_t^i \, f_i(y_t) \right],$$

and then apply the strategy displayed in Sects. 4–5, taking the whole term into brackets as the perturbation term. For sake of clarity, we have preferred not to include those considerations in the development of our method. However, it is easy to realize that the additional term $y_t \, dt$ doesn't raise any new technical difficulty in the reasoning, so that our main Theorem 1.1 remains true when replacing Δ with the above $\tilde{A}$.

Theorem 6.13 *Assume that x is a γ-Hölder process with $\gamma > 1/3$, which in addition allows the construction of a* 2*-rough path* $\mathbf{x}$. *Assume also that the coefficients a_{ij} satisfy* (H1)–(H4), *and that the vector field σ_i satisfies both conditions* (C1) *and* $(C2)_3$. *Then for any couple $(\kappa, p) \in (\frac{1}{3}, \gamma) \times \mathbb{N}^*$ such that $\gamma - \kappa > \frac{n}{2p}$, and any initial condition*

$\psi \in \mathcal{B}_{\kappa,p}$, the equation

$$y_0 = \psi, \quad dy_t(\xi) = \sum_{i,j=1}^{n} \partial_{\xi_i}(a_{ij} \cdot \partial_{\xi_j} y_t)(\xi) + \sum_{k=1}^{N} \sigma_k(\xi, y_t(\xi))\, dx_t^k, \quad \xi \in \mathbb{R}^n, \tag{108}$$

understood in the mild sense via Propositions 4.6 and 6.3, admits a unique solution $y \in \mathcal{C}^\kappa(\mathcal{B}_{\kappa,p})$ on an interval $[0, T]$, for a strictly positive time T which depends on x, $\mathbf{x^2}$ and ψ. Moreover, the continuity property (11) remains true for the solution of (108).

To conclude with, it may be worth mentioning that the rough paths approach often gives rise to (time-)discretization schemes for the solution without much additional effort. In the infinite-dimensional background at stake here, some space-discretization has to be performed, too, so as to retrieve an efficient scheme, following Galerkin's method for instance. The interested reader is referred to [11] for a detailed examination of some possible schemes derived from the constructions of this paper.

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