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FIRST-ORDER EULER SCHEME FOR SDES DRIVEN BY FRACTIONAL BROWNIAN MOTIONS: THE ROUGH CASE

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In this article, we consider the so-called modified Euler scheme for stochastic differential equations (SDEs) driven by fractional Brownian motions (fBm) with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$. This is a first-order time-discrete numerical approximation scheme, and has been introduced in [*Ann. Appl. Probab.* **26** (2016) 1147–1207] recently in order to generalize the classical Euler scheme for Itô SDEs to the case $H > \frac{1}{2}$. The current contribution generalizes the modified Euler scheme to the rough case $\frac{1}{3} < H < \frac{1}{2}$. Namely, we show a convergence rate of order $n^{\frac{1}{2}-2H}$ for the scheme, and we argue that this rate is exact. We also derive a central limit theorem for the renormalized error of the scheme, thanks to some new techniques for asymptotics of weighted random sums. Our main idea is based on the following observation: the triple of processes obtained by considering the fBm, the scheme process and the normalized error process, can be lifted to a new rough path. In addition, the Hölder norm of this new rough path has an estimate which is independent of the step-size of the scheme.

CONTENTS

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1. Introduction. This note is concerned with the following differential equation driven by a m-dimensional fractional Brownian motion (fBm in the sequel) B with Hurst parameter $\frac{1}{3} < H < \frac{1}{2}$:

$$
\begin{aligned}
&dy_t = b(y_t)\,dt + V(y_t)\,dB_t, \qquad t \in [0, T], \\
&y_0 = y \in \mathbb{R}^d.
\end{aligned}
\tag{1.1}
$$

Assuming that the collection of vector fields $b = (b^i)_{1\le i\le d}$ belongs to the space $C_b^2(\mathbb{R}^d, \mathbb{R}^d)$ and $V = (V_j^i)_{1\le i\le d, 1\le j\le m}$ sits in $C_b^3(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^d))$, the theory of rough paths gives a framework allowing to get existence and uniqueness results for equation (1.1). In addition, the unique solution y in the rough paths sense has γ-Hölder continuity for all $\gamma < H$. The reader is referred to [13, 14, 17] for further details.

In this paper, we are interested in the numerical approximation of equation (1.1) based on a discretization of the time parameter t. For simplicity, we are considering a finite time interval $[0, T]$ and we take the uniform partition $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ on $[0, T]$. Specifically, for $k = 0, \dots, n$ we have $t_k = kh$, where we denote $h = \frac{T}{n}$. Our generic approximation is called y^n, and it starts from the initial condition $y_0^n = y$. In order to introduce our numerical schemes, we shall also use the following notation.

NOTATION 1.1. Let $U = (U^1, \dots, U^d)$ and $V = (V^1, \dots, V^d)$ be two smooth vector fields defined on $\mathbb{R}^d$. We denote by ∂ the operator vector $\partial = (\partial_{x_1}, \dots, \partial_{x_d})$, that is, for $x \in \mathbb{R}^d$ we have $[\partial U(x)]^{kl} = \partial_{x_l} U^k(x)$. With the same matrix convention, the vector field $\partial U V$ is defined as $[\partial U V(x)]^k = \sum_{l=1}^d \partial_{x_l} U^k(x) V^l(x)$.

With those preliminaries in mind, the most classical numerical scheme for stochastic equations is the so-called Euler scheme (or first-order Taylor scheme), which is recursively defined as follows on the uniform partition:

$$
y_{t_{k+1}}^n = y_{t_k}^n + b(y_{t_k}^n)h + V(y_{t_k}^n)\delta B_{t_k t_{k+1}},
$$

where δf_{st} is defined as $f_t - f_s$ for a function f. However, it is easily seen (see, e.g., [8] for details) that the Euler scheme is divergent when the Hurst parameter H is less than $\frac{1}{2}$. To obtain a convergent numerical approximation in this rough situation, higher-order terms from the Taylor expansion need to be included in the scheme. Having the rough paths construction in mind, the simplest method of this kind is the Milstein scheme, or second-order Taylor scheme. It can be expressed recursively as

$$y^n_{t_{k+1}} = y^n_{t_k} + b(y^n_{t_k})h + V(y^n_{t_k})\delta B_{t_k t_{k+1}} + \sum_{i,j=1}^{m} \partial V_i V_j(y^n_{t_k})\mathbb{B}^{ij}_{t_k t_{k+1}}, \tag{1.2}$$

where we have used Notation 1.1 and where $\mathbb{B}$ designates the second-order iterated integral of B (see Section 3.1 for a proper definition). This numerical approximation has first been considered in [7], and has been shown to be convergent as long as $H > \frac{1}{3}$, with an almost sure convergence rate $n^{-(3H-1)+\kappa}$. Here and in the following, $\kappa > 0$ represents an arbitrarily small constant. An extension of the result to nth-order Taylor schemes and to an abstract rough path with arbitrary regularity is contained in [14]; see also [19] for the optimized nth-order Taylor scheme when $H > \frac{1}{2}$.

The nth-order Taylor schemes of the form (1.2) are, however, not implementable in general. This is due to the fact that when $i \neq j$ the terms $\mathbb{B}^{ij}_{t_k t_{k+1}}$ cannot be simulated exactly and have to be approximated on their own. We now mention some contributions giving implementable versions of (1.2) for stochastic differential equation (1.1). They all rely on some cancellation of the randomness in the error process $y - y^n$ related to our standing equation.

(i) The first second-order implementable scheme for (1.1) has been introduced in [8]. It can be expressed in the following form:

$$y^n_{t_{k+1}} = y^n_{t_k} + b(y^n_{t_k})h + V(y^n_{t_k})\delta B_{t_k t_{k+1}} + \frac{1}{2}\sum_{i,j=1}^{m} \partial V_i V_j(y^n_{t_k})\delta B^i_{t_k t_{k+1}}\delta B^j_{t_k t_{k+1}}. \tag{1.3}$$

This scheme has been shown to have convergence rate of order $n^{-(H-\frac{1}{3})+\kappa}$, and the proof relies on the fact that (1.3) is the second-order Taylor scheme for the Wong–Zakai approximation of equation (1.1). The approximation (1.3) has been extended in [1, 11] to a third-order scheme defined as follows:

$$\begin{aligned} y^n_{t_{k+1}} &= y^n_{t_k} + b(y^n_{t_k})h + V(y^n_{t_k})\delta B_{t_k t_{k+1}} + \frac{1}{2}\sum_{i,j=1}^{m} \partial V_i V_j(y^n_{t_k})\delta B^i_{t_k t_{k+1}}\delta B^j_{t_k t_{k+1}} \\ &\quad + \frac{1}{6}\sum_{i,j=1}^{m} \partial(\partial V_i V_j)V_k(y^n_{t_k})\delta B^i_{t_k t_{k+1}}\delta B^j_{t_k t_{k+1}}\delta B^k_{t_k t_{k+1}}. \end{aligned} \tag{1.4}$$

Thanks to a thorough analysis of differences of iterated integrals between two Gaussian processes, a convergence rate $n^{-(2H-\frac{1}{2})+\kappa}$ has been achieved for the scheme (1.4). One should also notice that [11] handles in fact very general Gaussian processes, as long as their covariance function is regular enough in the p-variation sense.

(ii) A different direction has been considered in [18], where the following first-order (meaning first-order with respect to the increments of B) scheme has been introduced:

$$y^n_{t_{k+1}} = y^n_{t_k} + b(y^n_{t_k})h + V(y^n_{t_k})\delta B_{t_k t_{k+1}} + \frac{1}{2}\sum_{j=1}^{m} \partial V_j V_j(y^n_{t_k})h^{2H}. \tag{1.5}$$

This approximation is called modified Euler scheme in [18]. As has been explained in [18], the modified Euler scheme is a natural generalization of the classical Euler scheme of the Stratonovich SDE to the rough SDE (1.1). For this reason, we will call (1.5) the Euler scheme from now on. As the reader might also see from relation (1.5), one gets the Euler scheme from the second-order Taylor scheme (1.2) by changing the terms $\mathbb{B}^{ij}_{t_k t_{k+1}}$ into their respective expected values. Note that since the Euler scheme does not involve products of increments of the underlying fBm, its computation cost is much lower than those of (1.3) and (1.4). In spite of this cost reduction, an exact rate of convergence $n^{-(2H-\frac{1}{2})}$ has been achieved in [18]. The asymptotic error distributions and the weak convergence of the scheme have also been considered, and those results heavily hinge on Malliavin calculus considerations. Notice however that the results in [18] are restricted to the case $H > \frac{1}{2}$.

Having recalled those previous results, the aim of the current paper is quite simple: we wish to extend the results concerning the Euler scheme (1.5) to a truly rough situation. Namely, we will consider equation (1.1) driven by a fractional Brownian motion B with $\frac{1}{3} < H < \frac{1}{2}$. For this equation, we show that the Euler scheme maintains the rate of convergence $n^{-(2H-\frac{1}{2})}$, which is the same as the third-order implementable scheme in (1.4). We also obtain some asymptotic results for the error distribution of the numerical scheme (1.5), which generalize the corresponding results in [18, 22] to the case $H < \frac{1}{2}$. More specifically, we will prove the following results (see Theorems 8.7 and 9.4 for more precise statements).

THEOREM 1.2. *Let y be the solution of equation* (1.1), *and consider the Euler approximation scheme y^n defined in* (1.5). *Then*:

(i) *For any arbitrarily small $\kappa > 0$, the following almost sure convergence holds true*:

$$n^{2H-\frac{1}{2}-\kappa} \sup_{t\in[0,T]} |y_t - y^n_t| \to 0 \qquad \textit{as } n \to \infty.$$

(ii) *The sequence of processes* $n^{2H-\frac{1}{2}}(y-y^n)$ *converges weakly in* $D([0,T])$ *to a process* U *which solves the following equation*:

$$U_t=\int_0^t \partial b(y_s)U_s\,ds+\sum_{j=1}^m\int_0^t \partial V_j(y_s)U_s\,dB_s^j+\sum_{i,j=1}^m\int_0^t \partial V_iV_j(y_s)\,dW_s^{ij}, \tag{1.6}$$

where $W=(W^{ij})$ *is a* $\mathbb{R}^{m\times m}$*-valued Brownian motion with correlated components*, *independent of* B.

We wish to mention again that our scheme (1.5) is numerically more efficient than the implementable schemes (1.3) and (1.4). Indeed:

(i) In (1.4), one has to consider third-order implementable schemes in order to reach the rate of convergence $n^{-(2H-\frac{1}{2})}$. This has to be compared to the modified Euler scheme, which is only first-order. As far as (1.3) is concerned, it yields a convergence rate which is slower than the one provided in the current contribution.

(ii) The implementable scheme (1.4) involves some products of increments of B. Since these are approximated quantities, the computation of their product yields an inconvenient propagation of numerical errors. This is in sharp contrast with the modified Euler scheme, for which the quantity h^{2H} in (1.5) has to be evaluated only once.

(iii) The high order derivatives involved in (1.4) is another source of computational cost that we can avoid in (1.5).

(iv) The algorithm complexity in order to simulate a fBm increment vector of the form $(\delta B_{t_0t_1},\dots,\delta B_{t_{n-1}t_n})$ is of order $n\log n$ (see, e.g., [10]). Therefore, all the aforementioned schemes will also inherit a complexity of order $n\log n$. However, it should be clear from our previous discussion that the modified Euler scheme leads to better constants.

Let us also highlight the fact that our approach does not rely on the special structure of the numerical scheme and does not require the analysis of the Wong–Zakai approximation. In fact, it provides a general procedure for studying time-discrete numerical approximations of RDEs, including the implementable schemes (1.3) and (1.4) we just mentioned, the backward Euler scheme, the Crank–Nicolson scheme and its modifications, Taylor schemes and their modifications introduced in [19] and so on.

Since the proof of our main Theorem 1.2 relies on long computations, we will explain briefly our strategy:

(1) Uniform bounds on the scheme process. Our first step in order to establish our main results is to get uniform boundedness in n for the numerical scheme y^n and its related processes as rough paths. This is done by considering y^n as the solution of a rough differential equation driven by (B,q) where q is the second chaos process given by (5.4), instead of just B. Then the uniform estimates are obtained thanks to some rough path techniques.

(2) Linearization of the error process. The starting point of our proof of the convergence for the scheme (1.5) is a linearization of the error process. Namely, let Φ be the Jacobian of equation (1.1), that is, the derivative of the solution with respect to the initial condition y [see equation (7.1)], and let Ψ designates the inverse matrix of Φ. Then we shall establish [see relation (8.10)] that the difference $y - y^n$ can be expressed as

$$\sum_{e=1}^{3} \Phi_t \int_s^t \Psi_u \, dA_u^e, \tag{1.7}$$

where the terms A^e are given as iterated integrals of the processes y^n, y, B and t^{2H}. Notice that this step is called linearization of the equation because the dynamics of Φ and Ψ are governed by a linear system.

(3) Determination of a main contribution. Next, we derive a decomposition of $\varepsilon = \Psi^n(y - y^n)$ based on (1.7) [see relations (7.2) and (7.3)]:

$$\delta\varepsilon = \delta\tilde{\varepsilon} + \delta\hat{\varepsilon} \qquad \text{with } \delta\hat{\varepsilon}_{st} = \sum_{i,j=1}^{m} \sum_{t_k=s}^{t} G_k^{ij} \left[\mathbb{B}_{t_k t_{k+1}}^{ij} - \frac{h^{2H}}{2} \mathbf{1}_{(i=j)} \right], \tag{1.8}$$

where $\delta\tilde{\varepsilon}$ is proved to be a remainder term and where G is a weight process which is specified later on. It should be noticed at this point that our rate of convergence $n^{-(2H-1/2)}$ in Theorem 1.2 comes from the main term $\hat{\varepsilon}$. Namely, it is a well-known fact (see [3]) that the unweighted sum

$$n^{2H-1/2} \sum_{t_k=0}^{t} \left[\mathbb{B}_{t_k t_{k+1}}^{ii} - \frac{h^{2H}}{2} \right]$$

converges in distribution to a Brownian motion. We shall prove that the weighted sum defining $\hat{\varepsilon}$ obeys the same law, which is one of our main technical steps.

In order to prove that $\tilde{\varepsilon}$ in (1.8) is a remainder term, a thorough estimation of the lift of (B, y^n, ε) and some related linear equations is required. This effort will be carried out in Section 7. Also observe that we implement a recursive procedure which has an interest in its own right. More specifically, we start from the basic estimate $y - y^n \sim n^{-\alpha}$ for some $\alpha > 0$, and we can show that $\delta\tilde{\varepsilon} \sim n^{1-3H-\alpha}$ thanks to some rough paths type expansions. Now combining the estimates for $\delta\tilde{\varepsilon}$ and $\delta\hat{\varepsilon}$ we arrive at a new estimate for ε, and thus $y - y^n$, namely $y - y^n \sim n^{(1-3H-\alpha)\wedge(1/2-2H)}$. Iterating this argument, we are able to improve the estimate of $y - y^n$ to the desired convergence rate of $n^{1/2-2H}$. Notice that the number of iterations is determined by the value of H.

(4) Limit theorems. As mentioned above, the limit theorems for our scheme are obtained by considering the asymptotic behavior of a rough linear equation describing the evolution of the error $y - y^n$, the center of which is a weighted-variation term in the second chaos. In our point of view, this weighted sum is a

"discrete" rough integral. Another substantial part of our efforts, summarized in Section 9, consists in deriving a central limit theorem for this kind of quantity.

Among the ingredients we have alluded to above, the asymptotic behavior of weighted variations has received a lot of attention in recent works; see, for example, [23, 25–28, 30]. Our approach to this problem relies on a combination of rough paths and Malliavin calculus tools, and might have an interest in its own right; see Theorem 4.10 for the precise statement. Indeed, with respect to the aforementioned results, it seems that we can reach a more general class of weights. We are also able to consider the variations for multidimensional fBms, thanks to a simple approximation argument on the simplex.

Eventually, let us stress the fact that, though we have restricted our analysis to equations driven by a fractional Brownian motion here for sake of simplicity, we believe that our results can be extended to a general class of Gaussian processes whose covariance function satisfies reasonable assumptions (such as the ones exhibited in [4, 12]). In this case, if X denotes the centered Gaussian process at stake, we expect the numerical scheme (1.5) to become

$$y_{t_{k+1}}^n = y_{t_k}^n + b(y_{t_k}^n)h + V(y_{t_k}^n)\delta X_{t_k t_{k+1}} + \frac{1}{2}\sum_{j=1}^m \partial V_j V_j(y_{t_k}^n) R(h),$$

where $R(h)$ is a deterministic constant defined by $\mathbb{E}(|X_h|^2)$ and where we have assumed that X has stationary increments.

REMARK 1.3. In spite of the fact that our Theorem 1.2 exhibits a rate of convergence $n^{-(2H-1/2)}$ for the numerical scheme (1.5), it should be noticed that in general the Euler scheme is divergent when one considers an equation driven by a fractional Brownian motion with Hurst parameter $\frac{1}{4} < H \le \frac{1}{3}$. As an intuitive illustration, consider (similar to what is done in [8]) the one-dimensional linear equation:

$$dy_t = y_t \, dB_t, \qquad y_0 = 1. \tag{1.9}$$

In this simple situation, the exact solution to (1.9) is given by

$$y_t = \exp(B_t), \qquad t \in [0, T]. \tag{1.10}$$

We can now compare the exact expression (1.10) to the modified Euler approximation. Indeed, for $t = T$ the modified Euler approximation can be written as

$$y_T^n = \prod_{k=0}^{n-1}\left(1 + \delta B_{t_k t_{k+1}} + \frac{1}{2}\Delta t^{2H}\right).$$

Hence for $n \in \mathbb{N}$ sufficiently large, a simple Taylor expansion argument shows that

$$
\begin{aligned}
y_T^n &= \exp\left(\sum_{k=0}^{n-1} \log\left(1 + \delta B_{t_k t_{k+1}} + \frac{1}{2}\Delta t^{2H}\right)\right) \\
&= \exp\left(B_T - \frac{1}{2}\sum_{k=0}^{n-1}\left(\delta B_{t_k t_{k+1}}^2 - \Delta t^{2H} + O\left(\delta B_{t_k t_{k+1}}^3\right)\right)\right).
\end{aligned} \tag{1.11}
$$

For $H > 1/3$, the right-hand side of (1.11) is easily seen to be convergent. Namely, in this case both terms $\sum_{k=0}^{n-1} O(\delta B_{t_k t_{k+1}}^3)$ and $\sum_{k=0}^{n-1}(\delta B_{t_k t_{k+1}}^2 - \Delta t^{2H})$ are converging to 0. This is in sharp contrast with the situation $H \leq 1/3$, for which it is well known that $\sum_{k=0}^{n-1} |\delta B_{t_k t_{k+1}}|^3 \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$, which implies that y_T^n is not uniformly bounded in n.

Here is how our paper is structured: In Section 2, we recall some results from the theory of rough paths, and prove a discrete version of the sewing map lemma. In Section 3, we consider the fractional Brownian motion as a rough path and derive some elementary results. In Section 4, we first develop some useful upper-bound estimates, and then we introduce a general limit theorem on the asymptotic behavior of weighted random sums. In Section 5, we consider the couple (y^n, B) as a rough path and show that it is uniformly bounded in n. In Section 6, we show that the Euler scheme y^n is convergent, and we derive our first result on the rate of strong convergence of y^n. We also derive some estimates on the error process $y - y^n$. Section 7 is devoted to an elaboration of the estimates for the error process under some new conditions. This leads, in Section 8, to consider the rate of strong convergence of the Euler scheme again, improving the results obtained in Section 6 up to an optimal rate. In Section 9, we prove our main result on the asymptotic error distribution of y^n. In the Appendix, we prove some auxiliary results.

Notation. Let $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ be a partition on $[0, T]$. Take $s, t \in [0, T]$. We write $[\![s, t]\!]$ for the discrete interval that consists of t_k's such that $t_k \in [s, t]$. We denote by $\mathcal{S}_k([s, t])$ the simplex $\{(t_1, \ldots, t_k) \in [s, t]^k; t_1 \leq \cdots \leq t_k\}$. In contrast, whenever we deal with a discrete interval, we set $\mathcal{S}_k([\![s, t]\!]) = \{(t_1, \ldots, t_k) \in [\![s, t]\!]^k; t_1 < \cdots < t_k\}$. For $t = t_k$, we denote $t- := t_{k-1}$, $t+ := t_{k+1}$.

Throughout the paper, we work on a probability space $(\Omega, \mathscr{F}, P)$. If X is a random variable, we denote by $\|X\|_p$ the L_p-norm of X. The letter K stands for a constant which can change from line to line, and $\lfloor a \rfloor$ denotes the integer part of a.

2. Elements of rough paths theory. This section is devoted to introducing the main rough paths notation which will be used in the sequel. We refer to [13, 14] for further details. We shall also state and prove a discrete sewing lemma which is a simplified version of an analogous result contained in [9].

2.1. *Hölder continuous rough paths and rough differential equations.* In this subsection, we introduce some basic concepts of the rough paths theory. Let $\frac{1}{3} < \gamma \leq \frac{1}{2}$, and call $T > 0$ a fixed finite time horizon. The following notation will prevail until the end of the paper: for a finite dimensional vector space $\mathcal{V}$ and two functions $f \in C([0, T], \mathcal{V})$ and $g \in C(\mathcal{S}_2([0, T]), \mathcal{V})$ we set

$$\delta f_{st} = f_t - f_s, \quad \text{and} \quad \delta g_{sut} = g_{st} - g_{su} - g_{ut}. \tag{2.1}$$

We start with the definition of some Hölder seminorms: consider here two paths $x \in C([0, T], \mathbb{R}^m)$ and $\mathbb{X} \in C(\mathcal{S}_2([0, T]), (\mathbb{R}^m)^{\otimes 2})$. Then we denote

$$\begin{aligned} \|x\|_{[s,t],\gamma} &:= \sup_{(u,v)\in\mathcal{S}_2([s,t])} \frac{|\delta x_{uv}|}{|v-u|^{\gamma}}, \\ \|\mathbb{X}\|_{[s,t],2\gamma} &:= \sup_{(u,v)\in\mathcal{S}_2([s,t])} \frac{|\mathbb{X}_{uv}|}{|v-u|^{2\gamma}}, \end{aligned} \tag{2.2}$$

where we stress the fact that the regularity of $\mathbb{X}$ is measured in terms of $|t-s|$. When the seminorms in (2.2) are finite, we say that x and $\mathbb{X}$ are respectively in $C^{\gamma}([0, T], \mathbb{R}^m)$ and $C^{2\gamma}(\mathcal{S}_2([0, T]), (\mathbb{R}^m)^{\otimes 2})$. For convenience, we denote $\|x\|_{\gamma} := \|x\|_{[0,T],\gamma}$ and $\|\mathbb{X}\|_{2\gamma} := \|\mathbb{X}\|_{[0,T],2\gamma}$.

With this preliminary notation in hand, we can now turn to the definition of rough path.

DEFINITION 2.1. Let $x \in C([0, T], \mathbb{R}^m)$, $\mathbb{X} \in C(\mathcal{S}_2([0, T]), (\mathbb{R}^m)^{\otimes 2})$, and $\frac{1}{3} < \gamma \leq \frac{1}{2}$. We call $S_2(x) := (x, \mathbb{X})$ a (second-order) γ-rough path if $\|x\|_{\gamma} < \infty$ and $\|\mathbb{X}\|_{2\gamma} < \infty$, and the following algebraic relation holds true:

$$\delta\mathbb{X}_{sut} = \mathbb{X}_{st} - \mathbb{X}_{su} - \mathbb{X}_{ut} = x_{su} \otimes x_{ut},$$

where we have invoked (2.1) for the definition of $\delta\mathbb{X}$. For a γ-rough path $S_2(x)$, we define a γ-Hölder seminorm as follows:

$$\|S_2(x)\|_{\gamma} := \|x\|_{\gamma} + \|\mathbb{X}\|_{2\gamma}^{\frac{1}{2}}. \tag{2.3}$$

An important subclass of rough paths are the so-called *geometric γ-Hölder rough paths*. A geometric γ-Hölder rough path is a rough path $(x, \mathbb{X})$ such that there exists a sequence of smooth $\mathbb{R}^d$-valued paths $(x^n, \mathbb{X}^n)$ verifying

$$\|x - x^n\|_{\gamma} + \|\mathbb{X} - \mathbb{X}^n\|_{2\gamma} \to 0 \qquad \text{as } n \to \infty. \tag{2.4}$$

We will mainly consider geometric rough paths in the remainder of the article.

In relation to (2.4), notice that when x is a smooth $\mathbb{R}^m$-valued path, we can choose

$$\mathbb{X}_{st} = \int_s^t \int_s^u dx_v \otimes dx_u. \tag{2.5}$$

It is then easily verified that $S_2(x) = (x, \mathbb{X})$, with $\mathbb{X}$ defined in (2.5), is a γ-rough path with $\gamma = 1$. In fact, this is also the unique way to lift a smooth path to a γ-rough path.

Recall now that we interpret equation (1.1) in the rough paths sense. That is, we shall consider the following general rough differential equation (RDE):

$$\begin{aligned} dy_t &= b(y_t)\,dt + V(y_t)\,dx_t, \\ y_0 &= y, \end{aligned} \tag{2.6}$$

where b and V are smooth enough coefficients and x is a rough path as given in Definition 2.1. We shall interpret equation (2.6) in a way introduced by Davie in [7], which is conveniently compatible with numerical approximations.

DEFINITION 2.2. We say that y is a solution of (2.6) on $[0, T]$ if $y_0 = y$ and there exists a constant $K > 0$ and $\mu > 1$ such that

$$\left| \delta y_{st} - \int_s^t b(y_u)\,du - V(y_s)\delta x_{st} - \sum_{i,j=1}^{m} \partial V_i V_j(y_s)\mathbb{X}_{st}^{ij} \right| \le K|t-s|^{\mu} \tag{2.7}$$

for all $(s,t) \in \mathcal{S}_2([0, T])$, where we recall that δy is defined by (2.1).

Notice that if y solves (2.6) according to Definition 2.2, then it is also a controlled process as defined in [13, 17]. Namely, if y satisfies relation (2.7), then we also have

$$\delta y_{st} = V(y_s)\delta x_{st} + r_{st}^{y}, \tag{2.8}$$

where $r^y \in C^{2\gamma}(\mathcal{S}_2([0, T]))$. We can thus define iterated integrals of y with respect to itself thanks to the sewing map; see Proposition 1 in [17]. This yields the following decomposition:

$$\left| \int_s^t y_u^i\,dy_u^j - y_s^i \delta y_{st}^j - \sum_{i',j'=1}^{m} V_{i'}^i V_{j'}^j(y_s)\mathbb{X}_{st}^{i'j'} \right| \le K(t-s)^{3\gamma},$$

for all $(s,t) \in \mathcal{S}_2([0, T])$ and $i, j = 1, \dots, d$. In other words, the signature type path $S_2(y) = (y, \mathbb{Y})$ defines a rough path according to Definition 2.1, where $\mathbb{Y}$ denotes the iterated integral of y.

We can now state an existence and uniqueness result for rough differential equations. The reader is referred to, for example, [14], Theorem 10.36 for further details.

THEOREM 2.3. *Assume that $V = (V_j)_{1 \le j \le m}$ is a collection of $C_b^{\lfloor 1/\gamma \rfloor + 1}$-vector fields on $\mathbb{R}^d$. Then there exists a unique RDE solution to equation* (2.6), *understood as in Definition* 2.2. *In addition, the unique solution y satisfies the following estimate*:

$$\left| S_2(y)_{st} \right| \le K\left(1 \vee \|S_2(x)\|_{\gamma,[s,t]}^{1/\gamma}\right)(t-s)^{\gamma}.$$

Whenever $V = (V_j)_{1\le j\le m}$ is a collection of linear vector fields, the existence and uniqueness results still hold, and we have the estimate:

$$\left|S_2(y)_{st}\right| \le K_1 \left\|S_2(x)\right\|_{\gamma,[s,t]} \exp\left(K_2 \left\|S_2(x)\right\|_{\gamma}^{1/\gamma}\right)(t-s)^{\gamma}.$$

2.2. *A discrete-time sewing map lemma*. In this subsection, we derive a discrete version of the sewing map lemma which will play a prominent role in the analysis of our numerical scheme. Let $\pi : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ be a generic partition of the interval $[0, T]$ for $n \in \mathbb{N}$. For $0 \le s < t \le T$, we denote by $[\![s, t]\!]$ the discrete interval $\{t_k : s \le t_k \le t\}$. We also label the following definition for further use.

DEFINITION 2.4. We denote by $\mathcal{C}_2(\pi, \mathcal{X})$ the collection of functions R on $\mathcal{S}_2([\![0, T]\!])$ with values in a Banach space $(\mathcal{X}, |\cdot|)$ such that $R_{t_k t_{k+1}} = 0$ for $k = 0, 1, \dots, n-1$. Similar to the continuous case [relations (2.1) and (2.2)], we define the operator δ and some Hölder seminorms on $\mathcal{C}_2(\pi, \mathcal{X})$ as follows:

$$\delta R_{sut} = R_{st} - R_{su} - R_{ut}, \quad \text{and} \quad \|R\|_{\mu} = \sup_{(u,v)\in\mathcal{S}_2([\![0,T]\!])} \frac{|R_{uv}|}{|u-v|^{\mu}}.$$

For $R \in \mathcal{C}_2(\pi, \mathcal{X})$ and $\mu > 0$, we also set

$$\|\delta R\|_{\mu} = \sup_{(s,u,t)\in\mathcal{S}_3([\![0,T]\!])} \frac{|\delta R_{sut}|}{|t-s|^{\mu}}. \tag{2.9}$$

The space of functions $R \in \mathcal{C}_2(\pi, \mathcal{X})$ equipped with the seminorm $\|\cdot\|_{\mu}$ is denoted by $\mathcal{C}_2^{\mu}(\pi, \mathcal{X})$.

We can now turn to our discrete version of the sewing map lemma. This result is inspired by [9], but is included here since our situation is simpler and leads to a straightforward proof.

LEMMA 2.5. *For a Banach space $\mathcal{X}$, an exponent $\mu > 1$ and $R \in \mathcal{C}_2^{\mu}(\pi, \mathcal{X})$ as in Definition 2.4, the following relation holds true*:

$$\|R\|_{\mu} \le K_{\mu} \|\delta R\|_{\mu} \qquad \text{where } K_{\mu} = 2^{\mu} \sum_{l=1}^{\infty} l^{-\mu}.$$

PROOF. Take $t_i, t_j \in \pi$. Let π_l, $l = 1, \dots, j-i$ be partitions on $[\![0, T]\!]$ defined recursively as follows: Set $\pi_1 = \{t_i, t_j\}$ and $\pi_{j-i} = [\![t_i, t_j]\!] \cap \pi$. Given a partition $\pi_l = \{t_i = t_0^l < \cdots < t_l^l = t_j\}$ on $[\![t_i, t_j]\!]$, $l = 2, \dots, j-i$, we can find $t_{k_l}^l \in \pi_l \setminus \{t_i, t_j\}$ such that

$$t_{k_l+1}^l - t_{k_l-1}^l \le \frac{2(t_j - t_i)}{l-1}. \tag{2.10}$$

We denote by π_{l-1} the partition $\pi_l \setminus \{t^l_{k_l}\}$. For $l = 1, \dots, j-i$, we also set

$$R^{\pi_l} = \sum_{k=0}^{l-1} R_{t^l_k t^l_{k+1}},$$

and we observe that $R^{\pi_1} = R_{t_i t_j}$, and $R^{\pi_{j-i}} = \sum_{k=i}^{j-1} R_{t_k t_{k+1}} = 0$, where the last relation is due to the fact that $R \in \mathcal{C}_2(\pi, \mathcal{X})$.

With those preliminaries in hand, we can decompose $R_{t_i t_j}$ as follows: we write

$$R_{t_i t_j} = R_{t_i t_j} - \sum_{k=i}^{j-1} R_{t_k t_{k+1}} = \sum_{l=2}^{j-i} \big(R^{\pi_{l-1}} - R^{\pi_l}\big). \tag{2.11}$$

Now, according to the definition of π_l, we have

$$\begin{aligned}\big|R^{\pi_{l-1}} - R^{\pi_l}\big| &= |\delta R_{t^l_{k_l-1} t^l_{k_l} t^l_{k_l+1}}| \le \|\delta R\|_\mu \big(t^l_{k_l+1} - t^l_{k_l-1}\big)^\mu \\ &\le \|\delta R\|_\mu \frac{2^\mu (t_j - t_i)^\mu}{(l-1)^\mu},\end{aligned}$$

where the first inequality follows from (2.9) and the second from (2.10). Applying the above estimate of $|R^{\pi_{l-1}} - R^{\pi_l}|$ to (2.11), we obtain

$$|R_{t_i t_j}| \le 2^\mu (t_j - t_i)^\mu \|\delta R\|_\mu \sum_{l=1}^{j-i-1} \frac{1}{l^\mu} \le K_\mu (t_j - t_i)^\mu \|\delta R\|_\mu.$$

Dividing both sides of the above inequality by $(t_j - t_i)^\mu$ and taking supremum over (t_i, t_j) in $\mathcal{S}_2([\![0, T]\!])$, we obtain the desired estimate. $\square$

3. Elements of fractional Brownian motions. In this section, we briefly recall the construction of a rough path above our fBm B. The reader is referred to [14] for further details. In the second part of the section, we turn to some estimates for the Lévy area of B on a discrete grid, which are essential in the analysis of our scheme.

3.1. *Enhanced fractional Brownian motion.* Let $B = (B^1, \dots, B^m)$ be a standard m-dimensional fBm on $[0, T]$ with Hurst parameter $H \in (\frac{1}{3}, \frac{1}{2})$. Recall that the covariance function of each coordinate of B is defined on $\mathcal{S}_2([0, T])$ by

$$R(s, t) = \frac{1}{2}\big[|s|^{2H} + |t|^{2H} - |t-s|^{2H}\big]. \tag{3.1}$$

We start by reviewing some properties of the covariance function of B considered as a function on $(\mathcal{S}_2([0, T]))^2$. Namely, take u, v, s, t in $[0, T]$ and set

$$R\begin{pmatrix} u & v \\ s & t \end{pmatrix} = \mathbb{E}\big[\delta B^j_{uv} \delta B^j_{st}\big]. \tag{3.2}$$

Then, whenever $H > 1/4$, it can be shown that the integral $\int R\, dR$ is well defined as a Young integral in the plane (see, e.g., [14], Section 6.4). Furthermore, if intervals $[u, v]$ and $[s, t]$ are disjoint, we have

$$R\begin{pmatrix} u & v \\ s & t \end{pmatrix} = \int_u^v \int_s^t \mu(dr'\, dr). \tag{3.3}$$

Here and in the following, we denote

$$\mu(dr'\, dr) = -H(1-2H)|r-r'|^{2H-2}\, dr'\, dr. \tag{3.4}$$

Using the elementary properties above, it is shown in [14], Chapter 15 that for any piecewise linear or mollifier approximation B^n to B, the geometric rough path $S_2(B^n)$ converges in the γ-Hölder seminorm (2.3) to a γ-geometric rough path $S_2(B) := (B, \mathbb{B})$ (given as in Definition 2.1) for $\frac{1}{3} < \gamma < H$. In addition, for $i \neq j$ the covariance of $\mathbb{B}^{ij}$ can be expressed in terms of a two-dimensional Young integral:

$$\mathbb{E}(\mathbb{B}^{ij}_{uv}\mathbb{B}^{ij}_{st}) = \int_u^v \int_s^t R\begin{pmatrix} u & r \\ s & r' \end{pmatrix} dR(r', r). \tag{3.5}$$

It is also established in [14], Chapter 15 that $S_2(B)$ enjoys the following integrability property.

PROPOSITION 3.1. *Let $S_2(B) := (B, \mathbb{B})$ be the rough path above B, and $\gamma \in (\frac{1}{3}, H)$. Then there exists a random variable $L_\gamma \in \bigcap_{p \geq 1} L^p(\Omega)$ such that $\|S_2(B)\|_\gamma \leq L_\gamma$, where $\|\cdot\|_\gamma$ is defined by* (2.3).

We now specialize (3.5) to a situation where (u, v) and (s, t) are disjoint intervals such that $u < v < s < t$. In this case relation (3.3) enables us to write

$$\mathbb{E}(\mathbb{B}^{ij}_{uv}\mathbb{B}^{ij}_{st}) = \int_u^v \int_s^t \int_u^r \int_s^{r'} \mu(dw'\, dw)\mu(dr'\, dr), \tag{3.6}$$

where μ is the measure given by (3.4). Note that the left-hand side of (3.6) converges to $\mathbb{E}(\mathbb{B}^{ij}_{us}\mathbb{B}^{ij}_{st})$ as $v \to s$. Therefore, the quadruple integral in (3.6) converges as $v \to s$. This implies that the quadruple integral exists and identity (3.6) still holds when $s = v$.

Having relation (3.6) in mind, let us label the following definition for further use.

DEFINITION 3.2. Denote by $\mathcal{E}_{[a,b]}$ the set of step functions on an interval $[a, b] \subset [0, T]$. We call $\mathcal{H}_{[a,b]}$ the Hilbert space defined as the closure of $\mathcal{E}_{[a,b]}$ with respect to the scalar product

$$\langle \mathbf{1}_{[u,v]}, \mathbf{1}_{[s,t]} \rangle_{\mathcal{H}_{[a,b]}} = R\begin{pmatrix} u & v \\ s & t \end{pmatrix}.$$

In order to alleviate notation, we will still write $\mathcal{H} = \mathcal{H}_{[a,b]}$ when $[a,b] = [0,T]$. Notice that the mapping $\mathbf{1}_{[s,t]} \to \delta B_{st}$ can be extended to an isometry between $\mathcal{H}_{[a,b]}$ and the Gaussian space associated with $\{B_t, t \in [a,b]\}$. We denote this isometry by $h \to \int_a^b h \delta B$. The random variable $\int_a^b h \delta B$ is called the (first-order) Wiener integral and is also denoted by $I_1(h)$.

Owing to the fact that $H < 1/2$ throughout this article, we have the following identity:

$$\|h\|_{\mathcal{H}_{[a,b]}} = \big\| d_H s^{\frac{1}{2}-H} \big(D_{T-}^{\frac{1}{2}-H} u^{H-\frac{1}{2}} h(u)\big)(s)\big\|_{L^2([a,b])}, \tag{3.7}$$

where d_H is a constant depending on H and $D_{T-}^{\frac{1}{2}-H}$ is the right-sided fractional differentiation operator; see (5.31) in [31]. With the help of (3.7), it is easy to derive the following relation for $1 > \kappa > 0$ and $\gamma > \frac{1}{2} - H$:

$$K_1 \|h\|_{L^{2-\kappa}([a,b])} \leq \|h\|_{\mathcal{H}_{[a,b]}} \leq K_2\Big(\sup_{t\in[a,b]} h(t) + \|h\|_{C^\gamma([a,b])}\Big). \tag{3.8}$$

Indeed, the lower-bound inequality can be obtained by the Hardy–Littlewood inequality, while the upper-bound estimate follows from the definition of the fractional derivative.

In order to generalize relations (3.3) to a more general situation, recall that for $h \in \mathcal{H}_{[a,b]}$ we have $h_n \in \mathcal{E}_{[a,b]}$ such that $h_n \to h$ in $\mathcal{H}_{[a,b]}$. We denote by h_n^e the extension of h_n on $[0,T]$ such that $h_n^e = h_n$ on $[a,b]$ and $h_n^e = 0$ on $[0,T] \setminus [a,b]$. Then $\int_0^T h_n^e \, dB = \int_{[a,b]} h_n \, dB$ is a Cauchy sequence in $L^2(\Omega)$, and thus so is h_n^e in $\mathcal{H}$. We denote the limit of h_n^e by h^e. It is easy to see that $h^e \in \mathcal{H}$ satisfies $h^e|_{[a,b]} = h$, $h^e|_{[a,b]^c} = 0$, and $\|h^e\|_{\mathcal{H}} = \|h\|_{\mathcal{H}_{[a,b]}}$.

LEMMA 3.3. *Take $f \in \mathcal{H}_{[a,b]}$ and $g \in \mathcal{H}_{[c,d]}$, where $[a,b]$ and $[c,d]$ are disjoint subintervals of $[0,T]$ such that $a < b < c < d$. Then the following identity holds true*:

$$\begin{aligned} \mathbb{E}\Big(\int_{[a,b]} f \delta B \int_{[c,d]} g \delta B\Big) &= \mathbb{E}\Big(\int_0^T f^e \delta B \int_0^T g^e \delta B\Big) \\ &= \int_{[a,b]} \int_{[c,d]} f_t g_s \mu(ds\, dt), \end{aligned} \tag{3.9}$$

where μ is the measure defined by (3.4).

PROOF. Take $f_n \in \mathcal{E}_{[a,b]}$ and $g_n \in \mathcal{E}_{[c,d]}$ such that $f_n \to f$ in $\mathcal{H}_{[a,b]}$ and $g_n \to g$ in $\mathcal{H}_{[c,d]}$. Then we have

$$\langle f_n^e, g_n^e \rangle_{\mathcal{H}} \to \langle f^e, g^e \rangle_{\mathcal{H}} \qquad \text{as } n \to \infty. \tag{3.10}$$

On the other hand, take $c-b>\kappa>0$, then owing to (3.4) we have

$$
\begin{aligned}
&-\int_{[a,b]}\int_{[c,d]} |f_t|\cdot|g_s|\mu(ds\,dt) \\
&\qquad \le H(1-2H)\kappa^{2H-2}\|f^e\|_{L^1([0,T])}\|g^e\|_{L^1([0,T])}.
\end{aligned}
\tag{3.11}
$$

In particular, the left-hand side of (3.11) is finite. Since

$$
\langle f_n^e, g_n^e\rangle_{\mathcal{H}} = \int_{[a,b]}\int_{[c,d]} f_n(t)g_n(s)\mu(ds\,dt),
$$

we can write

$$
\begin{aligned}
&\langle f_n^e, g_n^e\rangle_{\mathcal{H}} - \int_{[a,b]}\int_{[c,d]} f(t)g(s)\mu(ds\,dt) \\
&\quad = \int_{[a,b]}\int_{[c,d]} (f_n-f)(t)g_n(s)\mu(ds\,dt) \\
&\qquad + \int_{[a,b]}\int_{[c,d]} f(t)(g_n-g)(s)\mu(ds\,dt).
\end{aligned}
\tag{3.12}
$$

Applying (3.11) to the right-hand side of (3.12) and taking into account that $\|f_n - f\|_{L^1([a,b])}\to 0$ and $\|g_n-g\|_{L^1([c,d])}\to 0$ as $n\to\infty$, which follow from (3.8), we obtain

$$
\langle f_n^e, g_n^e\rangle_{\mathcal{H}} \to \int_{[a,b]}\int_{[c,d]} f_t g_s \mu(ds\,dt) \qquad \text{as } n\to\infty.
\tag{3.13}
$$

The identity (3.9) then follows from (3.10) and (3.13) and the uniqueness of the limit of $\langle f_n^e, g_n^e\rangle_{\mathcal{H}}$. $\square$

3.2. *Upper-bound estimates for a Lévy area type process.* Let $(B,\mathbb{B})$ be an enhanced fractional Brownian motion as in the previous subsection. We now go back to the discrete interval $[\![0,T]\!]$ considered in Section 2.2. We denote by F_t^{ij} the process on $[\![0,T]\!]$ such that

$$
F_0^{ij}=0, \qquad F_t^{ij} = \begin{cases} \displaystyle\sum_{t_k=0}^{t_-} \mathbb{B}^{ij}_{t_k,t_{k+1}}, & i\neq j, \\ \displaystyle\sum_{t_k=0}^{t_-}\left(\mathbb{B}^{ii}_{t_k t_{k+1}} - \frac{1}{2}h^{2H}\right), & i=j, \end{cases}
\tag{3.14}
$$

for $t>0$, where $t_-=t_{j-1}$ if $t=t_j$ and where we recall that $h=t_j-t_{j-1}=\frac{T}{n}$.

LEMMA 3.4. *For F^{ij} defined as in* (3.14), *we have the following estimate*:

$$
\|\delta F_{st}^{ij}\|_p \le K_p n^{\frac{1}{2}-2H}(t-s)^{\frac{1}{2}}, \qquad (s,t)\in\mathcal{S}_2([\![0,T]\!]),
\tag{3.15}
$$

where K_p is a constant depending on p, H and T, and $\|\cdot\|_p$ denotes the $L_p(\Omega)$-norm.

PROOF. We only consider the case $i = j$. The case $i \neq j$ can be considered similarly. Since δF^{ij}_{st} is a random variable in the second chaos of B, some hypercontractivity arguments (see, e.g., [31]) show that it suffices to consider the case $p = 2$ in (3.15). On the other hand, it is clear that $\mathbb{B}^{ii}_{t_k t_{k+1}} - \frac{1}{2}h^{2H}$ is equal to $\frac{1}{2}h^{2H} H_2(B_{k,k+1})$ in distribution, where $H_2(x) = x^2 - 1$. So we are reduced to estimate the following quantity:

$$\|F^{ij}_t\|_2 = \frac{1}{2}h^{2H} \left\| \sum_{t_k=0}^{t-} H_2(B_{k,k+1}) \right\|_2. \tag{3.16}$$

A direct computation of second moments shows that $\| \sum_{t_k=0}^{t-} H_2(B_{k,k+1})\|_2 \leq K(t-s)^{\frac{1}{2}}\sqrt{n}$; see, for example, the proof of Theorem 7.4.1 in [29]. Applying this relation to (3.16), we obtain the estimate (3.15). □

The following result provides a way to find a uniform almost sure upper-bound estimate for a sequence of stochastic processes.

LEMMA 3.5. *Let $\{X^n; n \in \mathbb{N}\}$ be a sequence of stochastic processes such that*

$$\|\delta X^n_{st}\|_p \leq K_p n^{-\alpha}(t-s)^{\beta} \tag{3.17}$$

for all $p \geq 1$, where K_p is a constant depending on p. Then for $0 < \gamma < \beta$ and $\kappa > 0$, we can find an integrable random variable $G_{\gamma,\kappa}$ independent of n such that

$$\|X^n\|_\gamma \leq G_{\gamma,\kappa} n^{-\alpha+\kappa}.$$

PROOF. Take $p \geq 1$ such that $0 < \gamma < \beta - 1/p$. The Garsia–Rodemich–Rumsey lemma (see [15]) implies that

$$\|X^n\|^p_\gamma \leq K_p \int_0^T \int_0^T \frac{|X^n_u - X^n_v|^p}{|u-v|^{2+p\gamma}}\, du\, dv.$$

Taking expectation on both sides and taking into account the inequality (3.17), we obtain

$$\begin{aligned}\mathbb{E}[\|X^n\|^p_\gamma] &\leq K_p \int_0^T \int_0^T \frac{\mathbb{E}[|X^n_u - X^n_v|^p]}{|u-v|^{2+p\gamma}}\, du\, dv \\ &\leq K_p n^{-p\alpha},\end{aligned}$$

and the last inequality can be recast as

$$\mathbb{E}[\|n^{\alpha-\kappa} X^n\|^p_\gamma] \leq K_p n^{-p\kappa}.$$

We now choose p such that $p > 1/\kappa$. Therefore, the above estimate implies that

$$\mathbb{E}\Big[\sup_{n\in\mathbb{N}} \|n^{\alpha-\kappa} X^n\|^p_\gamma\Big] \leq \mathbb{E}\Big[\sum_{n\in\mathbb{N}} \|n^{\alpha-\kappa} X^n\|^p_\gamma\Big] \leq K_p \sum_{n\in\mathbb{N}} n^{-p\kappa} < \infty.$$

In particular, we obtain that $\sup_{n\in\mathbb{N}} \|n^{\alpha-\kappa} X^n\|_\gamma^p$ is an integrable random variable. By taking $G_{\gamma,\kappa} = \sup_{n\in\mathbb{N}} \|n^{\alpha-\kappa} X^n\|_\gamma$, we obtain the desired estimate for $\|X^n\|_\gamma$. □

REMARK 3.6. One can improve the regularity of F in the following way. Let γ be a parameter such that $\frac{1}{3} < \gamma < H$. Starting from relation (3.15) and taking into account the fact that $|t-s| \geq \frac{T}{n}$ for all $(s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket)$, it is readily checked that the increment F introduced in Lemma 3.4 satisfies

$$\|\delta F_{st}\|_p \leq K_p n^{\beta-2H}(t-s)^\beta, \tag{3.18}$$

for all $2\gamma < \beta < 2H$ and $(s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket)$. By considering the linear interpolation of F on $[0,T]$, inequality (3.18) also holds for all $(s,t) \in \mathcal{S}_2([0,T])$. Owing to Lemma 3.5, we can thus find an integrable random variable G_γ such that for any $\gamma : \frac{1}{3} < \gamma < H$ we have

$$\left|\delta F_{st}^{ij}\right| \leq G_\gamma (t-s)^{2\gamma} \qquad \text{a.s.} \tag{3.19}$$

4. Weighted random sums via the rough path approach. In this section, we derive some useful upper-bound estimates for weighted random sums related to B. In the second part of the section, we prove a general limit theorem, which is our main result of this section.

4.1. *Upper-bound estimates for weighted random sums*. We now derive some estimates for weighted random sums. As has been mentioned in the Introduction, these results only require the weight function to satisfy some proper regularity conditions.

PROPOSITION 4.1. *Let f and g be paths on $\llbracket 0,T \rrbracket$ such that $|\delta f_{st}| \leq G(t-s)^\alpha$ and $|\delta g_{st}| \leq G(t-s)^\beta$, where $\alpha+\beta > 1$. We define an increment R on $\mathcal{S}_2(\llbracket 0,T \rrbracket)$ by*

$$R_{st} = \sum_{t_k=s}^{t-} \delta f_{st_k} \delta g_{t_k t_{k+1}}.$$

Then the following estimate holds true:

$$|R_{st}| \leq G^2 (t-s)^{\alpha+\beta} \qquad \textit{for all } (s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket).$$

PROOF. It is clear that $R_{t_k t_{k+1}} = \delta f_{t_k t_k} \delta g_{t_k t_{k+1}} = 0$. In addition, the following relation is readily checked, where we recall that δR is defined by (2.1):

$$\delta R_{sut} = \delta f_{su} \delta g_{ut} \qquad \text{for all } (s,u,t) \in \mathcal{S}_3(\llbracket 0,T \rrbracket).$$

Therefore, we have $|\delta R_{sut}| \leq G|t-s|^{\alpha+\beta}$. Since we have assumed $\alpha+\beta>1$, we can invoke the discrete sewing map Lemma 2.5, which yields

$$\|R\|_{\alpha+\beta} \leq K\|\delta R\|_{\alpha+\beta} \leq G^2.$$

The proposition then follows immediately. □

REMARK 4.2. The Riemann–Stieltjes sum $\sum_{t_k=s}^{t-} \delta f_{st_k} \delta g_{t_k t_{k+1}}$ in Proposition 4.1 can be thought of as a $\mathbb{R}$-valued "discrete" Young integral. One can also consider L_p-valued "discrete" Young integrals in a similar way by viewing f and g as functions with values in L_p. This will lead us to an L_p-estimate of $\sum_{t_k=s}^{t-} \delta f_{st_k} \delta g_{t_k t_{k+1}}$. Precisely, suppose that f and g are processes such that $\|\delta f_{st}\|_p \leq K(t-s)^{\alpha}$ and $\|\delta g_{st}\|_p \leq K(t-s)^{\beta}$ for all $p \geq 1$. Then we have

$$\left\| \sum_{t_k=s}^{t-} \delta f_{st_k} \delta g_{t_k t_{k+1}} \right\|_p \leq K(t-s)^{\alpha+\beta}.$$

In the sequel, we consider an application of Proposition 4.1 to third-order terms in our Taylor expansion for equation (1.1). Toward this aim, we first need the following estimate in $L_p(\Omega)$. They are somehow reminiscent of the estimates for triple integrals in [11], though our main focus here is on cumulative sums of triple integrals.

LEMMA 4.3. *Let B be an $\mathbb{R}^m$-valued fractional Brownian motion with Hurst parameter $H > \frac{1}{4}$. For a fixed set of coordinates $i, j, l \in \{1, \dots, m\}$, we define two increments $\zeta = \zeta^{ijl}$ and $\delta g = \delta g^{ijl}$ on $\mathcal{S}_2([\![0, T]\!])$ as follows*:

$$\zeta_{st}^{ijl} = \int_s^t \int_s^u \int_s^v dB_r^i \, dB_v^j \, dB_u^l, \quad \textit{and} \quad \delta g_{st} = \sum_{t_k=s}^{t-} \zeta_{t_k t_{k+1}}. \tag{4.1}$$

Then the following estimate is valid for $(s,t) \in \mathcal{S}_2([\![0, T]\!])$:

$$\|\delta g_{st}\|_p \leq K n^{\frac{1}{2}-3H} (t-s)^{\frac{1}{2}}. \tag{4.2}$$

PROOF. When $i = j = l$, we have $\zeta_{st} = \frac{1}{6}(\delta B_{st}^i)^3$ and (4.2) follows from the classical moment estimates results contained in [3, 16]. In the following, we consider the case when i, j, l are not all equal.

Let us further reduce our problem. First, since the fBm has stationary increment it suffices to prove the lemma for $s = 0$. Furthermore, by self-similarity of the fBm, we have the following equation in distribution:

$$\delta g_{0t} = T^{3H} n^{-3H} \sum_{k=0}^{\frac{nt}{T}-1} \zeta_{k,k+1}.$$

As a last preliminary step, note that ζ takes values in the third chaos of B, on which all L^p-norms are equivalent. Hence, our claim (4.2) boils down to prove

$$\left\| \sum_{k=0}^{\frac{nt}{T}-1} \zeta_{k,k+1} \right\|_2^2 \leq Knt. \tag{4.3}$$

We now focus on this inequality.

We first consider the case when i, j and l are different from each other. In order to prove relation (4.3), write

$$\left\| \sum_{k=0}^{\frac{nt}{T}-1} \zeta_{k,k+1} \right\|_2^2 = \sum_{|k-k'|\leq 1} \mathbb{E}(\zeta_{k,k+1}\zeta_{k',k'+1}) + \sum_{|k-k'|>1} \mathbb{E}(\zeta_{k,k+1}\zeta_{k',k'+1}). \tag{4.4}$$

For the sum $\sum_{|k-k'|>1}$ above, thanks to the independence of B^i, B^j and B^l, one can apply Lemma 3.3 twice in order to get

$$\mathbb{E}(\zeta_{k,k+1}\zeta_{k',k'+1}) = \int_k^{k+1}\int_{k'}^{k'+1}\int_k^u\int_{k'}^{u'}\int_k^v\int_{k'}^{v'} \mu(dr'\,dr)\mu(dv'\,dv)\mu(du'\,du).$$

In particular, since (3.4) reveals that μ is a negative measure whenever $H < 1/2$, we have

$$\sum_{|k-k'|>1} \mathbb{E}(\zeta_{k,k+1}\zeta_{k',k'+1}) \leq 0. \tag{4.5}$$

Moreover, since $\|\zeta_{k,k+1}\|_2 = \|\zeta_{0,1}\|_2$ and $\mathbb{E}(\zeta_{k,k+1}\zeta_{k+1,k+2}) = \mathbb{E}(\zeta_{0,1}\zeta_{1,2})$, the following bound is easily checked:

$$\sum_{|k-k'|\leq 1} \left|\mathbb{E}(\zeta_{k,k+1}\zeta_{k',k'+1})\right| \leq Knt. \tag{4.6}$$

Applying (4.5) and (4.6) to the right-hand side of (4.4), we obtain (4.3).

Assume now that $i = j \neq l$. Then

$$\zeta_{k,k+1} = \frac{1}{2}\int_k^{k+1} (\delta B^i_{ku})^2\, dB^l_u = \frac{1}{2}(\zeta^{1,il}_{k,k+1} + \zeta^{2,l}_{k,k+1}),$$

where we have set

$$\begin{aligned} \zeta^{1,il}_{k,k+1} &= \int_k^{k+1} \big[(\delta B^i_{ku})^2 - (u-k)^{2H}\big]\, dB^l_u, \quad \text{and} \\ \zeta^{2,l}_{k,k+1} &= \int_k^{k+1} (u-k)^{2H}\, dB^l_u. \end{aligned} \tag{4.7}$$

We now treat the terms $\zeta^{1,il}$ and $\zeta^{2,l}$ similar to the case of different indices i, j, l. Namely, we decompose $\|\sum_{k=0}^{\frac{nt}{T}-1} \zeta^{2,l}_{k,k+1}\|_2^2$ as in (4.4). Then in the same way as for (4.5) and (4.6), we can show that

$$\sum_{|k-k'|>1} \mathbb{E}(\zeta^{2,l}_{k,k+1}\zeta^{2,l}_{k',k'+1}) \leq 0, \quad \text{and} \quad \sum_{|k-k'|\leq 1} \mathbb{E}(\zeta^{2,l}_{k,k+1}\zeta^{2,l}_{k',k'+1}) \leq Knt.$$

One can thus easily show that $\zeta^{2,l}_{k,k+1}$ satisfies the inequality (4.3). The same argument can be applied to $\zeta^{1,il}_{k,k+1}$, which yields (4.3) for the case $i = j \neq l$. We let the patient reader check that the same inequality holds true in the case $i \neq j$, $i = l$ and the case $i \neq l = j$, resorting to the fact that the multiple integrals of ζ^{ijl} are exchangeable which follows from the way an enhanced fBm is constructed; see, for example, [14], Theorem 15.42. This completes the proof. □

REMARK 4.4. The estimate of $\zeta^{2,l}_{k,k+1}$ obtained in the proof of Lemma 4.3 implies that

$$\left\| \sum_{t_k=s}^{t-} \int_{t_k}^{t_{k+1}} (u - t_k)^{2H} \, dB_u \right\|_p \leq K n^{\frac{1}{2}-3H} (t-s)^{\frac{1}{2}}.$$

This inequality will be used below in order to prove Lemma 4.6.

We can now deliver a path-wise bound on weighted sums of the process ζ.

LEMMA 4.5. *Consider the increment ζ defined by (4.1). Let f be a process on $[0, T]$ such that, for any $\gamma < H$, there exists a random variable G such that $\|f\|_\gamma \leq G$. Then for any $\kappa > 0$ we have the estimate*

$$\left| \sum_{t_k=s}^{t-} f_{t_k} \zeta_{t_k t_{k+1}} \right| \leq G n^{1-4\gamma+2\kappa} (t-s)^{1-\gamma} \qquad \text{for all } (s,t) \in \mathcal{S}_2(\llbracket 0, T \rrbracket), \tag{4.8}$$

where G is an integrable random variable independent of n.

PROOF. Consider $(s,t) \in \mathcal{S}_2(\llbracket 0, T \rrbracket)$, and observe that the following decomposition holds true:

$$\left| \sum_{t_k=s}^{t-} f_{t_k} \zeta_{t_k t_{k+1}} \right| \leq \left| \sum_{t_k=s}^{t-} \delta f_{s t_k} \zeta_{t_k t_{k+1}} \right| + |f_s| \cdot |\delta g_{st}|, \tag{4.9}$$

where the increment of g has been defined by (4.1). In addition, thanks to Lemma 4.3, we obtain

$$\|\delta g_{st}\|_p \leq K n^{\frac{1}{2}-3H} (t-s)^{\frac{1}{2}} \leq K n^{1-4\gamma+\kappa} (t-s)^{1-\gamma+\kappa},$$

where the last inequality is due to the fact that $\frac{T}{n} \leq t - s$. Here, $\gamma < H$ and $\kappa > 0$. Applying Lemma 3.5 to a proper interpolation of g, we thus get

$$|\delta g_{st}| \leq G_1 n^{1-4\gamma+2\kappa} (t-s)^{1-\gamma}, \tag{4.10}$$

where G_1 is a random variable independent of n. Now observe that a direct application of Proposition 4.1 (notice that $\zeta_{t_k t_{k+1}} = \delta g_{t_k t_{k+1}}$ in the relation below) enables us to write

$$\left| \sum_{t_k=s}^{t-} \delta f_{s t_k} \zeta_{t_k t_{k+1}} \right| \leq G_2 n^{1-4\gamma+2\kappa} (t-s), \tag{4.11}$$

where G_2 is another integrable random variable independent of n. Plugging (4.10) and (4.11) into the right-hand side of (4.9), we obtain the desired estimate (4.8). $\square$

We now consider the case of a weighted sum involving a Wiener integral with respect to B.

LEMMA 4.6. *Let f be as in Lemma 4.5 and $\gamma < H$. Then the following estimate holds true*:

$$\left\| \sum_{t_k=s}^{t-} \delta f_{st_k} \otimes \int_{t_k}^{t_{k+1}} (u-t_k)^{2H}\, dB_u \right\|_p \leq K n^{1-4\gamma-2\kappa}(t-s), \qquad (s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket).$$

PROOF. The corollary is a direct application of Proposition 4.1 and is similar to the proof of Lemma 4.5. The details are omitted. $\square$

We turn to controls of weighted sums in cases involving rougher processes. They provide our first instances where we apply rough path methods for weighted sums, as announced in the Introduction. In the following, $\mathcal{V}$ and $\mathcal{V}'$ stands for some finite dimensional vector spaces.

PROPOSITION 4.7. *Let f, g be two processes defined on $[0,T]$ with values in $\mathcal{V}$ and $\mathcal{L}(\mathbb{R}^m, \mathcal{V})$, respectively, and h be a two-parameter path defined on $\mathcal{S}_2([0,T])$ with values in $\mathcal{V}'$ such that $h_{st} = h_{su} + h_{ut}$ for $(s,u,t) \in \mathcal{S}_3([0,T])$. Assume that there is a constant K and an exponent $\gamma > 0$ such that the following conditions are met for $(s,t) \in \mathcal{S}_2([0,T])$ and all $p \geq 1$*:

$$\begin{aligned} &\|f_t\|_p + \|g_t\|_p \leq K, \qquad \|\delta f_{st} - g_s \delta B_{st}\|_p \leq K(t-s)^{2\gamma}, \\ &\qquad\qquad \|\delta g_{st}\|_p \leq K(t-s)^{\gamma}. \end{aligned} \tag{4.12}$$

We also suppose that h satisfies

$$\|h_{st}\|_p \leq K(t-s)^{\alpha}, \quad \text{and} \quad \left\| \sum_{t_k=s}^{t-} \delta B_{st_k} \otimes h_{t_k t_{k+1}} \right\|_p \leq K(t-s)^{\gamma+\alpha}, \tag{4.13}$$

for $(s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket)$ and any $p \geq 1$, where α is such that $\alpha + 2\gamma > 1$. Then we have

$$\left\| \sum_{t_k=s}^{t-} \delta f_{st_k} \otimes h_{t_k t_{k+1}} \right\|_p \leq K(t-s)^{\gamma+\alpha} \tag{4.14}$$

and

$$\left\| \sum_{t_k=s+}^{t-} \sum_{t_{k'}=s}^{t_{k-1}} f_{t_{k'}} \otimes h_{t_{k'} t_{k'+1}} \otimes \delta B_{t_k t_{k+1}} \right\|_p \leq K(t-s)^{\gamma+\alpha}, \tag{4.15}$$

which are valid for $(s,t) \in \mathcal{S}_2([\![0,T]\!])$ *and all* $p \geq 1$. *Furthermore*, *set*

$$R_{st} = \sum_{t_k=s}^{t-} (\delta f_{st_k} - g_s \delta B_{st_k}) \otimes h_{t_k t_{k+1}}, \tag{4.16}$$

then we have the estimate

$$\|R_{st}\|_p \leq K(t-s)^{2\gamma+\alpha}. \tag{4.17}$$

PROOF. We start by proving inequality (4.14). To this aim, set $A_{st} = \sum_{t_k=s}^{t-} \delta f_{st_k} \otimes h_{t_k t_{k+1}}$ for $(s,t) \in \mathcal{S}_2([\![0,T]\!])$, and consider $p \geq 1$. We decompose the increment A into $A = M + R$, where R is the increment defined in (4.16) and M is defined by

$$M_{st} = g_s \sum_{t_k=s}^{t-} \delta B_{st_k} \otimes h_{t_k t_{k+1}}.$$

Then it is immediate from (4.12) and (4.13) that

$$\|M_{st}\|_p \leq K(t-s)^{\gamma+\alpha}. \tag{4.18}$$

In order to bound the increment R, we note that $R_{t_k t_{k+1}} = 0$. Let us now calculate δR: for $(s,u,t) \in \mathcal{S}_3([\![0,T]\!])$, it is readily checked that

$$\begin{aligned} \delta R_{sut} &= \delta f_{su} \otimes h_{ut} - \left(g_s \delta B_{su} \otimes h_{ut} - \delta g_{su} \sum_{t_k=u}^{t-} \delta B_{ut_k} \otimes h_{t_k t_{k+1}} \right) \\ &= (\delta f_{su} - g_s \delta B_{su}) \otimes h_{ut} + \delta g_{su} \sum_{t_k=u}^{t-} \delta B_{ut_k} \otimes h_{t_k t_{k+1}}. \end{aligned}$$

Therefore, invoking (4.12) and (4.13) again, we get

$$\|\delta R_{sut}\|_p \leq K(u-s)^{2\gamma}(t-u)^{\alpha} + K(u-s)^{\gamma}(t-u)^{\gamma+\alpha} \leq K(t-s)^{\mu},$$

where $\mu = 2\gamma + \alpha$, and where by assumption we have $\mu > 1$. Hence, owing to the discrete sewing Lemma 2.5 [applied to the Banach space $\mathcal{X} = L^p(\Omega)$] we have

$$\|R\|_{p,\mu} \leq K \|\delta R\|_{p,\mu} \leq K, \tag{4.19}$$

where $\|\cdot\|_{p,\mu}$ designates the μ-Hölder norm for $L^p(\Omega)$-valued functions. Putting together our estimates (4.18) and (4.19) on M and R, inequality (4.14) is proved.

In the following, we derive our second claim (4.15). The method is similar to the proof of (4.14), so that it will only be sketched for sake of conciseness. We resort to the following decomposition:

$$\tilde{A}_{st} \equiv \sum_{t_k=s+}^{t-} \sum_{t_{k'}=s}^{t_{k-1}} f_{t_{k'}} \otimes h_{t_{k'} t_{k'+1}} \otimes \delta B_{t_k t_{k+1}} = \tilde{M}_{st} + \tilde{R}_{st},$$

where

$$\tilde{M}_{st} = \sum_{t_k=s+}^{t-} \sum_{t_{k'}=s}^{t_{k-1}} f_s \otimes h_{t_{k'}t_{k'+1}} \otimes \delta B_{t_k t_{k+1}}$$

$$\text{and} \quad \tilde{R}_{st} = \sum_{t_k=s+}^{t-} \sum_{t_{k'}=s}^{t_{k-1}} \delta f_{st_{k'}} \otimes h_{t_{k'}t_{k'+1}} \otimes \delta B_{t_k t_{k+1}}.$$

In order to bound $\tilde{M}_{st}$, we change the order of summation, which allows to exhibit some terms of the form δB_{st_k}. Then we let the reader check that inequality (4.13) can be applied directly. As far as $\tilde{R}$ is concerned, notice again that $\tilde{R}_{t_k t_{k+1}} = 0$. It is then readily seen, as in the previous step, that our estimate boils down to a bound on $\delta\tilde{R}$. Furthermore, $\delta\tilde{R}$ can be computed as follows:

$$\delta\tilde{R}_{sut} = \left(\sum_{t_{k'}=s}^{u-} \delta f_{st_{k'}} \otimes h_{t_{k'}t_{k'+1}}\right) \otimes \delta B_{ut} + \delta f_{su} \otimes \sum_{t_k=u+}^{t-} \sum_{t_{k'}=u}^{t_{k-1}} h_{t_{k'}t_{k'+1}} \otimes \delta B_{t_k t_{k+1}}.$$

We can thus resort to (4.12), (4.14) and (4.13) in order to get

$$\|\delta\tilde{R}_{sut}\|_p \leq K(t-s)^{2\gamma+\alpha}.$$

The proof is now complete as for relation (4.14). □

REMARK 4.8. In Proposition 4.7, the weighted sum $\sum_{t_k=s}^{t-} \delta f_{st_k} \otimes h_{t_k t_{k+1}}$ is viewed as a L_p-valued "discrete" rough integral. Similarly as in Remark 4.2, a $\mathbb{R}$-valued "discrete" rough path can also be considered.

Proposition 4.7 can be applied to the sum F of Lévy area increments of B. This is the contents of the corollary below.

COROLLARY 4.9. *Let $\frac{1}{4} < \gamma < H$, f and g be as in Proposition 4.7. Let F be the process defined by (3.14), considered as a path taking values in $\mathcal{V}' = \mathbb{R}^{d\times d}$. Then the following estimates hold true for $(s,t) \in \mathcal{S}_2(\llbracket 0, T \rrbracket)$:*

$$\left\|\sum_{t_k=s}^{t-} f_{t_k} \otimes \delta F_{t_k t_{k+1}}\right\|_p \leq K n^{\frac{1}{2}-2H}(t-s)^{\frac{1}{2}}$$

and

$$\left\|\sum_{t_k=s+}^{t-} \sum_{t_{k'}=s}^{t_{k-1}} f_{t_{k'}} \otimes \delta F_{t_{k'}t_{k'+1}} \otimes \delta B_{t_k t_{k+1}}\right\|_p \leq K n^{\frac{1}{2}-2H}(t-s)^{H+\frac{1}{2}}.$$

Furthermore, set $R_{st} = \sum_{t_k=s}^{t-} (\delta f_{st_k} - g_s \delta B_{st_k}) \otimes F_{t_k t_{k+1}}$, then we have

$$\|R_{st}\|_p \leq K n^{\frac{1}{2}-2H}(t-s)^{2\gamma+\frac{1}{2}}. \tag{4.20}$$

PROOF. Take $h = n^{2H-\frac{1}{2}}F$ and $\alpha = \frac{1}{2}$. It follows from Lemma 3.4 and Lemma A.5 that h satisfies the conditions in Proposition 4.7. In addition, if $\gamma > \frac{1}{4}$, the condition $2\gamma + \frac{1}{2} > 1$ is trivially satisfied. The corollary then follows immediately from Proposition 4.7 and taking into account the decomposition $f_{t_k} = f_s + \delta f_{st_k}$. The estimate of R_{st} follows directly from relation (4.17). □

4.2. *Limiting theorem results via rough path approach.* Take two uniform partitions on $[0, T]$: $t_k = \frac{T}{n}k$ and $u_l = \frac{T}{\nu}l$, $n, \nu \in \mathbb{N}$ for $k = 0, \dots, n$ and $l = 0, \dots, \nu$. Let k_l be such that $t_{k_l+1} > u_l \geq t_{k_l}$. In the following, we set for each $t \in [0, T]$:

$$D_l = \{t_k : u_{l+1} > t_k \geq u_l, t \geq t_k\} \quad \text{and} \tag{4.21}$$
$$\tilde{D}_l = \{t_k : t_{k_{l+1}} > t_k \geq t_{k_l}, t \geq t_k\}.$$

Our main result in this section is a central limit theorem for sums weighted by a controlled process f.

THEOREM 4.10. *Let the assumptions in Proposition 4.7 prevail, and suppose that*

$$(h, B) \xrightarrow{\text{f.d.d.}} (W, B), \qquad n \to \infty, \tag{4.22}$$

where "$\xrightarrow{\text{f.d.d.}}$" stands for convergence of finite dimensional distributions and W is a Brownian motion independent of B. Set

$$\zeta_l^n = \sum_{t_k \in \tilde{D}_l} \delta B_{t_{k_l} t_k} \otimes h_{t_k t_{k+1}}, \tag{4.23}$$

and suppose that

$$\left\| \sum_{l=\frac{\nu r}{T}}^{\frac{\nu r'}{T}-1} \zeta_l^n \right\|_p \leq K \nu^{-\kappa} (r' - r)^{\alpha+\gamma-\kappa} \tag{4.24}$$

for $r, r' \in \{u_1, \dots, u_\nu\}$ and for an arbitrary $\kappa > 0$. Set $\Theta_t^n = \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} f_{t_k} \otimes h_{t_k t_{k+1}}$ and $\Theta_t = \int_0^t f_s \otimes dW_s$, where Θ_t should be understood as a Wiener integral conditionally on the process f. Then the following relation holds true:

$$(\Theta^n, B) \xrightarrow{\text{f.d.d.}} (\Theta, B) \qquad \textit{as } n \to \infty.$$

REMARK 4.11. There are several possible generalizations of the statement of Theorem 4.10. If one has the convergence of (h, B) in L_p instead of the weak convergence (4.22), then by a similar proof one can show that Θ^n converges to Θ in L^p. On the other hand, similar to what has been mentioned in Remark 4.8, if conditions (4.12), (4.13) and (4.24) are replaced by the corresponding almost sure upper-bound conditions, then one can show that Theorem 4.10 still holds true.

REMARK 4.12. In the case $\gamma > \frac{1}{2}$ and $\alpha \geq \frac{1}{2}$, conditions (4.12), (4.13) and (4.24) are reduced to $\|\delta f_{st}\|_p \leq K(t-s)^\gamma$ and $\|\delta h_{st}\|_p \leq K(t-s)^\alpha$. Theorem 4.10 then recovers the central and noncentral limit theorem results in [6] and [19].

REMARK 4.13. According to our proof of Theorem 4.10, in general the limit of the "Riemann sum" Θ^n is independent of the choices of the representative points. In the situation of Remark 4.12, this fact can be proved directly from the expression of Θ^n.

PROOF OF THEOREM 4.10. By definition of the f.d.d. convergence, it suffices to show the following weak convergence for $r_1, \dots, r_l \in [0, T]$:

$$\left(\Theta^n_{r_1}, \dots, \Theta^n_{r_l}, B_{r_1}, \dots, B_{r_l}\right) \xrightarrow{(d)} (\Theta_{r_1}, \dots, \Theta_{r_l}, B_{r_1}, \dots, B_{r_l}), \qquad n \to \infty.$$

Step 1: A coarse graining argument. Consider an extra parameter $\nu << n$ and take $\{u_0, \dots, u_\nu\}$ to be the uniform ν-step partition of $[0, T]$. We make the following decomposition:

$$\Theta^n_t = \tilde{\Theta}^n_t + \hat{\Theta}^n_t,$$

where

$$\tilde{\Theta}^n_t = \sum_{l=0}^{\nu-1} \sum_{t_k \in D_l} \delta f_{u_l t_k} \otimes h_{t_k t_{k+1}}, \qquad \hat{\Theta}^n_t = \sum_{l=0}^{\nu-1} \sum_{t_k \in D_l} f_{u_l} \otimes h_{t_k t_{k+1}}.$$

Let us first handle the convergence of $\hat{\Theta}^n$: by letting $n \to \infty$ and taking into account the convergence (4.22), and then letting $\nu \to \infty$, we easily obtain the weak convergence:

$$\left(\hat{\Theta}^n_{r_1}, \dots, \hat{\Theta}^n_{r_l}, B_{r_1}, \dots, B_{r_l}\right) \xrightarrow{(d)} (\Theta_{r_1}, \dots, \Theta_{r_l}, B_{r_1}, \dots, B_{r_l}).$$

Therefore, in order to prove our claim, we are reduced to show that for $t \in [0, T]$:

$$\lim_{\nu \to \infty} \limsup_{n \to \infty} \mathbb{E}\left(\left|\tilde{\Theta}^n_t\right|^2\right) = 0. \tag{4.25}$$

Step 2: First-order approximation of f. Let k_l be such that $t_{k_l+1} > u_l \geq t_{k_l}$. We compare the two sets: D_l defined by (4.21) and $\tilde{D}_l = \{t_k : t_{k_{l+1}} > t_k \geq t_{k_l}, t \geq t_k\}$. It is easy to see that $D_l \Delta \tilde{D}_l \subset \{t_{k_l}, t_{k_{l+1}}\}$. It is also readily checked from conditions (4.12) and (4.13) that $\|\delta f_{u_l t_k}\|_{L_4} \leq K\nu^{-\gamma}$ and $\|h_{t_k t_{k+1}}\|_{L_4} \leq Kn^{-\alpha}$. A simple use of the Cauchy–Schwarz inequality thus yields

$$\|\delta f_{u_l t_k} \otimes h_{t_k t_{k+1}}\|_2 \leq K\nu^{-\gamma} n^{-\alpha}, \tag{4.26}$$

for any $k = 0, 1, \dots, n-1$. In particular, (4.26) holds for $t_k \in D_l \Delta \tilde{D}_l$. Therefore, in order to show (4.25), it is sufficient to show that

$$\lim_{\nu \to \infty} \limsup_{n \to \infty} \mathbb{E}\left(\left|\sum_{l=0}^{\nu-1} R_{t_{k_l} t_{k_{l+1}}}\right|^2\right) = 0, \tag{4.27}$$

where

$$R_{t_{k_l}t_{k_{l+1}}} = \sum_{t_k \in \tilde{D}_l} \delta f_{u_l t_k} \otimes h_{t_k t_{k+1}}.$$

Now in order to get relation (4.27), consider the following decomposition for R:

$$R_{t_{k_l}t_{k_{l+1}}} = \bar{R}_{t_{k_l}t_{k_{l+1}}} + \tilde{R}_{t_{k_l}t_{k_{l+1}}} + \check{R}_{t_{k_l}t_{k_{l+1}}} + \hat{R}_{t_{k_l}t_{k_{l+1}}}, \tag{4.28}$$

where the increments $\bar{R}_{t_{k_l}t_{k_{l+1}}}$, $\tilde{R}_{t_{k_l}t_{k_{l+1}}}$, $\check{R}_{t_{k_l}t_{k_{l+1}}}$ and $\hat{R}_{t_{k_l}t_{k_{l+1}}}$ are defined by

$$\begin{aligned}\tilde{R}_{t_{k_l}t_{k_{l+1}}} &= \sum_{t_k \in \tilde{D}_l} \delta f_{u_l t_{k_l}} \otimes h_{t_k t_{k+1}},\\ \bar{R}_{t_{k_l}t_{k_{l+1}}} &= \sum_{t_k \in \tilde{D}_l} (\delta f_{t_{k_l} t_k} - g_{t_{k_l}} \otimes \delta B_{t_{k_l} t_k}) \otimes h_{t_k t_{k+1}}\end{aligned}$$

and

$$\begin{aligned}\check{R}_{t_{k_l}t_{k_{l+1}}} &= \delta g_{u_l t_{k_l}} \otimes \sum_{t_k \in \tilde{D}_l} \delta B_{t_{k_l} t_k} \otimes h_{t_k t_{k+1}},\\ \hat{R}_{t_{k_l}t_{k_{l+1}}} &= g_{u_l} \otimes \sum_{t_k \in \tilde{D}_l} \delta B_{t_{k_l} t_k} \otimes h_{t_k t_{k+1}}.\end{aligned} \tag{4.29}$$

It follows from (4.17) that

$$\|\bar{R}_{t_{k_l}t_{k_{l+1}}}\|_2 \le K\nu^{-2\gamma-\alpha}. \tag{4.30}$$

On the other hand, it follows from (4.12) and (4.13) that

$$\|\tilde{R}_{t_{k_l}t_{k_{l+1}}}\|_2 = \|\delta f_{u_l t_{k_l}} \otimes h_{t_{k_l}, t_{k_{l+1}} \wedge t+}\|_2 \le K n^{-\gamma} \nu^{-\alpha}, \tag{4.31}$$

where recall that we denote $t+ = t + \frac{T}{n}$ and $t_{k_{l+1}} \wedge t+ = \min(t_{k_{l+1}}, t+)$. Similarly, applying (4.12) and (4.13) we obtain

$$\|\check{R}_{t_{k_l}t_{k_{l+1}}}\|_2 \le K n^{-\gamma} \nu^{-\gamma-\alpha}. \tag{4.32}$$

It follows immediately from (4.30), (4.31) and (4.32) that

$$\lim_{\nu\to\infty} \limsup_{n\to\infty} \mathbb{E}\left(\left|\sum_{l=0}^{\nu-1} (\bar{R}_{t_{k_l}t_{k_{l+1}}} + \tilde{R}_{t_{k_l}t_{k_{l+1}}} + \check{R}_{t_{k_l}t_{k_{l+1}}})\right|^2\right) = 0. \tag{4.33}$$

In view of (4.33) and taking into account the decomposition (4.28), in order to show (4.27) it suffices to show that

$$\lim_{\nu\to\infty} \limsup_{n\to\infty} \mathbb{E}\left(\left|\sum_{l=0}^{\nu-1} \hat{R}_{t_{k_l}t_{k_{l+1}}}\right|^2\right) = 0. \tag{4.34}$$

Step 3: Study of $\hat{R}$. We will see that $\sum_{l=0}^{\nu-1} \hat{R}_{t_{k_l} t_{k_{l+1}}}$ can be considered as a discrete "Young" integral in L_2 in the sense of Remark 4.2 (see also Proposition 4.1), which then leads to the convergence (4.34). Namely, starting from the expression (4.29) of $\hat{R}$, let us first consider the "weight-free" sum

$$\hat{\zeta}_r^n := \sum_{l=0}^{\frac{\nu r}{T}-1} \zeta_l^n,$$

where $r \in \{u_1, \dots, u_\nu\}$ and recall that ζ_l^n is defined in (4.23). Observe that (4.29) can be recast as

$$\hat{R}_{t_{k_l} t_{k_{l+1}}} = g_{u_l} \otimes \zeta_l^n = g_{u_l} \delta \hat{\zeta}^n_{u_l u_{l+1}}, \tag{4.35}$$

for all $l = 0, \dots, \nu - 1$. According to (4.35), we have

$$\sum_{l=0}^{\nu-1} \hat{R}_{t_{k_l} t_{k_{l+1}}} = \sum_{l=0}^{\nu-1} g_{u_l} \otimes \delta \hat{\zeta}^n_{u_l u_{l+1}} = \sum_{l=0}^{\nu-1} \delta g_{0 u_l} \otimes \delta \hat{\zeta}^n_{u_l u_{l+1}} + \sum_{l=0}^{\nu-1} g_0 \otimes \delta \hat{\zeta}^n_{u_l u_{l+1}}.$$

Then our assumption (4.24) and the bound (4.12) ensures that we are in a position to apply Proposition 4.1. This immediately yields our claim (4.34), which concludes the proof. $\square$

5. Euler scheme process as a rough path. In this section, we consider a continuous time interpolation of the Euler scheme y^n given by (1.5). Namely, we introduce a sequence of processes y^n indexed by $[0, T]$ in the following way: for $t \in [t_k, t_{k+1})$, we set

$$y_t^n = y_{t_k}^n + b(y_{t_k}^n)(t - t_k) + V(y_{t_k}^n)\delta B_{t_k t} + \frac{1}{2} \sum_{j=1}^m \partial V_j V_j (y_{t_k}^n)(t - t_k)^{2H}, \tag{5.1}$$

where we recall that $\pi : 0 = t_0 < t_1 < \cdots < t_n = T$ designates the uniform partition of the interval $[0, T]$. The remainder of the section is devoted to getting some uniform bounds on y^n, and then to prove that the couple (y^n, B) can be lifted as a rough path. Throughout the section, we assume that $b \in C_b^2$ and $V \in C_b^4$.

5.1. *Hölder-type bounds for the Euler scheme.* Our main results on the Hölder regularity of the sequence y^n is summarized in the following proposition.

PROPOSITION 5.1. *Let y^n be the process defined by Euler scheme* (5.1). *Take $\frac{1}{3} < \gamma < H$. Then for all $(s, t) \in \mathcal{S}_2(\llbracket 0, T \rrbracket)$, the following relations are satisfied*:

$$\left|\delta y_{st}^n\right| \le G|t - s|^\gamma, \qquad \left|\delta y_{st}^n - V(y_s^n)\delta B_{st}\right| \le G(t - s)^{2\gamma}, \tag{5.2}$$

where G stands for an integrable random variable which is independent of the parameter n.

PROOF. We divide this proof into several steps. For the sake of conciseness, we omit the drift b in the proof, so that we analyze a scheme defined successively by

$$y_{t_{k+1}}^n = y_{t_k}^n + V(y_{t_k}^n)\delta B_{t_k t_{k+1}} + \frac{1}{2}\sum_{j=1}^m \partial V_j V_j(y_{t_k}^n)h^{2H}. \tag{5.3}$$

Step 1: Definition of the remainder. We first define some increments of interest for the analysis of the scheme given by (5.1). Let us start with a 2nd-order increment q defined by

$$q_{st} = \sum_{i,j=1}^m (\partial V_i V_j)(y_s^n)\delta F_{st}^{ij}, \qquad \mathcal{S}_2([\![0,T]\!]), \tag{5.4}$$

where recall that F^{ij} is defined in (3.14). Next, our remainder term for (5.1) is given by

$$R_{st} = \delta y_{st}^n - V(y_s^n)\delta B_{st} - \sum_{i,j=1}^m (\partial V_i V_j)(y_s^n)\mathbb{B}_{st}^{ij} + q_{st}, \qquad \mathcal{S}_2([\![0,T]\!]). \tag{5.5}$$

Since R is expected to be regular in $|t-s|$ and $R_{t_k t_{k+1}} = 0$ by the very definition of y^n, we will analyze R through an application of the discrete sewing Lemma 2.5. To this aim, we calculate δR, which is easily decomposed as follows:

$$\delta R_{sut} = A_1 + A_2 + A_3 + A_4, \tag{5.6}$$

where for $(s,u,t) \in \mathcal{S}_3([\![0,T]\!])$, the quantities A_1, A_2 are given by

$$A_1 = \delta V(y^n)_{su}\delta B_{ut}, \qquad A_2 = \sum_{i,j=1}^m \delta(\partial V_i V_j)(y^n)_{su}\mathbb{B}_{ut}^{ij},$$

and where A_3, A_4 are defined by

$$A_3 = -\sum_{i,j=1}^m (\partial V_i V_j)(y_s^n)\delta B_{su}^j \delta B_{ut}^i, \quad \text{and} \quad A_4 = -\sum_{i,j=1}^m \delta(\partial V_i V_j)(y_\cdot^n)_{su}\delta F_{ut}^{ij}.$$

Observe that in order to compute A_e, $e = 1,2,3,4$ we have used the fact that $\delta\delta B = 0$, $\delta\delta F = 0$, and $\delta\mathbb{B} = \delta B \otimes \delta B$. Moreover, note that owing to an elementary Taylor-type expansion we have

$$\delta V(y^n)_{su}\delta B_{ut} = \sum_{i=1}^m [\partial V_i(y^n)]_{su}\delta y_{su}^n \delta B_{ut}^i,$$

where we denote $[\partial V_i(y^n)]_{su} = \int_0^1 \partial V_i(y_s^n + \lambda\delta y_{su}^n)\,d\lambda$. So invoking relation (5.5), we can further decompose A_1 as follows:

$$A_1 = \delta V(y^n)_{su}\delta B_{ut} = A_{11} + A_{12} + A_{13} + A_{14},$$

where A_{11} and A_{12} are defined by

$$A_{11}=\sum_{i=1}^{m}\big[\partial V_i\big(y^n\big)\big]_{su}V\big(y_s^n\big)\delta B_{su}\delta B_{ut}^i,$$

$$A_{12}=\sum_{i=1}^{m}\big[\partial V_i\big(y^n\big)\big]_{su}\sum_{i',j'=1}^{m}(\partial V_{i'}V_{j'})\big(y_s^n\big)\mathbb{B}_{su}^{i'j'}\delta B_{ut}^i,$$

and where

$$A_{13}=-\sum_{i=1}^{m}\big[\partial V_i\big(y^n\big)\big]_{su}q_{su}\delta B_{ut}^i,\quad \text{and} \quad A_{14}=\sum_{i=1}^{m}\big[\partial V_i\big(y^n\big)\big]_{su}R_{su}\delta B_{ut}^i.$$

We will bound those terms separately.

Step 2: Upper-bound for y and R on small intervals. Consider the following deterministic constant:

$$K_V=\big(1+\|V\|_\infty+\|\partial V\|_\infty+\big\|\partial^2 V\big\|_\infty+K_{3\gamma}\big)^3,$$

where $K_{3\gamma}$ is the constant appearing in Lemma 2.5, and $\|V\|_\infty$ for a vector-valued field V denotes the supnorm of the function $|V|$. We also introduce the following random variable:

$$G=G_\gamma+L_\gamma+1, \tag{5.7}$$

where G_γ is defined in (3.19) and L_γ is introduced in Proposition 3.1. Assume that n is large enough so that

$$Gn^{-\gamma}\le\big(8K_V^2\big)^{-1}. \tag{5.8}$$

In this step, we show by induction that for $(s,t)\in\mathcal{S}_2([\![0,T]\!])$ such that

$$G(t-s)^\gamma\le\big(8K_V^2\big)^{-1}, \tag{5.9}$$

we have

$$\big\|y^n\big\|_{[s,t],\gamma,n}\le 2K_V G,\qquad \|R\|_{[s,t],3\gamma,n}\le 8K_V^3G^3. \tag{5.10}$$

Notice that here and in the following, we adopt the notation

$$\big\|y^n\big\|_{[s,t],\alpha,n}:=\sup_{(u,v)\in\mathcal{S}_2([\![s,t]\!])}\frac{|\delta y_{uv}^n|}{|v-u|^\alpha},\qquad \|R\|_{[s,t],\alpha,n}:=\sup_{(u,v)\in\mathcal{S}_2([\![s,t]\!])}\frac{|R_{uv}|}{|v-u|^\alpha}$$

for $\alpha>0$. The relations (5.10) will be achieved by bounding successively the terms in (5.6).

Specifically, we assume that relation (5.10) holds true when $(s,u,t)\in\mathcal{S}_3([\![0,(N-1)h]\!])$ and verify (5.9). Our aim is to extend this inequality to $\mathcal{S}_3([\![0,Nh]\!])$. We thus start from our induction assumption, and we consider $(s,u,t)\in\mathcal{S}_3([\![0,$

$Nh]\!])$ such that (5.9) is satisfied and $t = Nh$. Then we start by bounding the terms A_{11} and A_3 as follows:

$$A_{11} + A_3 = \sum_{i=1}^{m} [\partial V_i(y^n)]_{su} V(y_s^n) \delta B_{su} \delta B_{ut}^i - \sum_{i,j=1}^{m} (\partial V_i V_j)(y_s^n) \delta B_{su}^j \delta B_{ut}^i$$
$$= \sum_{i,j=1}^{m} ([\partial V_i(y^n)]_{su} - \partial V_i(y_s^n)) V_j(y_s^n) \delta B_{su}^j \delta B_{ut}^i$$

By the induction assumption (5.10) on $\|y^n\|_{[s,t],\gamma,n}$ and the definition (5.7) of our random variable G, we thus have

$$\|A_{11} + A_3\|_{[s,t],3\gamma,n} \leq 2K_V^2 G^3.$$

Along the same lines, since $\|\mathbb{B}\|_\gamma \leq G$ and invoking the induction assumption again we obtain

$$\|A_2\|_{[s,t],3\gamma,n} \leq 2K_V^2 G^3 \quad \text{and} \quad \|A_{12}\|_{[s,t],3\gamma,n} \leq K_V G^3.$$

Similarly, the estimate (3.19) and the induction assumption implies that

$$\|A_4\|_{[s,t],3\gamma,n} \leq 2K_V^2 G^3 \quad \text{and} \quad \|A_{13}\|_{[s,t],3\gamma,n} \leq K_V G^3.$$

Finally, by the induction assumption (5.10) on R we obtain

$$\|A_{14}\|_{[s,t],3\gamma,n} \leq 8K_V^4 G^4 (t-s)^\gamma \leq K_V^2 G^3,$$

where we have used the assumption (5.9) for the second inequality. Applying the above estimates on $A_1, \ldots, A_4$ to (5.6), we have thus obtained

$$\|\delta R\|_{[s,t],3\gamma,n} \leq 8K_V^2 G^3.$$

Since $R_{t_k t_{k+1}} = 0$ and $3\gamma > 1$, we are now in a position to apply the discrete sewing Lemma 2.5. This yields

$$\|R\|_{[s,t],3\gamma,n} \leq 8K_V^3 G^3. \tag{5.11}$$

Otherwise stated, our induction assumption (5.10) is propagated for the term R.

Let us turn to the propagation of the induction assumption (5.10) for the norm of y. Plugging the bound (5.11) into relation (5.5), taking onto account the definition of the random variable G and recalling relation (5.9), it is readily checked that

$$\begin{aligned} &|\delta y_{st}^n - V(y_s^n) \delta B_{st}| \\ &\quad \leq K_V G^2 (t-s)^{2\gamma} + K_V G^2 (t-s)^{2\gamma} + 8K_V^3 G^3 (t-s)^{3\gamma} \\ &\quad \leq 3K_V G^2 (t-s)^{2\gamma}. \end{aligned} \tag{5.12}$$

Therefore, since we have $\|B\|_\gamma \le G$, we obtain

$$\begin{aligned}|\delta y^n_{st}| &\le K_V G(t-s)^\gamma + 2K_V G^2(t-s)^{2\gamma} + 8K_V^3 G^3(t-s)^{3\gamma}\\ &\le 2K_V G(t-s)^\gamma,\end{aligned} \tag{5.13}$$

where we have invoked our hypothesis (5.9) again. This achieves the propagation of the induction (5.10) for the term $\|y^n\|_\gamma$.

Step 3: Upper bound estimates on $[\![0,T]\!]$. Recall that we have proved relation (5.10) on small intervals $[\![s,t]\!]$ satisfying (5.9). In order to extend this result to the whole interval $[\![0,T]\!]$, we use a partition of the form $[\![kT_0,(k+1)T_0]\!]$. Namely, consider $T_0 \in [\![0,T]\!]$ such that

$$\left(2^\gamma 8K_V^2\right)^{-1} \le GT_0^\gamma \le \left(8K_V^2\right)^{-1}. \tag{5.14}$$

Also consider $s,t\in[0,T]$ such that $t-s>T_0$, and denote $k=\lfloor\frac{t-s}{T_0}\rfloor$ and $s_i = s+iT_0$, $i=0,\dots,k$. Then we obviously have

$$|\delta y^n_{st}| \le |\delta y^n_{s_0s_1}| + |\delta y^n_{s_1s_2}| + \cdots + |\delta y^n_{s_kt}|$$

Furthermore, on each subinterval $[s_j,s_{j+1}]$ one can apply (5.10) in order to get

$$|\delta y^n_{st}| \le 2K_V GkT_0^\gamma + 2K_V G(t-s-kT_0)^\gamma.$$

Now resort to the fact that $k\le\frac{t-s}{T_0}$ and inequality (5.14). This yields, for another K,

$$|\delta y^n_{st}| \le 4K_V G\frac{t-s}{T_0^{1-\gamma}} \le KG^{\frac{1}{\gamma}}(t-s). \tag{5.15}$$

Hence, gathering our estimates (5.13) and (5.15), we end up with

$$|\delta y^n_{st}| \le KG^{\frac{1}{\gamma}}(t-s)^\gamma, \tag{5.16}$$

for $(s,t)\in\mathcal{S}_2([\![0,T]\!])$. That is, we have extended the first part of (5.10) to the whole interval $[\![0,T]\!]$, and thus we have proved the first relation in (5.2) when n satisfies (5.8).

We now prove the second-order estimate in (5.2) when n satisfies (5.8). We start by a new decomposition of the form

$$|\delta y^n_{st} - V(y^n_s)\delta B_{st}| \le |r_1| + |r_2|, \tag{5.17}$$

where

$$r_1 = \delta y^n_{st} - \sum_{i=0}^k V(y^n_{s_i})\delta B_{s_i,t\wedge s_{i+1}}, \qquad r_2 = \sum_{i=0}^k V(y^n_{s_i})\delta B_{s_i,t\wedge s_{i+1}} - V(y^n_s)\delta B_{st}.$$

Now the term r_1 can be bounded as follows:

$$|r_1| \le \sum_{i=0}^k |\delta y^n_{s_i,t\wedge s_{i+1}} - V(y^n_{s_i})\delta B_{s_i,t\wedge s_{i+1}}|.$$

Therefore, by (5.12) we obtain

$$|r_1| \le 3(k+1)K_V G^2 T_0^{2\gamma}.$$

Moreover, since $k \le \frac{t-s}{T_0}$ and $\frac{1}{T_0} \le (2^\gamma 8K_V^2 G)^{\frac{1}{\gamma}}$ owing to (5.14), we can recast the previous equation as

$$|r_1| \le 4K_V\big(2^\gamma 8K_V^2\big)^{\frac{1}{\gamma}-2} G^{\frac{1}{\gamma}}(t-s). \tag{5.18}$$

In order to bound r_2, observe that we have

$$|r_2| \le \sum_{i=0}^{k} \big|V\big(y_{s_i}^n\big) - V\big(y_s^n\big)\big| \cdot |\delta B_{s_i, t\wedge s_{i+1}}|.$$

Thanks to (5.16), we thus have

$$|r_2| \le 2kK_V\big(KG^{\frac{1}{\gamma}}(t-s)^\gamma\big)GT_0^\gamma.$$

Invoking again the inequalities $k \le \frac{t-s}{T_0}$ and $\frac{1}{T_0} \le (2^\gamma 8K_V^2 G)^{\frac{1}{\gamma}}$, we thus get

$$|r_2| \le 2K_V K\big(2^\gamma 8K_V^2\big)^{\frac{1}{\gamma}-1}\big(G^{\frac{2}{\gamma}}\big)(t-s)^{1+\gamma}. \tag{5.19}$$

Applying (5.18) and (5.19) to relation (5.17), this yields

$$\big|\delta y_{st}^n - V\big(y_s^n\big)\delta B_{st}\big| \le KG^{\frac{2}{\gamma}}(t-s)^{2\gamma}. \tag{5.20}$$

We have now proved (5.2) under the assumption (5.8).

Step 4: Upper-bound estimate for small n. We are now reduced to prove inequalities (5.2) when (5.8) is not satisfied. Namely, we assume in this step that

$$Gn^{-\gamma} > \big(8K_V^2\big)^{-1}, \quad \text{that is,} \quad n < \big(8K_V^2 G\big)^{\frac{1}{\gamma}}. \tag{5.21}$$

For $(s,t) \in \mathcal{S}_2([\![0,T]\!])$, we will also resort to the same partition $t_0, \dots, t_{k+1}$ as in the previous step. In this case, due to the very definition (5.1) of y^n, it is readily checked that

$$\big|\delta y_{t_k t_{k+1}}^n\big| \le K_V G\left(\frac{T}{n}\right)^H + K_V\left(\frac{T}{n}\right)^{2H}. \tag{5.22}$$

Therefore, summing (5.22) between s and t for $(s,t) \in \mathcal{S}_2([\![0,T]\!])$ we get

$$\big|\delta y_{st}^n\big| = \left|\sum_{t_k=s}^{t-} \delta y_{t_k t_{k+1}}^n\right| \le n(t-s)T^{-1}\left[K_V G\left(\frac{T}{n}\right)^H + K_V\left(\frac{T}{n}\right)^{2H}\right].$$

Taking into account the estimate of n in (5.21), this yields

$$\big|\delta y_{st}^n\big| \le KG^{\frac{1}{\gamma}}|t-s|, \tag{5.23}$$

for $(s,t) \in \mathcal{S}_2([\![0,T]\!])$. We have thus proved the first relation in (5.2) when (5.8) is not met.

In order to handle the second relation in (5.2) for n small, just decompose the increment at stake along our partition $t_0, \dots, t_{k+1}$:

$$\left|V\left(y_s^n\right)\delta B_{st}\right| \le \left|V\left(y_s^n\right)\right| \sum_{t_k=s}^{t-} |\delta B_{t_k t_{k+1}}| \le K G n^{1-\gamma}(t-s) \le K G^{\frac{1}{\gamma}}(t-s).$$

Taking into account inequality (5.23), we thus easily get

$$\left|\delta y_{st}^n - V\left(y_s^n\right)\delta B_{st}\right| \le K G^{\frac{1}{\gamma}}(t-s), \tag{5.24}$$

which achieves the second relation in (5.2) for small n.

Step 5: Conclusion. Gathering the estimates (5.16), (5.23), we have obtained the desired estimate for $\|y^n\|_\gamma$ on $[\![0,T]\!]$, for all n. In the same way, putting (5.20) and (5.24) together implies the second estimate in (5.2) on $[\![0,T]\!]$ and for any n. The proof is complete. □

5.2. *The couple (y^n, B) as a rough path.* Our next aim is to prove that (y^n, B) can be lifted as a rough path, which amounts to a proper definition of the signature $S_2(y^n, B)$ as given in Definition 2.1. The result below, providing an estimate of the integral $\int_s^t \delta y_{su}^n \otimes dB_u$, can be seen as an important step in this direction. Note that on each interval $[t_k, t_{k+1}]$, the process y^n is a controlled process with respect to B, as alluded to in (2.8). For each n, the integral $\int_s^t \delta y_{su}^n \otimes dB_u$ is thus defined as

$$\int_s^t \delta y_{su}^n \otimes dB_u = \sum_{t_k=s}^{t_-} \int_{t_k}^{t_{k+1}} \delta y_{su}^n \otimes dB_u, \tag{5.25}$$

thanks to classical rough paths considerations.

LEMMA 5.2. *Let y^n be the process defined by the Euler scheme* (5.1), *and consider $\frac{1}{3} < \gamma < H$. Then we can find a random variable $G \in \bigcap_{p \ge 1} L^p(\Omega)$ independent of n, such that for the integral $\int_s^t \delta y_{su}^n \otimes dB_u$ in the sense of* (5.25), *we have the estimate*

$$\left|\int_s^t \delta y_{su}^n \otimes dB_u\right| \le G(t-s)^{2\gamma} \qquad \text{for } (s,t) \in \mathcal{S}_2\left([\![0,T]\!]\right). \tag{5.26}$$

PROOF. Similar to what we did for Proposition 5.1, we will assume that $b = 0$ for this proof, and analyze the scheme given by (5.3). Next, in order to bound the integral $\int_s^t \delta y_{su}^n \otimes dB_u$, let us define two increments: first, just as in the definition (4.7), we set

$$\zeta_{st}^2 = \int_s^t (u-s)^{2H} \, dB_u.$$

Then we define a remainder type increment $\tilde{R}$ on $\mathcal{S}_2([\![0,T]\!])$ by

$$\tilde{R}_{st} = \int_s^t \delta y_{su}^n \otimes dB_u - V(y_s^n)\mathbb{B}_{st} - \frac{1}{2}\sum_{j=1}^m \partial V_j V_j(y_s^n) \otimes \zeta_{st}^2. \tag{5.27}$$

According to the definition (5.1) of our scheme, it is clear that $\tilde{R}_{t_k t_{k+1}} = 0$, for all $k = 0, 1, \dots, n-1$. Moreover, applying δ to $\tilde{R}$ and recalling the elementary rule $\delta(\int dy \otimes dB) = \delta y \otimes \delta B$, we obtain

$$\begin{aligned}\delta\tilde{R}_{srt} &= (\delta y_{sr}^n - V(y_s^n)\delta B_{sr}) \otimes \delta B_{rt} + \delta V(y_\cdot^n)_{sr}\mathbb{B}_{rt} \\ &\quad - \frac{1}{2}\sum_{j=1}^m \partial V_j V_j(y_s^n) \otimes \delta\zeta_{srt}^2 + \frac{1}{2}\sum_{j=1}^m \delta(\partial V_j V_j(y^n))_{sr} \otimes \zeta_{rt}^2,\end{aligned}$$

where we remark that $\delta\zeta_{srt}^2 = \int_r^t [(u-s)^{2H} - (u-r)^{2H}]\, dB_u$. Starting from the above expression, one can thus apply Proposition 3.1 and Proposition 5.1 in order to get

$$\|\delta\tilde{R}\|_{3\gamma} \le G.$$

Note that for the Young integral $\delta\zeta^2$ we used the following estimate, valid for $(s,r,t) \in \mathcal{S}_3([0,T])$,

$$\delta\zeta_{srt}^2 \le \left|\int_r^t (u-s)^{2H}\, dB_u\right| + \left|\int_r^t (u-r)^{2H}\, dB_u\right| \le \|B\|_\gamma (t-s)^{2H+\gamma}.$$

Therefore, since $\|\delta\tilde{R}\|_{3\gamma} \le G$, it follows from the sewing Lemma 2.5 that

$$|\tilde{R}_{st}| \le G(t-s)^{3\gamma}.$$

Our claim (5.26) is now easily deduced from the above estimate of $\tilde{R}$ and expression (5.27). $\square$

Now we provide some estimates for the iterated integral $\int_s^t \delta y_{su}^n \otimes dy_u^n$, which is also part of the rough path above (y^n, B). Note that $\int_s^t \delta y_{su}^n \otimes dy_u^n$ is defined as

$$\int_s^t \delta y_{su}^n \otimes dy_u^n = \sum_{t_k=s}^{t-} \int_{t_k}^{t_{k+1}} \delta y_{su}^n \otimes dy_u^n,$$

in the same way as for the integral $\int_s^t \delta y_{su}^n \otimes dB_u$.

LEMMA 5.3. *Let the assumptions be as in Lemma 5.2. Then the following estimate holds true*:

$$\left|\int_s^t \delta y_{su}^n \otimes dy_u^n\right| \le G(t-s)^{2\gamma} \qquad \textit{for } (s,t) \in \mathcal{S}_2([\![0,T]\!]),$$

where G is a random variable in $\bigcap_{p\ge1} L^p(\Omega)$, independent of n.

PROOF. We proceed similarly as in the proof of Lemma 5.2. Namely, we still assume $b=0$ for the sake of conciseness, and the existence of the integral $\int_s^t \delta y^n_{su} \otimes dy^n_u$ is justified as for (5.25). Next, we define a remainder type increment $\bar{R}$ on $\mathcal{S}_2([\![0,T]\!])$ by

$$\begin{aligned} \bar{R}_{st} &= \int_s^t \delta y^n_{su} \otimes dy^n_u - V(y^n_s) \int_s^t \delta B_{su} \otimes dy^n_u \\ &\quad - \frac{1}{2} \sum_{j=1}^m \partial V_j V_j(y^n_s) \otimes \int_s^t (u-s)^{2H} \, dy^n_u. \end{aligned} \tag{5.28}$$

As previously, it is clear that $\bar{R}_{t_k t_{k+1}} = 0$. In the same way as in Lemma 5.2, we can also show that $\|\delta \bar{R}\|_{3\gamma} \le G$, so by Lemma 2.5 we obtain

$$|\bar{R}_{st}| \le G(t-s)^{3\gamma}.$$

Applying this estimate to (5.28), we obtain the desired estimate for $\int_s^t \delta y^n_{su} \otimes dy^n_u$. □

We can now conclude and get a uniform bound on (y^n, B) as a rough path.

PROPOSITION 5.4. *Let y be the solution of equation* (1.1) *and y^n be the solution of the Euler scheme* (5.1). *Consider $\frac{1}{3} < \gamma < H < \frac{1}{2}$, and set*

$$S_2(B, y^n)_{st} = \left((\delta B_{st}, \delta y^n_{st}), \int_s^t (\delta B_{su}, \delta y^n_{su}) \otimes d(\delta B_u, \delta y^n_u) \right).$$

Then $S_2(B, y^n)$ can be considered as a γ-rough path according to Definition 2.1. *In addition, there exists a random variable $G \in \bigcap_{p \ge 1} L^p(\Omega)$ independent of n such that*

$$\|S_2(B, y^n)\|_\gamma \le G,$$

where $\|\cdot\|_\gamma$ is defined by (2.3).

PROOF. Putting together the results of Proposition 5.1, Lemma 5.2 and Lemma 5.3, we easily get the definition of $S_2(B, y^n)$, together with the bound:

$$|S_2(B, y^n)_{st}| \le G(t-s)^\gamma \qquad \text{for } (s,t) \in \mathcal{S}_2([\![0,T]\!]).$$

On the other hand, by the definition of y^n it is clear the same estimate holds for $s, t \in [t_k, t_{k+1}]$, $k = 0, \dots, n-1$. The proposition then follows by applying Lemma A.1 to $S_2(B, y^n)$. □

6. Almost sure convergence of the Euler scheme. We now take advantage of the information gathered up to now, and show the almost sure convergence of the Euler scheme (5.1). Notice however that the convergence rate obtained in this section is not optimal, and has to be seen as a preliminary step; see Section 8.2 for a more accurate result.

REMARK 6.1. The approximation process y^n is discrete by nature, and the reader might wonder why we have spent some effort trying to show that (y^n, B) is a rough path. The answer will be clearer within the landmark of the current section. Indeed, our analysis of the numerical scheme mainly hinges on the fact that the renormalized error satisfies a linear equation driven by both y^n and B. The best way we have found to properly define this equation is by showing that (y^n, B) can be seen as a rough path. Let us mention however two alternative ways to get the same kind of result:

(i) We could have relied on the fact that y^n is a controlled process with respect to B; see (2.8) and [13, 17] for the notion of controlled process. However, due to the fact that y^n is defined on a discrete grid, we have not been able to find a satisfactory way to see y^n as a continuous time controlled process.

(ii) We could also have dealt with a discrete version of the linear equation, which governs the error process on our discrete grid. Nevertheless, we believe that the continuous time version exhibited below is more elegant, and this is why we have stuck to the continuous time strategy.

With Remark 6.1 in mind, we will now introduce the linear equation which will govern the error process, and then analyze the Euler scheme. Throughout the section, we assume that $b \in C_b^2$ and $V \in C_b^4$.

6.1. *A linear rough differential equation.* Recall that we are dealing with the unique solution y to the following equation:

$$dy_t = b(y_t)\,dt + V(y_t)\,dB_t, \qquad t \in [0, T]. \tag{6.1}$$

Its numerical approximation y^n is given by the Euler scheme (5.1). As we shall see later in the paper, the error process is governed by a kind of discrete equivalent of the Jacobian for equation (6.1). Specifically, we consider the following linear equation:

$$\Phi_t^n = \mathrm{Id} + \int_0^t \{\partial b(y^n)\}_s \Phi_s^n\,ds + \sum_{j=1}^m \int_0^t \{\partial V_j(y^n)\}_s \Phi_s^n\,dB_s^j, \tag{6.2}$$

where Id is the $d \times d$ identity matrix, and where we have set

$$\{\partial V_j(y^n)\}_s = \int_0^1 \partial V_j(y_s^n + \lambda(y_s - y_s^n))\,d\lambda,$$

$$\{\partial b(y^n)\}_s = \int_0^1 \partial b(y_s^n + \lambda(y_s - y_s^n))\,d\lambda.$$

In this subsection, we derive an upper-bound estimate of Φ^n and its inverse Ψ^n based on Proposition 5.4.

PROPOSITION 6.2. *The linear equation* (6.2) *has a unique solution* Φ^n, *and there exists an integrable random variable* G *such that the following estimate holds true*:

$$|S_2(\Phi^n, z^n)_{st}| \le e^G (t-s)^\gamma, \tag{6.3}$$

where we recall that $z^n = (y^n, B)$ *and the signature* S_2 *is introduced in Definition* 2.1. *Furthermore*, Φ^n *admits an inverse process* $\Psi^n \equiv (\Phi^n)^{-1}$, *where* $(\Phi_t^n)^{-1}$ *stands for the inverse matrix of* Φ_t^n, *and estimate* (6.3) *also holds for* Ψ^n.

PROOF. Define two $\mathbb{R}^{d\times d}$-valued processes θ and ξ, respectively, by

$$\theta_t^{il} = \sum_{j=1}^m \int_0^t \{\partial_i V_j^l(y^n)\}_s \, dB_s^j \quad \text{and} \quad \xi_t^{il} = \int_0^t \{\partial_i b^l(y^n)\}_s \, ds.$$

Then we can easily recast equation (6.2) as

$$\Phi_t^{n,l} = \delta_l + \sum_{i=1}^d \int_0^t \Phi_s^{n,i} \, d\xi_s^{il} + \sum_{i=1}^d \int_0^t \Phi_s^{n,i} \, d\theta_s^{il}. \tag{6.4}$$

Here, δ_l is a vector in $\mathbb{R}^d$ with the lth entry equal to 1 and the other entries equal to 0. In particular, Φ^n satisfies a linear equation driven by $\zeta = \{\theta^{il}, \xi^{il}; i, l = 1, \dots, d\}$. By the estimate of $\|S_2(z^n)\|_\gamma$ contained in Proposition 5.4, we can show that for $S_2(\zeta, z^n)$ we have $\|S_2(\zeta, z^n)\|_\gamma \le G$, where G is an integrable random variable independent of n. So applying Theorem 2.3 to equation (6.4), we obtain

$$|S_2(\Phi^n, z^n)_{st}| \le K_1 \|S_2(\zeta, z^n)\|_\gamma (t-s)^\gamma \exp(K_2 \|S_2(\zeta, z^n)\|_\gamma^{1/\gamma}),$$

and the estimate (6.3) then follows. Note that by Lemma A.6 the inverse Ψ^n of Φ^n satisfies a linear equation similar to (6.4), and the estimate of the Ψ^n can be obtained in the same way. □

REMARK 6.3. Note that from the proof of Proposition 6.2, it is not clear that the random variable e^G in (6.3) is integrable. However, the almost sure bound (6.3) will be enough for our use in deriving the almost sure convergence rate of the Euler scheme (5.1). Let us mention that the methodology adopted in [5] in order to get the integrability of the Jacobian of a RDE driven by Gaussian processes does not apply to equation (6.2). This is due to the fact that (6.2) involves the process y^n, which is the solution of a "discrete" RDE driven by both B and F [recall that F is defined in (3.14)]. We believe that a discrete strategy in order to bound Φ^n would lead to the integrability of $|S_2(\Phi^n)_{st}|$, but we have not delved deeper into this direction for sake of conciseness.

6.2. *Error process as a rough path.* In this subsection, we derive some estimates on the error process of the Euler scheme. To this aim, we will first write the process y^n as the solution of a differential equation in continuous time. Namely, it is readily checked that one can recast equation (5.1) as follows:

$$y_t^n = y_0 + \int_0^t b\big(y_{\eta(s)}^n\big)\,ds + \int_0^t V\big(y_{\eta(s)}^n\big)\,dB_s - A_t^1, \tag{6.5}$$

where we have set

$$\eta(s) = t_{\lfloor \frac{ns}{T} \rfloor}, \quad \text{and} \quad A_t^1 = -\frac{1}{2}\sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor}\sum_{j=1}^{m} \partial V_j V_j\big(y_{t_k}^n\big)(t \wedge t_{k+1} - t_k)^{2H}. \tag{6.6}$$

Note that the dependence of A^1 on n is omitted for simplicity. With this simple algebraic decomposition in hand, we can state the following bound on the error process.

LEMMA 6.4. *Let y, y^n and Φ^n be the solution of equations* (6.1), (5.1) *and* (6.2), *respectively, and Ψ^n be the inverse process of Φ^n. Consider the path ε defined by*

$$\varepsilon_t = \Psi_t^n\big(y_t - y_t^n\big). \tag{6.7}$$

Then for all $\frac{1}{3} < \gamma < H < \frac{1}{2}$, we can find an almost surely finite random variable G independent of n such that

$$|\delta\varepsilon_{st}| \le G(t-s)^{1-\gamma} n^{1-3\gamma}, \qquad (s,t) \in \mathcal{S}_2\big([\![0,T]\!]\big). \tag{6.8}$$

PROOF. Putting together equations (6.1) and (6.5), it is easily seen that

$$y_t - y_t^n = \int_0^t \big(b(y_s) - b\big(y_{\eta(s)}^n\big)\big)\,ds + \int_0^t \big(V(y_s) - V\big(y_{\eta(s)}^n\big)\big)\,dB_s + A_t^1.$$

In addition, the chain rule for rough integrals enables us to write

$$\varphi\big(y_s^n\big) - \varphi\big(y_{\eta(s)}^n\big) = \int_{\eta(s)}^s \partial\varphi\big(y_u^n\big)\,dy_u^n,$$

for any $\varphi \in C^{1/\gamma}(\mathbb{R}^d;\mathbb{R}^d)$ and $s \in [0,T]$, and where $\partial\varphi$ designates the gradient of φ. Owing to this relation, applied successively to b and V, we get

$$y_t - y_t^n = \int_0^t \big(b(y_s) - b\big(y_s^n\big)\big)\,ds + \int_0^t \big(V(y_s) - V\big(y_s^n\big)\big)\,dB_s + \sum_{e=1}^{3} A_t^e,$$

where we recall that A^1 is defined by (6.6), and where we have set

$$A_t^2 = \int_0^t \int_{\eta(s)}^s \partial b\big(y_u^n\big)\,dy_u^n\,ds, \quad \text{and} \quad A_t^3 = \sum_{j=1}^{m}\int_0^t \int_{\eta(s)}^s \partial V_j\big(y_u^n\big)\,dy_u^n\,dB_s^j. \tag{6.9}$$

Notice that A_t^2 and A_t^3 above can be considered as a rough integral, thanks to Proposition 5.4. Taking into account the identity

$$
\begin{aligned}
b(y_s) - b(y_s^n) &= \{\partial b(y^n)\}_s (y_s - y_s^n), \\
V_j(y_s) - V_j(y_s^n) &= \{\partial V_j(y^n)\}_s (y_s - y_s^n),
\end{aligned} \tag{6.10}
$$

we have

$$
\begin{aligned}
y_t - y_t^n = {} & \int_0^t \{\partial b(y^n)\}_s (y_s - y_s^n)\, ds \\
& + \int_0^t \{\partial V_j(y^n)\}_s (y_s - y_s^n)\, dB_s + \sum_{e=1}^{3} A_t^e.
\end{aligned} \tag{6.11}
$$

Now starting from expression (6.11) and applying the variation of parameter method to the equation (6.2) governing Φ^n, it is easy to verify that

$$
y_t - y_t^n = \sum_{e=1}^{3} \Phi_t^n \int_0^t \Psi_s^n \, dA_s^e. \tag{6.12}
$$

Therefore, we can also write

$$
\varepsilon_t = \Psi_t^n (y_t - y_t^n) = \sum_{e=1}^{3} \int_0^t \Psi_s^n \, dA_s^e, \qquad t \in [0, T]. \tag{6.13}
$$

Our claim (6.8) thus follows from Proposition 6.2, together with Lemma 6.5 below. □

LEMMA 6.5. *Let A^e, $e = 1, 2, 3$, be as in (6.6) and (6.9). Let f be a continuous function with values in a finite dimensional vector space $\mathcal{V}$ such that the path*

$$
S_2(f, B) := \left((f_t, B_t), \int_0^t (f_s, B_s) \otimes d(f_s, B_s) \right)
$$

is well defined and assume that $f_0 = 0$. We also assume that there exists an a.s finite random variable G satisfying the upper bound $\|S_2(f, B)\|_\gamma \le G$ for any $\frac{1}{3} < \gamma < H < \frac{1}{2}$. Then for all $(s, t) \in \mathcal{S}_2(\llbracket 0, T \rrbracket)$ we have

$$
\left| \sum_{e=1}^{3} \int_s^t f_u \otimes dA_u^e \right| \le G (t - s)^{1-\gamma} n^{1-3\gamma}.
$$

PROOF. We divide this proof in several steps.

Step 1: Decomposition of A^e. Applying the chain rule to A^3, we obtain

$$
A_t^3 = A_t^{31} + R_t^2,
$$

where the paths A_t^{31} and R_t^2 are respectively defined by

$$A_t^{31} = \sum_{j=1}^{m} \int_0^t \int_{\eta(s)}^s \partial V_j\big(y_{\eta(s)}^n\big)\, dy_u^n \, dB_s^j,$$

$$R_t^2 = \sum_{j,j'=1}^{m} \int_0^t \int_{\eta(s)}^s \int_{\eta(s)}^u \partial_{j'} \partial V_j\big(y_v^n\big)\, dy_v^{n,j'} \, dy_u^n \, dB_s^j.$$

Moreover, recalling the equation (6.5) governing y^n, we obtain

$$A_t^{31} = A_t^{310} + R_t^3 + R_t^4,$$

where we have set

$$A_t^{310} = \sum_{i,j=1}^{m} \int_0^t \int_{\eta(s)}^s \partial V_j\big(y_{\eta(s)}^n\big) V_i\big(y_{\eta(s)}^n\big)\, dB_u^i \, dB_s^j,$$

$$R_t^3 = \sum_{j=1}^{m} \int_0^t \int_{\eta(s)}^s \partial V_j\big(y_{\eta(s)}^n\big) b\big(y_{\eta(s)}^n\big)\, du \, dB_s^j,$$

and where

$$R_t^4 = \frac{1}{2} \sum_{j,j'=1}^{m} \int_0^t \int_{\eta(s)}^s \partial V_j\big(y_{\eta(s)}^n\big) \partial V_{j'} V_{j'}\big(y_{\eta(s)}^n\big)\, d\big(u - \eta(s)\big)^{2H} \, dB_s^j.$$

Summarizing our decomposition up to now, we have found that $A_t^3 = A_t^{310} + R_t^2 + R_t^3 + R_t^4$. Denoting $R_t^5 = A_t^{310} + A_t^1$, and $R_t^1 = A_t^2$, we can now express our driving process $\sum_{e=1}^3 A^e$ as a sum of remainder type terms:

$$\sum_{e=1}^{3} A_t^e = \sum_{e=1}^{5} R_t^e. \tag{6.14}$$

Step 2: Estimation procedure. We will now upper bound the terms R^e given in our decomposition (6.14). For the term R^2, observe that (due to Proposition 5.4) the couple (B, y^n) can be seen as a γ-Hölder rough path. Taking into account all the time increments defining R^2, we obtain $|\delta R_{t_k t_{k+1}}^2| \le G n^{-3\gamma}$ for all $t_k = s, \dots, t_-$. Therefore,

$$\big|\delta R_{st}^2\big| \le \sum_{t_k=s}^{t-} \big|\delta R_{t_k t_{k+1}}^2\big| \le G(t-s) n^{1-3\gamma}, \qquad (s,t) \in \mathcal{S}_2\big([\![0,T]\!]\big). \tag{6.15}$$

In the same way we can show that estimate (6.15) holds for R^4. In order to bound R^5, note that for $t \in [\![0,T]\!]$ we have

$$R_t^5 = \sum_{i,j=1}^{m} \sum_{k=0}^{\frac{nt}{T}-1} (\partial V_j V_i)\big(y_{t_k}^n\big) \delta F_{t_k t_{k+1}}^{ij}.$$

Furthermore, by the two inequalities of Proposition 5.1 one can show that the process $f=[\partial V_j V_i](y^n)$ and $g=\partial(\partial V_j V_i)V(y^n)$ satisfies the conditions of Proposition 4.7. Hence, combining Corollary 4.9 and Lemma 3.5, we end up with the following inequality for $\kappa>0$ arbitrarily:

$$\begin{aligned}|\delta R^5_{st}| &\le G(t-s)^{\frac{1}{2}-\kappa} n^{\frac{1}{2}-2H+\kappa}\\ &\le G(t-s)^{1-\gamma-2\kappa} n^{1-3\gamma}, \qquad (s,t)\in\mathcal{S}_2(\llbracket 0,T\rrbracket),\end{aligned}\tag{6.16}$$

where we have used the fact that $t-s\ge\frac{T}{n}$ for the last step. The terms R^1 and R^3 are bounded along the same lines, in a slightly easier way due to the presence of Lebesgue type integrals. We get

$$|\delta R^e_{st}| \le G(t-s)n^{-\gamma}, \qquad (s,t)\in\mathcal{S}_2(\llbracket 0,T\rrbracket)\tag{6.17}$$

for $e=1,3$. In summary of the estimates (6.15), (6.16) and (6.17), we have obtained

$$\sum_{e=1}^{5}|\delta R^e_{st}| \le G(t-s)^{1-\gamma-2\kappa} n^{1-3\gamma}, \qquad (s,t)\in\mathcal{S}_2(\llbracket 0,T\rrbracket).\tag{6.18}$$

Step 3: Conclusion. Thanks to our decomposition (6.14), we can write

$$\begin{aligned}\sum_{e=1}^{3}\int_s^t f_u\otimes dA^e_u &= \sum_{e=1}^{3}\int_s^t \delta f_{\eta(u),u}\otimes dA^e_u + \sum_{e=1}^{5}\int_s^t f_{\eta(u)}\otimes dR^e_u\\ &\equiv B^1_{st}+B^2_{st}.\end{aligned}\tag{6.19}$$

Let us start by bounding the term B^2. Similar to relation (4.9), we can decompose B^2 as

$$B^2_{st}=\sum_{e=1}^{5}\sum_{t_k=s}^{t-} f_s\otimes\delta R^e_{t_k t_{k+1}} + \sum_{e=1}^{5}\sum_{t_k=s}^{t-}\delta f_{st_k}\otimes\delta R^e_{t_k t_{k+1}}.$$

Then recall the assumption $f\in C^{\gamma'}$ for any $\frac{1}{3}<\gamma'<H$. By choosing γ' and κ such that $\gamma'+1-\gamma-2\kappa>1$, we can apply Proposition 4.1. Taking into account (6.18) this yields

$$|B^2_{st}|\le G(t-s)^{1-\gamma}n^{1-3\gamma},\tag{6.20}$$

for $(s,t)\in\mathcal{S}_2(\llbracket 0,T\rrbracket)$. As far as the term B^1 above is concerned, we get

$$|B^1_{st}|\le\sum_{t_k=s}^{t-}\left|\sum_{e=1}^{3}\int_{t_k}^{t_{k+1}}\delta f_{t_k u}\otimes dA^e_u\right|.\tag{6.21}$$

Moreover, note that $\int_{t_k}^{t_{k+1}}\delta f_{t_k u}\otimes dA^e_u$ is a third-order integral of the process (f,B) on the interval $[t_k,t_{k+1}]$, for all k. Therefore, since we have assumed

$\|\mathcal{S}_2(f,B)\|_\gamma \le G$, we easily obtain the estimate $|\int_{t_k}^{t_{k+1}} \delta f_{t_k u} \otimes dA_u^e| \le G n^{-3\gamma}$. Applying this inequality to (6.21) yields

$$(6.22) \qquad |B_{st}^1| \le G(t-s) n^{1-3\gamma}.$$

The lemma follows by applying (6.20) and (6.22) to (6.19). $\square$

We now wish, as in the case of y^n, to consider the error process ε as a rough path. As a first step, let us label the following regularity assumption for further use.

HYPOTHESIS 6.6. Let ε be the process defined by (6.7). We suppose that there exists an exponent $\alpha < 2H - \frac{1}{2}$ and an almost surely finite random variable G such that the error process ε satisfies

$$(6.23) \qquad |\delta\varepsilon_{st}| \le \frac{G}{n^\alpha}(t-s)^{\frac{1}{2}}, \qquad (s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket).$$

REMARK 6.7. It follows from Lemma 6.4 that Hypothesis 6.6 holds true for $\alpha = 3\gamma - 1$. We will see later on how to improve it to larger values of α.

We are now ready to define and estimate the double iterated integrals of ε, which are a fundamental part of the rough path above ε.

LEMMA 6.8. *Let ε be the process defined by (6.7) and assume that Hypothesis 6.6 is satisfied for some $\alpha < 2H - \frac{1}{2}$. Then for any $\kappa > 0$ we have*

$$(6.24) \qquad \left| \int_s^t \delta\varepsilon_{su} \otimes d\varepsilon_u \right| \le G n^{-2\alpha+2\kappa}(t-s), \qquad (s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket),$$

where G is a random variable such that $G \in \bigcap_{p\ge1} L^p(\Omega)$.

PROOF. Observe that the double integral $\int_s^t \delta\varepsilon_{su} \otimes d\varepsilon_u$ is well defined, since (y, y^n, B) admits a rough path lift. Next, take $(s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket)$. We can write

$$(6.25) \qquad \int_s^t \delta\varepsilon_{su} \otimes d\varepsilon_u = \int_s^t \delta\varepsilon_{s\eta(u)} \otimes d\varepsilon_u + \int_s^t \delta\varepsilon_{\eta(u)u} \otimes d\varepsilon_u \equiv D_{st}^1 + D_{st}^2,$$

where recall that $\eta(u)$ is defined in (6.6). Let us bound those two terms separately.

The term D^1 above can be expressed in a more elementary way as $D_{st}^1 = \sum_{t_k=s}^{t-} \delta\varepsilon_{st_k} \otimes \delta\varepsilon_{t_k t_{k+1}}$. Moreover, thanks to (6.23) we have for any $\kappa < \alpha$ and $(s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket)$:

$$|\delta\varepsilon_{st}| \le G n^{-\alpha+\kappa}(t-s)^{\frac{1}{2}+\kappa}.$$

Taking this estimate into account and applying Proposition 4.1, we get

$$(6.26) \qquad |D_{st}^1| \le G^2 n^{-2\alpha+2\kappa}(t-s)^{1+2\kappa}.$$

On the other hand, owing to identity (6.13) for ε, we have

$$D^2_{st} = \sum_{e,e'=1}^{3} \int_s^t \int_{\eta(u)}^u \Psi^n_v \, dA^{e'}_v \otimes \Psi^n_u \, dA^e_u. \tag{6.27}$$

Take γ such that $\alpha < 2\gamma - 1/2 < 2H - 1/2$. By the definition of A_e, $e = 1, 2, 3$ in (6.6) and (6.9), it is easy to see that each of the nine terms on the right-hand side is bounded by $Gn^{1-4\gamma}(t-s)$. Indeed, for the term corresponding to $e = e' = 3$, we use (6.9) to write

$$\int_s^t \int_{\eta(u)}^u \Psi^n_v \, dA^3_v \otimes \Psi^n_u \, dA^3_u = \sum_{j,j'=1}^{m} \sum_{t_k=s}^{t-} A^{33,jj'}_{t_k t_{k+1}},$$

where

$$A^{33,jj'}_{t_k t_{k+1}} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \Psi^n_v \int_{t_k}^{v} \partial V_j(y^n_r) \, dy^n_r \, dB^j_v \otimes \Psi^n_u \int_{t_k}^{u} \partial V_{j'}(y^n_{r'}) \, dy^n_{r'} \, dB^{j'}_u.$$

It follows from Proposition 6.2 and the Lyon's lift map theorem (see, e.g., [14]) that

$$\left| S_4(B, y^n, \Psi^n)_{t_k, t_{k+1}} \right| \le G\left(\frac{T}{n}\right)^{\gamma},$$

so for all $j, j' \le m$ and $(s,t) \in \mathcal{S}_2([\![0,T]\!])$ we have

$$\left| A^{33,jj'}_{t_k t_{k+1}} \right| \le G\left(\frac{T}{n}\right)^{4\gamma}.$$

Therefore, summing this bound over j, j' and t_k we end up with

$$\left| \int_s^t \int_{\eta(u)}^u \Psi^n_v \, dA^3_v \otimes \Psi^n_u \, dA^3_u \right| \le \sum_{j,j'=1}^{m} \sum_{t_k=s}^{t-} \left| A^{33,jj'}_{t_k t_{k+1}} \right| \le G\left(\frac{T}{n}\right)^{4\gamma-1} (t-s). \tag{6.28}$$

The other terms of the form $\int_s^t \int_{\eta(u)}^u \Psi^n_v \, dA^e_v \otimes \Psi^n_u \, dA^{e'}_u$ on the right-hand side of (6.27) can be estimated in the same way. Therefore, we obtain the estimate

$$\left| D^2_{st} \right| \le Gn^{1-4\gamma}(t-s). \tag{6.29}$$

In conclusion, plugging (6.26) and (6.29) into (6.25), we obtain the desired estimate (6.24). $\square$

Recall that we wish to construct a rough path above (ε, B, y^n). In the previous lemma, we have analyzed the double integral $\int_s^t \delta\varepsilon_{su} \otimes \delta\varepsilon_u$. We now consider the integral $\int_s^t \delta\varepsilon_{su} \otimes d(B_u, y^n_u)$.

LEMMA 6.9. *Denote $z^n = (B, y^n)$, and let the assumptions of Lemma 6.4 prevail. The following estimate holds true*:

$$\left| \int_s^t \delta\varepsilon_{su} \otimes dz^n_u \right| \le G(t-s)n^{1-3\gamma}, \qquad (s,t) \in \mathcal{S}_2([\![0,T]\!]).$$

PROOF. We use the same kind of decomposition as in Lemma 6.8:

$$\int_s^t \delta\varepsilon_{su} \otimes dz_u^n = \int_s^t \delta\varepsilon_{s\eta(u)} \otimes dz_u^n + \int_s^t \delta\varepsilon_{\eta(u)u} \otimes dz_u^n := \hat{D}_{st}^1 + \hat{D}_{st}^2. \tag{6.30}$$

For the first term on the right-hand side of (6.30), we have the following expression:

$$\hat{D}_{st}^1 = \sum_{t_k=s}^{t-} \delta\varepsilon_{st_k} \otimes \delta z_{t_k t_{k+1}}^n, \qquad (s,t) \in \mathcal{S}_2(\llbracket 0, T\rrbracket).$$

Moreover, it follows from Lemma 6.4 and Proposition 5.4 that for all $(s,t) \in \mathcal{S}_2(\llbracket 0, T\rrbracket)$ the following bounds holds true:

$$|\delta\varepsilon_{st}| \le G(t-s)^{1-\gamma} n^{1-3\gamma} \quad \text{and} \quad |\delta z_{st}| \le G(t-s)^{\gamma+\kappa},$$

where $\kappa < H - \gamma$, so by Proposition 4.1 we have

$$|\hat{D}_{st}^1| \le G(t-s) n^{1-3\gamma}.$$

On the other hand, in the same way as the estimate of D_{st}^2 in (6.29), we can show that

$$|\hat{D}_{st}^2| \le G(t-s) n^{1-3\gamma}.$$

The lemma follows from applying the above two inequalities for $\hat{D}_{st}^1$ and $\hat{D}_{st}^2$ to (6.30). □

The next result provides further estimates of the rough path above the path (ε, B, y^n) for $(s,t) \in \mathcal{S}_2([t_k, t_{k+1}])$.

LEMMA 6.10. *Let ε be the process defined by (6.7) and recall that we have set $z^n = (B, y^n)$. Take $\gamma < H$, $(s,t) \in \mathcal{S}_2([t_k, t_{k+1}])$ and $k = 0, 1, \dots, n-1$. Then the following estimate for the first-order increments of ε holds true*:

$$|\delta\varepsilon_{st}| \le G(t-s)^{\gamma} n^{-\gamma}. \tag{6.31}$$

In addition, the second-order iterated integrals of ε and z^n satisfy

$$\begin{aligned} \left|\int_s^t \delta z_{su}^n \otimes d\varepsilon_u\right| &\le G(t-s)^{2\gamma} n^{-\gamma}, \\ \left|\int_s^t \delta\varepsilon_{su} \otimes d\varepsilon_u\right| &\le G(t-s)^{2\gamma} n^{-2\gamma}. \end{aligned} \tag{6.32}$$

PROOF. The estimate (6.31) follows by showing that the three terms on the right-hand side of (6.13) are all bounded by $G(t-s)^{\gamma} n^{-\gamma}$. As before, we will focus on a bound for the increment $\int_s^t \Psi_s^n \, dA_s^3$. In fact, owing to (6.9) it is easily seen that $\int \Psi^n \, dA^3$ can be decomposed as a sum of double iterated integrals:

$$\begin{aligned}\int_s^t \Psi_s^n \, dA_s^3 &= \sum_{j=1}^m \int_s^t \Psi_u^n \int_{t_k}^u \partial V_j(y_v^n) \, dy_v^n \, dB_u^j \\ &= \sum_{j=1}^m \int_s^t \Psi_u^n \int_s^u \partial V_j(y_v^n) \, dy_v^n \, dB_u^j + \sum_{j=1}^m \int_{t_k}^s \partial V_j(y_v^n) \, dy_v^n \int_s^t \Psi_u^n \, dB_u^j.\end{aligned}$$

One can easily bound the two terms above, thanks to the fact that (y^n, B) is a rough path. We obtain

$$\int_s^t \Psi_s^n \, dA_s^3 \le G(t-s)^\gamma n^{-\gamma}.$$

In the same way, we can show that the same estimate holds for the term $\sum_{e=1}^2 \int_s^t \Psi_s^n \, dA_s^e$ on the right-hand side of (6.13). This proves our claim (6.31).

In order to prove (6.32), let us invoke (6.13) again, which yields

$$\int_s^t \delta z_{su}^n \otimes d\varepsilon_u = \sum_{e=1}^3 \int_s^t \delta z_{su}^n \otimes \Psi_u^n \, dA_u^e.$$

The estimate (6.32) then follows from a similar argument as for the estimate of (6.31). The estimate of integral $\int_s^t \varepsilon_{su} \otimes d\varepsilon_u$ can be shown in a similar way. This completes the proof. $\square$

The following is the main result of this section. Recall that $\varepsilon = \Psi_t^n(y_t - y_t^n)$ is defined in (6.13) and $z^n = (y^n, B)$, and $S_2(n^{3\gamma-1}\varepsilon, z^n)$ denotes the lift of the process $(n^{3\gamma-1}\varepsilon, z^n)$, that is,

$$S_2(n^{3\gamma-1}\varepsilon, z^n)_{st} = \left((n^{3\gamma-1}\delta\varepsilon_{st}, \delta z_{st}^n), \int_s^t (n^{3\gamma-1}\delta\varepsilon_{su}, \delta z_{su}^n) \otimes (n^{3\gamma-1}\varepsilon_u, z_u^n) \right),$$

PROPOSITION 6.11. *Let y be the solution of equation (1.1) and y^n be the solution of the Euler scheme (5.1). Take $\frac{1}{3} < \gamma < H < \frac{1}{2}$. Then we have the estimate*

$$\| S_2(n^{3\gamma-1}\varepsilon, y^n, B) \|_\gamma \le G,$$

where G is a random variable independent of n.

PROOF. In summary of Lemmas 6.4, 6.8, 6.9 and Proposition 5.4, we have

$$|S_2(n^{3\gamma-1}\varepsilon, y^n, B)_{st}| \le G(t-s)^\gamma \tag{6.33}$$

for $(s,t) \in \mathcal{S}_2([\![0,T]\!])$. On the other hand, Lemma 6.10 implies that relation (6.33) still holds true for $s,t \in [t_k, t_{k+1}]$. The lemma then follows by applying Lemma A.1 to $S_2(n^{3\gamma-1}\varepsilon, y^n, B)$. $\square$

7. The error process as a rough path under new conditions. In this section, we derive an improved upper-bound estimate of the error process under new conditions. Our considerations relies on the following process, solution of a linear RDE:

$$\Phi_t = \mathrm{Id} + \int_0^t \partial b(y_s)\Phi_s \, ds + \sum_{j=1}^m \int_0^t \partial V_j(y_s)\Phi_s \, dB_s^j. \tag{7.1}$$

Throughout the section, we assume that $b \in C_b^2$ and $V \in C_b^4$. The reader might have noticed that Φ is simply the Jacobian related to equation (1.1). The process Φ is also the limit of the process Φ^n defined in (6.2), in a sense which will be made clear in the next section. We denote by Ψ_t the inverse matrix of Φ_t. Let us introduce the following process on $\mathcal{S}_2([\![0, T]\!])$:

$$\delta\tilde{\varepsilon}_{st} = \delta\varepsilon_{st} - \delta\hat{\varepsilon}_{st}, \tag{7.2}$$

where ε is defined by (6.7) and

$$\delta\hat{\varepsilon}_{st} = \sum_{j,j'=1}^m \sum_{t_k=s}^{t-} \Psi_{t_k}\partial V_j V_{j'}(y_{t_k})\delta F_{t_k t_{k+1}}^{jj'}. \tag{7.3}$$

We shall now assume some a priori bounds on $\tilde{\varepsilon}$, similar to what we did in Hypothesis 6.6.

HYPOTHESIS 7.1. The process $\tilde{\varepsilon}$ defined in (7.2) satisfies the following relation for some $\alpha > 0$:

$$|\delta\tilde{\varepsilon}_{st}| \le G n^{-\alpha}(t-s)^{1-\gamma}, \qquad (s,t) \in \mathcal{S}_2([\![0, T]\!]).$$

Our aim is to get a new bound on the rough path above (ε, z^n) under Hypothesis 7.1, similar to what has been obtained in Proposition 6.11. Let us first consider $\int_s^t \delta\varepsilon_{su} \otimes dB_u$.

LEMMA 7.2. *Suppose that Hypothesis 7.1 is met for some $\alpha : 0 < \alpha < 2H - \frac{1}{2}$. Then the following estimate holds true for all $(s,t) \in \mathcal{S}_2([\![0, T]\!])$:*

$$\left| \int_s^t \delta\varepsilon_{su} \otimes dB_u \right| \le G\big(n^{-\alpha} + n^{\frac{1}{2}-2\gamma}\big)(t-s)^{2\gamma}. \tag{7.4}$$

PROOF. As in Lemma 6.9, we first write

$$\int_s^t \delta\varepsilon_{su} \otimes dB_u = \int_s^t \delta\varepsilon_{s\eta(u)} \otimes dB_u + \int_s^t \delta\varepsilon_{\eta(u)u} \otimes dB_u := \tilde{D}_{st}^1 + \tilde{D}_{st}^2. \tag{7.5}$$

Furthermore, invoking decomposition (6.13) we have

$$\tilde{D}_{st}^2 = \sum_{e=1}^3 \int_s^t \int_{\eta(u)}^u \Psi_v^n \, dA_v^e \otimes dB_u := \sum_{e=1}^3 I_{st}^e. \tag{7.6}$$

Note that, recalling expression (6.9) for A^2, I^2 can be seen as a triple iterated integral which is interpreted in the Young sense. Then similar to (6.28) it is easy to show that

$$|I_{st}^2| \le G n^{-2\gamma}(t-s). \tag{7.7}$$

As far as I_1 is concerned, some elementary computations reveal that for $(s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket)$:

$$I_{st}^1 = -\frac{1}{2}\sum_{j=1}^{m}\sum_{t_k=s}^{t-} \partial V_j^i V_j(y_{t_k}^n)\int_{t_k}^{t_{k+1}}\int_{t_k}^{u} \Psi_v^{n,i}\, d(v-t_k)^{2H} \otimes dB_u.$$

Then it follows from (A.2) in Lemma A.2 that

$$|I_{st}^1| \le G n^{1-4\gamma}(t-s)^{1-\gamma} \le G n^{-\gamma}(t-s)^{2\gamma}, \tag{7.8}$$

where we use the fact that $t-s \ge \frac{T}{n}$ for the second inequality. For I_3, we start from relation (6.9). Then due to the expression (5.1) for y^n, we have

$$I_{st}^3 = I_{st}^{31} + I_{st}^{32} + I_{st}^{33},$$

where

$$I_{st}^{31} = \sum_{t_k=s}^{t-}\int_{t_k}^{t_{k+1}}\int_{t_k}^{u} \Psi_v^n \sum_{j=1}^{m}\int_{t_k}^{v} \partial V_j(y_r^n) V(y_{t_k}^n)\, dB_r\, dB_v^j \otimes dB_u,$$

$$I_{st}^{32} = \sum_{t_k=s}^{t-}\int_{t_k}^{t_{k+1}}\int_{t_k}^{u} \Psi_v^n \sum_{j=1}^{m}\int_{t_k}^{v} \partial V_j(y_r^n)\frac{1}{2}\sum_{j'=1}^{m}\partial V_{j'}V_{j'}(y_{t_k}^n)$$
$$\cdot\, d(r-t_k)^{2H}\, dB_v^j \otimes dB_u$$

$$I_{st}^{33} = \sum_{t_k=s}^{t-}\int_{t_k}^{t_{k+1}}\int_{t_k}^{u} \Psi_v^n \sum_{j=1}^{m}\int_{t_k}^{v} \partial V_j(y_r^n) b(y_{t_k}^n)\, dr\, dB_v^j \otimes dB_u.$$

As for I^2, it is easy to show that $|I_{st}^{32}|$ and $|I_{st}^{33}|$ are bounded by $Gn^{-2\gamma}(t-s)$ and $Gn^{1-4\gamma}(t-s)$, respectively. On the other hand, it follows from (A.8) in Lemma A.2 that for any $\kappa > 0$ we can find a random variable G such that $|I_{st}^{31}|$ is less than $Gn^{1-4\gamma+2\kappa}(t-s)^{1-\gamma}$. We now choose $\kappa > 0$ small enough such that $1-4\gamma+2\kappa < -\gamma$. Then summarizing our estimates of I^{31}, I^{32} and I^{33}, we have

$$|I_{st}^3| \le G n^{1-4\gamma+2\kappa}(t-s)^{1-\gamma} \le G n^{-\gamma}(t-s)^{2\gamma}. \tag{7.9}$$

Applying (7.7), (7.8) and (7.9) to (7.6), we have thus obtained

$$|\tilde{D}_{st}^2| \le G n^{-\gamma}(t-s)^{2\gamma}. \tag{7.10}$$

We now turn to the term $\tilde{D}_{st}^1$ on the right-hand side of (7.5). Write

$$\tilde{D}_{st}^1 = \int_s^t (\delta\varepsilon_{s\eta(u)} - \delta\tilde{\varepsilon}_{s\eta(u)}) \otimes dB_u + \int_s^t \delta\tilde{\varepsilon}_{s\eta(u)} \otimes dB_u := \tilde{D}_{st}^{11} + \tilde{D}_{st}^{12},$$

where we recall that $\tilde{\varepsilon}$ is defined in (7.2). Applying Proposition 4.1 and taking into account Hypothesis 7.1 and the fact that $H > \gamma$, we obtain

$$|\tilde{D}_{st}^{12}| \leq G n^{-\alpha}(t-s). \tag{7.11}$$

On the other hand, by (7.2) and (7.3) we have

$$\begin{aligned}\tilde{D}_{st}^{11} &= \int_s^t \sum_{j,j'=1}^m \int_s^{\eta(u)} \Psi_{\eta(v)} \partial V_j V_{j'}(y_{\eta(v)})\, dF_{\eta(v)v}^{jj'} \otimes dB_u \\ &= \sum_{j,j'=1}^m \sum_{t_k=s+}^{t-} \sum_{t_{k'}=s}^{t_{k-1}} \Psi_{t_{k'}} \partial V_j V_{j'}(y_{t_{k'}}) F_{t_{k'}t_{k'+1}}^{jj'} \otimes \delta B_{t_k t_{k+1}}.\end{aligned}$$

So applying Corollary 4.9 to $\tilde{D}_{st}^{11}$ we obtain

$$\|\tilde{D}_{st}^{11}\|_p \leq K n^{\frac{1}{2}-2H}(t-s)^{H+\frac{1}{2}} \qquad \text{for all } p \geq 1.$$

Taking into account Lemma 3.5, we thus get

$$|\tilde{D}_{st}^{11}| \leq G n^{\frac{1}{2}-2\gamma}(t-s)^{2\gamma}. \tag{7.12}$$

In summary of (7.10), (7.11) and (7.12), we end up with

$$\left|\int_s^t \delta\varepsilon_{su} \otimes dB_u\right| \leq G n^{-\gamma}(t-s)^{2\gamma} + G n^{-\alpha}(t-s) + G n^{\frac{1}{2}-2\gamma}(t-s)^{2\gamma},$$

from which our claim (7.4) is easily deduced. □

In order to complete the study of the rough path above (ε, y^n, B), let us turn to the integral $\int_s^t \delta y_{su}^n \otimes d\varepsilon_u$.

LEMMA 7.3. *Suppose that Hypotheses 6.6 and 7.1 are met for some $\alpha \in (0, 2H - \frac{1}{2})$. Take $\gamma < H$. Then the integral $\int_s^t \delta y_{su}^n \otimes d\varepsilon_u$ satisfies the following relation for all $(s,t) \in \mathcal{S}_2([\![0,T]\!])$:*

$$\left|\int_s^t \delta y_{su}^n \otimes d\varepsilon_u\right| \leq G n^{-\alpha}(t-s)^{2\gamma}. \tag{7.13}$$

PROOF. We consider a remainder term R defined for $(s,t) \in \mathcal{S}_2([\![0,T]\!])$ by

$$R_{st} = \int_s^t \delta y_{su}^n \otimes d\varepsilon_u - V(y_s^n) \int_s^t \delta B_{su} \otimes d\varepsilon_u. \tag{7.14}$$

According to the basic rules of action of δ on products of increments, we have

$$\delta R_{sut} = \left(\delta y_{su}^n - V(y_s^n)\delta B_{su}\right) \otimes \delta\varepsilon_{ut} + \delta V(y_\cdot^n)_{su} \int_u^t \delta B_{uv} \otimes d\varepsilon_v, \tag{7.15}$$

for all $(s,u,t) \in \mathcal{S}_3(\llbracket 0,T \rrbracket)$. Applying the second inequality of (5.2) and Hypothesis 6.6 to the first term on the right-hand side of (7.15) and invoking Lemma 7.2 for the the second term, we obtain

$$|\delta R_{sut}| \le G n^{-\alpha}(u-s)^{2\gamma}(t-u)^{\frac{1}{2}} + G n^{-\alpha}(u-s)^{\gamma}(t-u)^{2\gamma} \le G n^{-\alpha}(t-s)^{3\gamma}.$$

Since $3\gamma > 1$, we are in a position to apply the discrete sewing Lemma 2.5, which yields

$$\|R\|_{3\gamma} \le K \|\delta R\|_{3\gamma} \le G n^{-\alpha}. \tag{7.16}$$

We now recast (7.14) as follows:

$$\int_s^t \delta y^n_{su} \otimes d\varepsilon_u = V\big(y^n_s\big) \int_s^t \delta B_{su} \otimes d\varepsilon_u + R_{st}. \tag{7.17}$$

Then resorting to (7.16) and Lemma 7.2 for (7.17), our claim (7.13) easily follows. □

We can now state the main result of this section, giving a full estimate of the rough path above $(z^n, n^{\alpha-\kappa}\varepsilon)$. Recall that z^n designates the couple (B, y^n).

PROPOSITION 7.4. *Let $\tilde{\varepsilon}$ be defined in (7.2) and suppose that Hypothesis 7.1 holds true for some $0 < \alpha < 2H - \frac{1}{2}$. Take $\gamma < H$. Then for any $\kappa \in (0,\alpha)$ and $(s,t) \in \mathcal{S}_2([0,T])$ we have*

$$\big|S_2\big(z^n, n^{\alpha-\kappa}\varepsilon\big)_{st}\big| \le G(t-s)^{\gamma}. \tag{7.18}$$

PROOF. We start by analyzing the first-order increments of $S_2(z^n, n^{\alpha-\kappa}\varepsilon)$. First, notice that δz^n is controlled by Proposition 5.4. Furthermore, according to relation (7.2), we have

$$\delta\varepsilon = \delta\hat{\varepsilon} + \delta\tilde{\varepsilon}. \tag{7.19}$$

As in the proof of Lemma 7.2, equation (7.3) also asserts that Corollary 4.9 can be applied to $\delta\hat{\varepsilon}$, yielding an inequality of the form

$$\|\delta\hat{\varepsilon}_{st}\|_p \le K n^{\frac{1}{2}-2H}(t-s)^{\frac{1}{2}} \tag{7.20}$$

for $(s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket)$. Applying Lemma 3.5 to relation (7.20), plugging this information into (7.19) and invoking Hypothesis 7.1, we obtain

$$|\delta\varepsilon_{st}| \le G n^{-\alpha}(t-s)^{\frac{1}{2}} \le G n^{-\alpha+\kappa}(t-s)^{\frac{1}{2}+\kappa}, \tag{7.21}$$

for all $(s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket)$. This is compatible with our claim (7.18).

Let us now handle the second-order increments of $S_2(z^n, n^{\alpha-\kappa}\varepsilon)$. According to Lemma 6.8,

$$\left|\int_s^t \delta\varepsilon_{su} \otimes d\varepsilon_u\right| \le G n^{-2\alpha+2\kappa}(t-s), \qquad (s,t) \in \mathcal{S}_2(\llbracket 0,T \rrbracket). \tag{7.22}$$

In the same way, gathering Lemma 7.2, Lemma 7.3 together with (7.21), we get that

$$\left|\int_s^t \delta\varepsilon_{su} \otimes dz_u^n\right| \le G n^{-\alpha+\kappa}(t-s)^{2\gamma}, \qquad (s,t) \in \mathcal{S}_2(\llbracket 0,T\rrbracket). \tag{7.23}$$

Hence, putting together inequalities (7.22) and (7.23) and adding the estimate of $S_2(z^n)$ in Proposition 5.4, we obtain that on the grid $\mathcal{S}_2(\llbracket 0,T\rrbracket)$:

$$\left|S_2\left(z^n, n^{\alpha-\kappa}\varepsilon\right)_{st}\right| \le G(t-s)^{\gamma} \tag{7.24}$$

On the other hand, Lemma 6.10 implies that (7.24) also holds true for $s,t \in [t_k, t_{k+1}]$. Therefore, applying Lemma A.1 to $S_2(z^n, n^{\alpha-\kappa}\varepsilon)$ we obtain the desired estimate (7.18). □

8. Rate of convergence for the Euler scheme.

In this section, we take another look at the strong convergence of the Euler scheme. Thanks to the information we have gathered on the error process, we shall reach optimality for the convergence rate of the scheme. However, before we can state this optimal result, let us give some preliminaries about the Jacobian Φ of equation (1.1).

8.1. *Rate of convergence for the Jacobian.* As mentioned in Section 7, the Jacobian Φ of equation (1.1) should be seen as the limit of the process Φ^n. In the current section we shall quantify this convergence. We start by an algebraic identity which is stated as a lemma.

LEMMA 8.1. *Let Φ and Φ^n be the solutions of equations (7.1) and (6.2), respectively. Set*

$$\mathcal{E}_t = \Psi_t\left(\Phi_t - \Phi_t^n\right) \tag{8.1}$$

for $t \in [0,T]$. Then $\mathcal{E}$ satisfies the following equation on $[0,T]$:

$$\begin{aligned}\mathcal{E}_t &= \int_0^t \Psi_s \sum_{i=1}^d \{\partial_i \partial b(y^n)\}_s (\Phi_s^n \varepsilon_s)^i \Phi_s^n \, ds \\ &\quad + \sum_{j=1}^m \int_0^t \Psi_s \sum_{i=1}^d \{\partial_i \partial V_j(y^n)\}_s (\Phi_s^n \varepsilon_s)^i \Phi_s^n \, dB_s^j,\end{aligned} \tag{8.2}$$

where the processes $\{\partial_i \partial b(y^n)\}_s$ and $\{\partial_i \partial V_j(y^n)\}_s$ are defined by

$$\begin{aligned}\{\partial_i \partial b(y^n)\}_s &= \int_0^1 \int_0^1 \partial_i \partial b\left(y_s + (1-\mu)(1-\lambda)\left(y_s^n - y_s\right)\right) \\ &\quad \times (1-\lambda)\, d\mu \, d\lambda,\end{aligned} \tag{8.3}$$

$$\begin{aligned}\{\partial_i \partial V_j(y^n)\}_s &= \int_0^1 \int_0^1 \partial_i \partial V_j\left(y_s + (1-\mu)(1-\lambda)\left(y_s^n - y_s\right)\right) \\ &\quad \times (1-\lambda)\, d\mu \, d\lambda.\end{aligned} \tag{8.4}$$

If we define $\tilde{\mathcal{E}}_t = \Phi_t(\Psi_t - \Psi_t^n)$, then a similar expression can be derived for $\tilde{\mathcal{E}}_t$.

PROOF. Subtracting (6.2) from (7.1), it is easily seen that

$$\Phi_t - \Phi_t^n = \int_0^t \partial b(y_s)\big(\Phi_s - \Phi_s^n\big)\, ds + \sum_{j=1}^m \int_0^t \partial V_j(y_s)\big(\Phi_s - \Phi_s^n\big)\, dB_s^j + L_t^1 + L_t^2,$$

where

$$L_t^1 = \int_0^t \big(\partial b(y_s) - \{\partial b(y^n)\}_s\big)\Phi_s^n\, ds,$$

$$L_t^2 = \sum_{j=1}^m \int_0^t \big(\partial V_j(y_s) - \{\partial V_j(y^n)\}_s\big)\Phi_s^n\, dB_s^j.$$

Let now $\Psi = \Phi^{-1}$ be the inverse of Φ. By means of the variation of the constant method, one can verify that

$$\Phi_t - \Phi_t^n = \sum_{e=1,2} \Phi_t \int_0^t \Psi_s\, dL_s^e.$$

Hence, for $\mathcal{E}$ defined by (8.1), we have

$$\mathcal{E}_t = \sum_{e=1,2} \int_0^t \Psi_s\, dL_s^e. \tag{8.5}$$

In addition, observe that with (8.3) and (8.4) in mind, the following identities hold true:

$$\partial b(y_s) - \{\partial b(y^n)\}_s = \sum_{i=1}^d \{\partial_i \partial b(y^n)\}_s \big(y_s^i - y_s^{n,i}\big) = \sum_{i=1}^d \{\partial_i \partial b(y^n)\}_s \big(\Psi_s^n \varepsilon_s\big)^i,$$

and

$$\partial V_j(y_s) - \{\partial V_j(y^n)\}_s = \sum_{i=1}^d \{\partial_i \partial V_j(y^n)\}_s \big(\Psi_s^n \varepsilon_s\big)^i.$$

Plugging these relations into definitions of L^1 and L^2, our claim (8.2) easily stems from relation (8.5).

Note that according to Lemma A.6, (Ψ, Ψ^n) and (Φ, Φ^n) satisfies similar linear equations, and a similar expression can thus be derived for $\tilde{\mathcal{E}}_t = \Phi_t(\Psi_t - \Psi_t^n)$. $\square$

We shall now assume some a priori bounds on the lift of $(z^n, n^\alpha \varepsilon)$.

HYPOTHESIS 8.2. The processes $z^n = (y^n, B)$ and ε defined in (6.7) satisfy the following inequality for some $\alpha > 0$ and $\gamma < H$:

$$\big|S_2\big(z^n, n^\alpha \varepsilon\big)_{st}\big| \le G(t-s)^\gamma, \qquad (s,t) \in \mathcal{S}_2\big([0,T]\big).$$

Notice that Hypothesis 8.2 is a version of Hypothesis 6.6 for the lift of $(z^n, n^\alpha \varepsilon)$. Thanks to the previous lemma, we can now consider $(y^n, B, n^\alpha \varepsilon, n^\alpha \mathcal{E}, n^\alpha \tilde{\mathcal{E}})$ as a single rough path. This is achieved in the following lemma.

LEMMA 8.3. *Suppose that $b \in C_b^2$, $V \in C_b^4$, and Hypothesis 8.2 is met for $\gamma > \frac{1}{3}$ and $\alpha < 2H - \frac{1}{2}$. We also consider the processes $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as defined in Lemma 8.1. Then the vector $(z^n, n^\alpha \varepsilon, n^\alpha \mathcal{E}, n^\alpha \tilde{\mathcal{E}})$ satisfies the following upper bound*:

$$\left\| S_2\left(z^n, n^\alpha \varepsilon, n^\alpha \mathcal{E}, n^\alpha \tilde{\mathcal{E}}\right) \right\|_\gamma \le G. \tag{8.6}$$

PROOF. Note that Φ, Ψ, Φ^n, Ψ^n are solutions of equations driven by z^n. Furthermore, owing to relation (8.2), it is easily seen that $n^\alpha \mathcal{E}$ and $n^\alpha \tilde{\mathcal{E}}$ as rough paths are solutions of equations driven by $(z^n, n^\alpha \varepsilon)$. Thus (8.6) is a direct consequence of Theorem 2.3 (linear part) and of Hypothesis 8.2. □

REMARK 8.4. Roughly speaking, Lemma 8.3 shows that if the convergence rate of the numerical scheme y^n to y is $n^{-\alpha}$, then so is that of (Φ^n, Ψ^n) to (Φ, Ψ) as $n \to \infty$.

8.2. *Optimal rate of convergence.* Recall that ε is defined by (6.7) and $\delta\tilde{\varepsilon}_{st} = \delta\varepsilon_{st} - \delta\hat{\varepsilon}_{st}$ is defined in (7.2). With the preliminary results obtained in Section 8.1, we can now go further in our analysis of the error process ε.

PROPOSITION 8.5. *Consider the process $z^n = (y^n, B)$ and the error process ε defined in (6.7). Assume that $b \in C_b^2$, $V \in C_b^4$. As in Lemma 8.3, suppose that Hypothesis 8.2 is met for some exponents α, γ such that $\frac{1}{3} < \gamma < H$ and $\alpha < 2H - \frac{1}{2}$. Take $\kappa > 0$ arbitrarily small. Then the following estimate holds true for $(s, t) \in \mathcal{S}_2([\![0, T]\!])$:*

$$|\delta\tilde{\varepsilon}_{st}| \le G\left(n^{1-3\gamma-\alpha+2\kappa} + n^{-\gamma}\right)(t-s)^{1-\gamma}. \tag{8.7}$$

In addition, for all $(s, t) \in \mathcal{S}_2([\![0, T]\!])$ we also have

$$|\delta\varepsilon_{st}| \le G\left(n^{1-3\gamma-\alpha+2\kappa} + n^{\frac{1}{2}-2\gamma}\right)(t-s)^{\frac{1}{2}-\kappa}. \tag{8.8}$$

REMARK 8.6. In Proposition 8.5, we prove that Hypothesis 7.1 is satisfied for $\tilde{\varepsilon}$, with α replaced by $(3\gamma - 1 + \alpha) \wedge \gamma$. We also prove that Hypothesis 6.6 for ε is fulfilled with an improved exponent $\alpha_1 = (\alpha + 3\gamma - 1) \wedge 2\gamma - \frac{1}{2}$, which satisfies $\alpha_1 > \alpha$.

PROOF OF PROPOSITION 8.5. This proof is divided into several steps.

Step 1: Decomposition of $\delta\varepsilon$. Starting from the decomposition (6.13) of ε, one can write

$$\delta\varepsilon_{st} = U^1_{st} + U^2_{st}, \tag{8.9}$$

with

$$\begin{aligned} U^1_{st} &= \sum_{e=1}^{3} \int_s^t \Psi_u \, dA^e_u \\ U^2_{st} &= \sum_{e=1}^{3} \int_s^t (\Psi^n_u - \Psi_u) \, dA^e_u = -\sum_{e=1}^{3} \int_s^t \Psi_u \tilde{\mathcal{E}}_u \, dA^e_u, \end{aligned} \tag{8.10}$$

where we recall that $\tilde{\mathcal{E}}$ has been introduced in Lemma 8.1. Moreover, the term U^2 above is easily bounded. Indeed, applying Lemma 6.5 and taking into account the estimate in Lemma 8.3 we obtain for all $(s,t) \in \mathcal{S}_2([\![0,T]\!])$ the estimate:

$$|U^2_{st}| \le G(t-s)^{1-\gamma} n^{1-3\gamma-\alpha}. \tag{8.11}$$

Step 2: Decomposition of U^1. We turn to the quantity U^1_{st} given by (8.10). First, from the expression of A^2 in (6.9) and a discrete-time decomposition similar to the estimate of (6.19) it is clear that

$$\left| \int_s^t \Psi_u \, dA^2_u \right| \le G n^{-\gamma} (t-s). \tag{8.12}$$

In the case $e = 1$, recall expression (6.6) for A^1. Then one can decompose $\int_s^t \Psi_u \, dA^1_u$ into

$$\int_s^t \Psi_u \, dA^1_u = M^1_{st} + M^2_{st}, \tag{8.13}$$

where M^1 and M^2 are defined by

$$\begin{aligned} M^1_{st} &= -\frac{1}{2} \sum_{j=1}^{m} \int_s^t \Psi_{\eta(u)} \partial V_j V_j \big(y^n_{\eta(u)}\big) \, d\big(u - \eta(u)\big)^{2H} \\ M^2_{st} &= -\frac{1}{2} \sum_{j=1}^{m} \int_s^t \delta\Psi_{\eta(u)u} \partial V_j V_j \big(y^n_{\eta(u)}\big) \, d\big(u - \eta(u)\big)^{2H}. \end{aligned} \tag{8.14}$$

We defer the evaluation for M^1 to the end of the proof, but M^2 is easily controlled. Indeed, by (A.1) in Lemma A.2 applied to $f = \partial V_j V_j(y^n)$ and $g = \Psi$, we have

$$|M^2_{st}| \le G n^{1-4\gamma} (t-s)^{1-\gamma}. \tag{8.15}$$

Note that according to Lemma A.6 Ψ admits the decomposition: $\delta\Psi_{st} = -\int_s^t \Psi_u \times \partial b(y_u) \, du - \sum_{j=1}^m \int_s^t \Psi_u \partial V_j(y_u) \, dB^j_u$.

We now decompose the term $\int_s^t \Psi_u \, dA_u^3$ in (8.10): it is readily checked that owing to (6.9), one can write

$$\int_s^t \Psi_u \, dA_u^3 = \sum_{j=1}^m \int_s^t \Psi_u \int_{\eta(u)}^u \partial V_j\big(y_v^n\big) \, dy_v^n \, dB_u^j.$$

Hence, plugging the equation (5.1) followed by y^n into this relation we can write

$$\int_s^t \Psi_u \, dA_u^3 = I_{st}^1 + I_{st}^2 + I_{st}^3, \tag{8.16}$$

where I^1, I^2, I^3 are given by

$$I_{st}^1 = \sum_{j=1}^m \int_s^t \Psi_u \int_{\eta(u)}^u \partial V_j\big(y_v^n\big) b\big(y_{\eta(v)}^n\big) \, dv \, dB_u^j,$$

$$I_{st}^2 = \sum_{j=1}^m \int_s^t \Psi_u \int_{\eta(u)}^u \partial V_j\big(y_v^n\big) V\big(y_{\eta(v)}^n\big) \, dB_v \, dB_u^j, \tag{8.17}$$

$$I_{st}^3 = \frac{1}{2} \sum_{j,j'=1}^m \int_s^t \Psi_u \int_{\eta(u)}^u \partial V_j\big(y_v^n\big) \partial V_{j'} V_{j'}\big(y_{\eta(v)}^n\big) \, d\big(v - \eta(v)\big)^{2H} \, dB_u^j.$$

Step 3: Estimate of I^1, I^2, I^3. We will now evaluate I^1, I^2, I^3 separately. First, invoking a discrete-time decomposition as in (6.19) again, we get

$$\big|I_{st}^1\big| \le G(t-s)n^{-\gamma}. \tag{8.18}$$

On the other hand, applying (A.2) in Lemma A.2 to I_{st}^3 we obtain

$$\big|I_{st}^3\big| \le Gn^{1-4\gamma}(t-s)^{1-\gamma}. \tag{8.19}$$

Let us now consider the term I^2 defined by (8.17). To this aim, set

$$J_{st}^2 = \sum_{j=1}^m \int_s^t \Psi_{\eta(u)} \int_{\eta(u)}^u \partial V_j\big(y_{\eta(v)}^n\big) V\big(y_{\eta(v)}^n\big) \, dB_v \, dB_u^j. \tag{8.20}$$

By (A.3) in Lemma A.2, the patient reader can check that for any $\kappa > 0$ we have

$$\big|I_{st}^2 - J_{st}^2\big| \le Gn^{1-4\gamma+2\kappa}(t-s)^{1-\gamma}, \qquad (s,t) \in \mathcal{S}_2\big([0,T]\big). \tag{8.21}$$

In addition, we may consider $\kappa > 0$ such that $1 - 4\gamma + 2\kappa < -\gamma$. In this case, the previous bound becomes

$$\big|I_{st}^2 - J_{st}^2\big| \le Gn^{-\gamma}(t-s)^{1-\gamma}. \tag{8.22}$$

Step 4: Conclusion. So far we have made a sequence of decompositions for ε in (8.9), (8.10), (8.13), (8.16). Taking into account the estimates (8.11), (8.12), (8.15), (8.18), (8.19), (8.22), it is clear that to prove the estimate (8.7) for $\tilde{\varepsilon}$ it suffices to show that

$$|\delta\bar{\varepsilon}_{st}| \le G n^{1-3\gamma-\alpha+2\kappa}(t-s)^{1-\gamma} \tag{8.23}$$

for any $\kappa > 0$, where the increment $\delta\bar{\varepsilon}$ is defined by

$$\delta\bar{\varepsilon}_{st} = J^2_{st} + M^1_{st} - \delta\hat{\varepsilon}_{st}$$

with J^2 given by (8.20) and M^1 given by (8.14). We also recall that $\hat{\varepsilon}$ has been introduced in (7.3) and is given by the following expression:

$$\delta\hat{\varepsilon}_{st} = \sum_{j,j'=1}^{m} \sum_{t_k=s}^{t-} \Psi_{t_k} \partial V_j V_{j'}(y_{t_k}) \delta F^{jj'}_{t_k t_{k+1}}. \tag{8.24}$$

In order to prove (8.23), let us now observe that

$$\delta\bar{\varepsilon}_{st} = \sum_{j,j'=1}^{m} \sum_{t_k=s}^{t-} \Psi_{t_k}\big(\partial V_j V_{j'}\big(y^n_{t_k}\big) - \partial V_j V_{j'}(y_{t_k})\big)\delta F^{jj'}_{t_k t_{k+1}}. \tag{8.25}$$

Further, we note that similarly to (6.10), the following identity holds true:

$$\begin{aligned}
&\partial V_j V_{j'}\big(y^n_{t_k}\big) - \partial V_j V_{j'}(y_{t_k}) \\
&\quad = -\sum_{i=1}^{d} \int_0^1 (\partial_i \partial V_j V_{j'})\big(\lambda y^n_{t_k} + (1-\lambda) y_{t_k}\big)\big)\, d\lambda \cdot \big(\Phi^n_{t_k} \varepsilon_{t_k}\big)^i.
\end{aligned}$$

According to Hypothesis 8.2, y^n, Ψ and $n^\alpha \varepsilon$ are γ-Hölder continuous functions. Hence,

$$n^\alpha \big\| \Psi\big(\partial V_j V_{j'}\big(y^n\big) - \partial V_j V_{j'}(y)\big)\big\|_\gamma \le G. \tag{8.26}$$

In order to bound the right-hand side of (8.25), let us apply a bound on weighted sums of the process F as in relation (6.16). Taking into account (8.26), this yields

$$|\delta\bar{\varepsilon}_{st}| \le G n^{1-3\gamma-\alpha}(t-s)^{1-\gamma-2\kappa} \le G n^{1-3\gamma-\alpha+2\kappa}(t-s)^{1-\gamma}$$

for an arbitrary $\kappa > 0$, which is our claim (8.23). The proof of (8.7) is now complete.

In order to get (8.8) from (8.7), we recall once again relation (7.19) and we just analyze the term $\delta\hat{\varepsilon}$. This can be done in a similar way as in (7.20) and (7.21). Our proof is complete. $\square$

THEOREM 8.7. *Let ε be given by* (6.7) *and $\tilde{\varepsilon}$ be defined in* (7.2). *Suppose that $b \in C^2_b$, $V \in C^4_b$. Then the following statements hold true*:

(i) *There exists a constant* $\kappa_H > 0$ *depending on* H *such that*

$$n^{2H-\frac{1}{2}}|\delta\tilde{\varepsilon}_{st}| \le G n^{-\kappa_H}|t-s|^{1-\gamma}, \tag{8.27}$$

for $(s,t) \in \mathcal{S}_2(\llbracket 0,T\rrbracket)$. *In particular, we have the following almost sure convergence*:

$$\lim_{n\to\infty} n^{2H-\frac{1}{2}}\delta\tilde{\varepsilon}_{st} = 0.$$

(ii) *Take a constant* $\kappa > 0$. *The error process* $y - y^n$ *satisfies*

$$n^{2H-\frac{1}{2}-\kappa}\sup_{t\in[0,T]}|y_t - y_t^n| \to 0 \qquad \text{as } n\to\infty, \tag{8.28}$$

meaning that the Euler scheme has a rate of convergence $n^{\frac{1}{2}-2H+\kappa}$ *for an arbitrary* $\kappa > 0$.

PROOF. Item (i): Take $\frac{1}{3} < \gamma < H$. According to Proposition 6.11 for $S_2(n^{3\gamma-1}\varepsilon, z^n)$, Hypothesis 8.2 holds with $\alpha = 3\gamma - 1$. Hence, one can apply Proposition 8.5 in order to get

$$|\delta\tilde{\varepsilon}_{st}| \le G\big(n^{2(1-3\gamma)+2\kappa} + n^{-\gamma}\big)(t-s)^{1-\gamma}. \tag{8.29}$$

In the case $\frac{3}{8} < H < \frac{1}{2}$, it is easy to see that $3H-1 < 2H-\frac{1}{2}$ and $2(3H-1) > 2H-\frac{1}{2}$. Take $\frac{3}{8} < \gamma < H$ such that $2(3\gamma-1) - 2\kappa > 2H - \frac{1}{2}$ and $\gamma > 2H - \frac{1}{2}$. Then (8.29) implies that for $\frac{3}{8} < H < \frac{1}{2}$ we have $n^{2H-\frac{1}{2}}|\delta\tilde{\varepsilon}_{st}| \le Gn^{-\kappa_H}|t-s|^{1-\gamma}$ for $\kappa_H = ((6\gamma - 2 - 2\kappa)\wedge\gamma) - (2H - \frac{1}{2})$. This proves our claim (8.27) for $\frac{3}{8} < H < \frac{1}{2}$.

Let us now handle the case $\frac{1}{3} < H \le \frac{3}{8}$. To this aim, set $H_k = \frac{2k-1}{6k-4}$, $k \ge 2$. We choose $k \ge 2$ such that $H_{k+1} < H \le H_k$ holds. It is easy to verify that $k(3H-1) < 2H - \frac{1}{2}$ and $(k+1)(3H-1) > 2H - \frac{1}{2}$. We can thus choose $H_{k+1} < \gamma < H$ and $\kappa > 0$ such that $(k+1)(3\gamma - 1) - 3k\kappa > 2H - \frac{1}{2}$ and $\gamma > 2H - \frac{1}{2}$. It follows from inequality (8.29) that

$$|\delta\tilde{\varepsilon}_{st}| \le Gn^{2(1-3\gamma)+2\kappa}(t-s)^{1-\gamma}.$$

We can now iterate this bound in the following way: apply Proposition 7.4 which gives

$$\big|S_2\big(z^n, n^{2(3\gamma-1)-3\kappa}\varepsilon\big)_{st}\big| \le G(t-s)^{\gamma}. \tag{8.30}$$

Then invoke Proposition 8.5 again. Taking into account the estimate (8.30), we obtain

$$|\delta\tilde{\varepsilon}_{st}| \le G\big(n^{3(1-3\gamma)+5\kappa} + n^{-\gamma}\big)(t-s)^{1-\gamma}.$$

We can now repeat the application of Proposition 7.4 and 8.5 in order to get

$$|\delta\tilde{\varepsilon}_{st}| \le G\big(n^{(k+1)(1-3\gamma)+3k\kappa} + n^{-\gamma}\big)(t-s)^{1-\gamma}.$$

This implies that $n^{2H-\frac{1}{2}}|\delta\tilde{\varepsilon}_{st}| \le Gn^{-\kappa_H}|t-s|^{1-\gamma}$ for

$$\kappa_H = \big((k+1)(3\gamma-1) - 3k\kappa\big) \wedge \gamma - \left(2H - \frac{1}{2}\right).$$

Item (ii): Recall $\delta\varepsilon = \delta\hat{\varepsilon} + \delta\tilde{\varepsilon}$ given by relation (7.2). With item (i) in hand and the fact that Φ^n is uniformly bounded thanks to Proposition 6.2, our claim (8.28) is reduced to prove that

$$\lim_{n\to\infty} n^{2H-\frac{1}{2}-\kappa} \sup_{(s,t)\in\mathcal{S}_2([\![0,T]\!])} |\delta\hat{\varepsilon}_{st}| = 0. \tag{8.31}$$

In order to prove (8.31), recall the expression (8.24) for $\delta\hat{\varepsilon}_{st}$ as a weighted sum of the increment δF. We can thus apply Corollary 4.9 with f equal to $\Psi\partial V_j V_{j'}(y)$. Indeed, one can easily see that f satisfies the assumptions of Proposition 4.7: both Ψ and $\partial V_j V_{j'}(y)$ are controlled processes admitting moments of any order (see [5] for the integrability of Ψ). Applying Corollary 4.9 we thus get

$$\|\delta\hat{\varepsilon}_{st}\|_p \le Kn^{\frac{1}{2}-2H}(t-s)^{\frac{1}{2}}.$$

Then, invoking Lemma 3.5, we end up with

$$|\delta\hat{\varepsilon}_{st}| \le Gn^{\frac{1}{2}-2H+\frac{\kappa}{2}},$$

which completes the proof. □

9. Asymptotic error distributions. In this section, we first review a central limit theorem from [24] (see also [19]), then in the second part, we prove the asymptotic error distribution of the Euler scheme.

9.1. *A central limit theorem for the Lévy area process.* In this subsection, we recall a central limit theorem for the process F. Let us first define some parameters that will appear in the limit distribution of F. Namely, for $k \in \mathbb{Z}$, we set

$$\begin{aligned} Q(k) &= \int_{[0,1]^2} R\begin{pmatrix} 0 & s' \\ k & k+s \end{pmatrix} dR(s,s'), \\ P(k) &= \int_{[0,1]^2} R\begin{pmatrix} 0 & s' \\ k+s & k+1 \end{pmatrix} dR(s,s'), \end{aligned} \tag{9.1}$$

where we recall that R is the covariance function defined by (3.1), whose rectangular increments are given by formula (3.2). We now state a slight elaboration of [24], Theorem 3 and [19], Proposition 5.1.

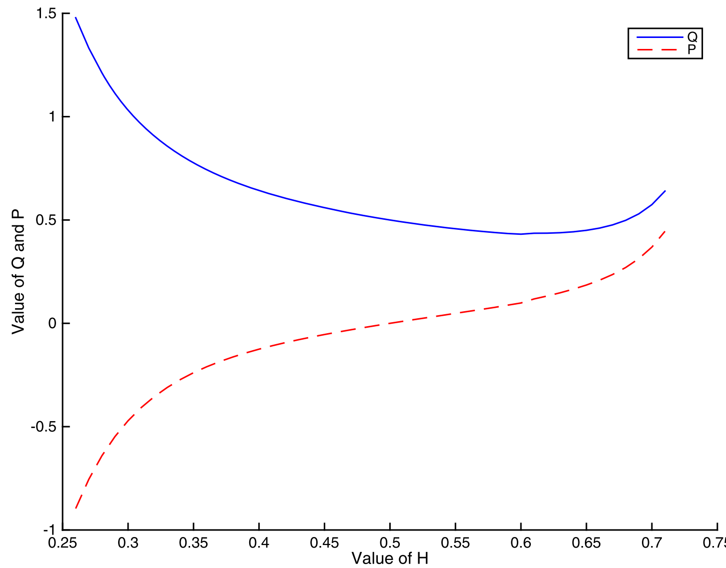

FIG. 1. *Q and R.*

PROPOSITION 9.1. *Let $B=(B^1,\dots,B^m)$ be a m-dimensional standard fBm with Hurst parameter $\frac{1}{4}<H<\frac{1}{2}$. Let $\bar{F}_t=F_{\eta(t)}$ for $t\in[0,T]$, where we recall that the process F is defined by (3.14) and η is given by (6.6). Then the finite dimensional distributions of $\{n^{2H-\frac{1}{2}}\bar{F},B\}$ converge weakly to those of (W,B), where $W=(W^{ij})$ is an $m\times m$-dimensional Brownian motion, independent of B, such that*

$$\mathbb{E}[W_t^{ij}W_s^{i'j'}]=T^{4H-1}(Q\delta_{ii'}\delta_{jj'}+P\delta_{ij'}\delta_{ji'})(t\wedge s). \tag{9.2}$$

In formula (9.2), we have set $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ if $i\neq j$, and $Q=\sum_{k\in\mathbb{Z}}Q(k)$, $P=\sum_{k\in\mathbb{Z}}P(k)$.

REMARK 9.2. Proposition 9.1 shows that the process $n^{2H-\frac{1}{2}}\bar{F}$ converges stably to W when $\frac{1}{4}<H<\frac{1}{2}$. We refer the reader to Chapter 8 in [21] for the definition of stable convergence and its equivalent conditions.

REMARK 9.3. The following plot of constants Q and P shows that Q is strictly larger than P for $H\in(\frac{1}{4},\frac{3}{4})$. In particular, this implies that the $m\times m$ random matrix W defined in Proposition 9.1 is not symmetric. Let us also mention that as has been observed in [20], the fact that $Q>P$ results in different features of the Crank–Nicolson scheme and the numerical schemes (1.3) and (1.4) between the scalar case and the multidimensional cases.

9.2. *Asymptotic error distributions*. We can now prove the convergence of a renormalized version of the error process $y - y^n$ related to the Euler-type scheme y^n. Namely, we prove the following central limit theorem.

THEOREM 9.4. *Let y^n be the Euler scheme defined in* (1.5). *Suppose $b \in C_b^2$ and $V \in C_b^4$. Then the sequence of processes $(n^{2H-\frac{1}{2}}(y - y^n), B)$ converges weakly in $D([0,T])$ to the couple (U, B) as $n \to \infty$, where U is the solution of the linear SDE*

$$U_t = \int_0^t \partial b(y_s) U_s \, ds + \sum_{j=1}^m \int_0^t \partial V_j(y_s) U_s \, dB_s^j + \sum_{i,j=1}^m \int_0^t \partial V_i V_j(y_s) \, dW_s^{ij}, \tag{9.3}$$

and where W is the Wiener process obtained in Proposition 9.1.

PROOF. Recall that $y - y^n = \Phi^n \varepsilon$, we consider the following decomposition:

$$\begin{aligned} y_t - y_t^n = {} & \Phi_{\eta(t)}^n \tilde{\varepsilon}_{\eta(t)} + (\Phi_{\eta(t)}^n - \Phi_{\eta(t)}) \hat{\varepsilon}_{\eta(t)} + \Phi_{\eta(t)} \hat{\varepsilon}_{\eta(t)} \\ & + (y_t - y_{\eta(t)}) - (y_t^n - y_{\eta(t)}^n), \end{aligned} \tag{9.4}$$

where recall that ε is defined by (6.7), and $\tilde{\varepsilon}$, $\hat{\varepsilon}$ are respectively introduced in (7.2) and (7.3).

Note that, thanks to Theorem 8.7, Lemma 8.3 and Corollary 4.9 and taking into account the relation $\Phi_t - \Phi_t^n = \Phi_t \mathcal{E}_t$, we have almost surely

$$\lim_{n \to \infty} \sup_{t \in [0,T]} n^{2H-\frac{1}{2}} (\Phi_{\eta(t)}^n \tilde{\varepsilon}_{\eta(t)} + (\Phi_{\eta(t)}^n - \Phi_{\eta(t)}) \hat{\varepsilon}_{\eta(t)}) = 0. \tag{9.5}$$

On the other hand, thanks to Theorem 2.3 and equation (5.1), governing y^n it is clear that

$$|y_t - y_{\eta(t)}| + |y_t^n - y_{\eta(t)}^n| \leq G n^{-\gamma} \leq G n^{\frac{1}{2} - 2H - \kappa} \tag{9.6}$$

for $G = K(1 + \|B\|_\gamma^{1/\gamma})$ and for any $\gamma < H$, where we use the fact that $H > 2H - \frac{1}{2}$ for the last inequality and we take $\kappa = \gamma - 2H + \frac{1}{2} > 0$. Therefore, going back to (9.4) the convergence of the finite dimensional distributions of $(n^{2H-\frac{1}{2}}(y - y^n), B)$ can be reduced to the convergence of $(n^{2H-\frac{1}{2}} \Phi_{\eta(t)} \hat{\varepsilon}_{\eta(t)}, B_t, t \in [0,T])$. Furthermore, Proposition 9.5 delivers a central limit theorem for general weighted sums of the process F. Taking into account the expression (7.3) for $\hat{\varepsilon}$, it can be applied in order to get the convergence of the finite dimensional distributions of $(n^{2H-\frac{1}{2}} \Phi_{\eta(\cdot)} \hat{\varepsilon}_{\eta(\cdot)}, B)$ to (U, B), where

$$U_t = \sum_{jj'=1}^m \Phi_t \int_0^t \Psi_u \partial V_j V_{j'}(y_u) \, dW^{jj'}, \qquad t \in [0,T] \tag{9.7}$$

as $n \to \infty$. Similar to (6.12), an easy variation of parameter argument shows that U defined by (9.7) solves the linear SDE (9.3). Summarizing our considerations so far, we have obtained the finite dimensional distribution convergence of $(n^{2H-\frac{1}{2}}(y-y^n), B)$ to (U, B).

It remains to show the tightness of the error $n^{2H-\frac{1}{2}}(y-y^n)$. To this end, we apply Lemma 3.31 in Chapter 6 [21] to our decomposition (9.4). It then suffices to show the tightness of $n^{2H-\frac{1}{2}}\Phi_{\eta(\cdot)}\hat{\varepsilon}_{\eta(\cdot)}$, and that the supremum of the other terms of (9.4) in $[0,t]$ converges in probability to zero. The convergence of the first two and the last two terms on the right-hand side of (9.4) follows from relations (9.5) and (9.6), respectively. The tightness of $n^{2H-\frac{1}{2}}\Phi_{\eta(\cdot)}\hat{\varepsilon}_{\eta(\cdot)}$ follows from Corollary 4.9, the fact that Ψ admits moments of all orders thanks to the integrability results in [5], and a tightness criterion in (13.14) of [2]. The proof is now complete. □

We now state the limit theorem on which Theorem 9.4 relies.

PROPOSITION 9.5. *Let f, g be processes defined as in Proposition 4.7 and W be the Brownian motion defined in Proposition 9.1. Set*

$$\Theta_t^n = n^{2H-\frac{1}{2}} \sum_{k=0}^{\lfloor \frac{nt}{T} \rfloor} f_{t_k} \otimes \delta F_{t_k t_{k+1}} \quad \textit{and} \quad \Theta_t = \int_0^t f_s \otimes dW_s.$$

Then the following relation holds true as $n \to \infty$:

$$(\Theta^n, B) \xrightarrow{\text{f.d.d.}} (\Theta, B) \qquad \textit{as } n \to \infty.$$

PROOF. The proposition is an application of Theorem 4.10. As in Corollary 4.9, we take $\gamma > \frac{1}{3}$, $h = n^{2H-\frac{1}{2}}F$ and $\alpha = \frac{1}{2}$. It suffices to verify the conditions (4.13), (4.22) and (4.24). According to Corollary 4.9 and Proposition 9.1, conditions (4.13) and (4.22) hold true for our h. Applying Lemma A.4 and taking $\kappa : \frac{1}{2} + H - \kappa + \gamma > 1$ we obtain the relation (4.24), which concludes our proof. □

APPENDIX

A.1. Estimates for the Hölder seminorm of a rough path. The following lemma is convenient while deriving upper bound estimates for the Hölder seminorm of a rough path.

LEMMA A.1. *Let X and Y be functions on $[0,T]$ and $\mathbb{Z}$ be a two parameter path on $\mathcal{S}_2([0,T])$ such that $\delta\mathbb{Z}_{sut} = \delta X_{su} \otimes \delta Y_{ut}$. We recall the notation*

$(t_0, t_1, \dots, t_n)$ *for a partition of* $[0, T]$ *and* $[\![s, t]\!]$ *for discrete intervals given in the* *Introduction*. *Suppose that*

$$|\delta X_{st}| + |\delta Y_{st}| + |\mathbb{Z}_{st}|^{\frac{1}{2}} \leq \begin{cases} K|t-s|^{\beta}, & (s,t) \in \mathcal{S}_2([\![0,T]\!]), \\ K|t-s|^{\beta}, & s,t \in [t_k, t_{k+1}], k = 0, 1, \dots, n-1 \end{cases}$$

for some $\beta > 0$ *and* $K > 0$. *Then the following relations hold for all* $(s, t) \in \mathcal{S}_2([0, T])$:

$$|\delta X_{st}| + |\delta Y_{st}| \leq K(t-s)^{\beta}, \qquad |\mathbb{Z}_{st}| \leq K(t-s)^{2\beta}.$$

PROOF. We first consider δX and δY. Take $t_{k-1} \leq s \leq t_k \leq t_{k'} \leq t \leq t_{k'+1}$. We have

$$\begin{aligned} |\delta X_{st}| &\leq |\delta X_{t_{k'}t}| + |\delta X_{t_k t_{k'}}| + |\delta X_{st_k}| \\ &\leq K\big((t-t_{k'})^{\beta} + (t_{k'} - t_k)^{\beta} + (t_k - s)^{\beta}\big) \\ &\leq K(t-s)^{\beta}. \end{aligned}$$

The same estimate holds for δY. We now turn to the estimate for $\mathbb{Z}_{st}$. We consider $t_{k-1} \leq s \leq t_k \leq t_{k'} \leq t \leq t_{k'+1}$ again, and we have

$$\begin{aligned} |\mathbb{Z}_{st}| &= |\mathbb{Z}_{st_k} + \mathbb{Z}_{t_k t_{k'}} + \mathbb{Z}_{t_{k'}t} + \delta X_{t_k t_{k'}} \otimes \delta Y_{t_{k'}t} + \delta X_{st_k} \otimes \delta Y_{t_k t}| \\ &\leq K\big((t_k - s)^{2\beta} + (t - t_{k'})^{2\beta} + (t_{k'} - t_k)^{2\beta}\big) \\ &\quad + K^2\big((t_{k'} - t_k)^{\beta}(t - t_{k'})^{\beta} + (t_k - s)^{\beta}(t - t_k)^{\beta}\big) \\ &\leq K(t-s)^{2\beta}. \end{aligned}$$

The proof is complete. $\square$

A.2. Estimates for some iterated integrals. This section summarizes some estimates for weighted sums involving double or triple iterated integrals.

LEMMA A.2. *Let B be our m-dimensional fBm with Hurst parameter $H > \frac{1}{3}$. Let f be a real-valued path on $[0, T]$ such that $\|f\|_{\gamma} \leq K$ for $\frac{1}{3} < \gamma < H$. Suppose that g is another real-valued path and $\tilde{g} = (\tilde{g}^1, \tilde{g}^2, \tilde{g}^3)$ is a continuous path in $\mathcal{L}(\mathbb{R}^m, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}$, such that $S_2(g, \tilde{g}, B)$ is well defined as a γ-rough path, and that δg can be decomposed as*

$$\delta g_{st} = \int_s^t \tilde{g}_u^1 \, dB_u + \int_s^t \tilde{g}_u^2 \, du + \int_s^t \tilde{g}_u^3 \, d\big(u - \eta(u)\big)^{2H}$$

for all $(s, t) \in \mathcal{S}_2([\![0, T]\!])$. Consider an arbitrarily small parameter $\kappa > 0$. Then the following inequalities hold true for $(s, t) \in \mathcal{S}_2([\![0, T]\!])$:

$$\left| \sum_{t_k = s}^{t-} f_{t_k} \int_{t_k}^{t_{k+1}} \delta g_{t_k u} \, d(u - t_k)^{2H} \right| \leq G n^{1-4\gamma+2\kappa} (t-s)^{1-\gamma}, \tag{A.1}$$

$$\left|\sum_{t_k=s}^{t-} f_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} g_v \, d(v-t_k)^{2H} \, dB_u\right| \le G n^{1-4\gamma+2\kappa} (t-s)^{1-\gamma}, \tag{A.2}$$

$$\left|\sum_{t_k=s}^{t-} f_{t_k} \int_{t_k}^{t_{k+1}} \delta g_{t_k u} \int_{t_k}^{u} dB_v \otimes dB_u\right| \le G n^{1-4\gamma+2\kappa} (t-s)^{1-\gamma}. \tag{A.3}$$

PROOF. For the sake of clarity, we will only prove our claims for g whose increments can be written as $\delta g_{st} = \int_s^t \tilde{g}_s \, dB_s$, where $S_2(g, \tilde{g}, B)$ defines a rough path. We also focus on the inequality (A.1). For convenience, we denote by D the following increment:

$$D_{st} = \sum_{t_k=s}^{t-} f_{t_k} \int_{t_k}^{t_{k+1}} \delta g_{t_k u} \, d(u-t_k)^{2H}$$

defined on $\mathcal{S}_2(\llbracket 0, T \rrbracket)$. Since we have assumed that $\delta g_{st} = \int_s^t \tilde{g} \, dB$, we can write

$$D_{st} = \sum_{t_k=s}^{t-} f_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \tilde{g}_v \, dB_v \, d(u-t_k)^{2H}.$$

We now consider the following decomposition of D_{st}:

$$D_{st} = \sum_{t_k=s}^{t} (D_k^1 + D_k^2), \tag{A.4}$$

where D_k^1 and D_k^2 are given by

$$D_k^1 = f_{t_k} \tilde{g}_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} dB_v \, d(u-t_k)^{2H},$$

$$D_k^2 = f_{t_k} \int_{t_k}^{t_{k+1}} \left(\int_{t_k}^{u} \int_{t_k}^{v} d\tilde{g}_r \, dB_v \right) d(u-t_k)^{2H}.$$

Both D_k^1 and D_k^2 are easily bounded. Indeed, one can note that D_k^2 is a Young integral, and since $(\tilde{g}, B)$ admits a lift as a γ-rough path it is easy to show that

$$|D_k^2| \le G n^{-2H-2\gamma}. \tag{A.5}$$

Therefore, summing up both sides of (A.5) from s to t we obtain

$$\left|\sum_{t_k=s}^{t-} D_k^2\right| \le G n^{1-4\gamma} (t-s). \tag{A.6}$$

In order to bound D_k^1, we apply a change of variable formula for Young integrals, which yields

$$D_k^1 = f_{t_k} \tilde{g}_{t_k} \delta B_{t_k t_{k+1}} (t_{k+1}-t_k)^{2H} - f_{t_k} \tilde{g}_{t_k} \int_{t_k}^{t_{k+1}} (u-t_k)^{2H} \, dB_u \equiv D_k^{11} + D_k^{12}.$$

Then $\sum_{t_k=s}^{t-} D_k^{11}$ is bounded by elementary considerations. The terms D_k^{12} are handled by decomposing $f_{t_k}\tilde{g}_{t_k}$ as $\delta(f\tilde{g})_{st_k} + f_s\tilde{g}_s$ and by a direct application of Lemma 4.6. It is thus readily seen that

$$\left|\sum_{t_k=s}^{t-} D_k^1\right| \leq Gn^{1-4\gamma+2\kappa}(t-s)^{1-\gamma}. \tag{A.7}$$

The estimate (A.1) follows by applying (A.6) and (A.7) to (A.4).

Inequalities (A.2) and (A.3) can be shown in a similar way by invoking Lemma 4.6 and Lemma 4.3, respectively. The proof is omitted. □

The following lemma considers almost sure bounds of a triple integral. It can be shown along the same lines as for Lemma A.2. The proof, which hinges on Lemma 4.3, is omitted for sake of conciseness.

LEMMA A.3. *Let f, κ be as in Lemma A.2. Let $h = (h^1, h^2, h^3)$ and $\tilde{h} = (\tilde{h}^1, \tilde{h}^2, \tilde{h}^3)$ be continuous paths such that h^e takes values in $\mathbb{R}$ and $\tilde{h}^e$ takes values in $\mathcal{L}(\mathbb{R}^m, \mathbb{R})$. We also assume that $S_2(h, \tilde{h}, B)$ is a γ-rough path for $H > \gamma > \frac{1}{3}$, and that $\delta h_{st}^e = \int_s^t \tilde{h}^e\, dB$ for $(s,t) \in \mathcal{S}_2([0,T])$ and $e = 1, 2, 3$. Then we have the following estimate for all $(s,t) \in \mathcal{S}_2([\![0,T]\!])$:*

$$\left|\sum_{t_k=s}^{t-} f_{t_k} \int_{t_k}^{t_{k+1}} \int_{t_k}^{u} \int_{t_k}^{v} h_r^1\, dB_r \otimes h_v^2\, dB_v \otimes h_u^3\, dB_u\right| \leq Gn^{1-4\gamma+2\kappa}(t-s)^{1-\gamma}. \tag{A.8}$$

The following results provide some upper-bound estimates for the L_p-norm of a "discrete" rough double integral. Recall that $0 = t_0 < \cdots < t_n = T$ and $0 = u_0 < \cdots < u_\nu = T$ are two uniform partitions on $[0,T]$.

LEMMA A.4. *Let F be defined in (3.14) and $H > \frac{1}{4}$. We set*

$$\hat{\zeta}_r^n := \sum_{l=0}^{\frac{\nu r}{T}-1} \zeta_l^n \qquad \text{with } \zeta_l^n := \zeta_l^{n,ijj'} = n^{2H-\frac{1}{2}} \sum_{t_k\in\tilde{D}_l} \delta B_{t_{k_l}t_k}^i \delta F_{t_kt_{k+1}}^{jj'},$$

where $r \in \{u_1, \dots, u_\nu\}$. Then the following estimate holds true for all $p \geq 1$:

$$\|\delta\hat{\zeta}_{r,r'}^n\|_p \leq K\nu^{-H} n^{\frac{1}{2}-2H}(r'-r)^{\frac{1}{2}}, \qquad r, r' \in \{u_0, \dots, u_\nu\}. \tag{A.9}$$

PROOF. Let us treat the special case $i = j \neq j'$. Then we can expand the variance of ζ_l^n as

$$\mathbb{E}\left(\left|\sum_{l=0}^{\frac{\nu r}{T}-1} \zeta_l^n\right|^2\right) = n^{4H-1} \sum_{l,l'=0}^{\frac{\nu r}{T}-1} \sum_{t_k\in\tilde{D}_l} \sum_{t_{k'}\in\tilde{D}_{l'}} J(k, k', t_{k_l}, t_{k_{l'}}), \tag{A.10}$$

where for all k, k', u and v we have set

$$J(k,k',u,v) = \mathbb{E}\big(\delta B^i_{ut_k}\delta F^{jj'}_{t_kt_{k+1}}\delta B^i_{vt_{k'}}\delta F^{jj'}_{t_{k'}t_{k'+1}}\big). \tag{A.11}$$

In order to evaluate (A.11), we first condition the expected value on $B^{j'}$. We are thus left with the expected value of a product of four centered Gaussian random variables, for which we can use the Gaussian identity, and the isometry property stated in Definition 3.2. We use the same isometry again in order to integrate with respect to $B^{j'}$, which yields

$$J(k,k',u,v) = \sum_{e=1}^{3} J_e(k,k',u,v), \tag{A.12}$$

where we have

$$J_1(k,k',u,v) = \langle \mathbf{1}_{[u,t_k]}, \mathbf{1}_{[v,t_{k'}]}\rangle_{\mathcal{H}} \langle \beta_k, \beta_{k'}\rangle_{\mathcal{H}^{\otimes 2}}, \tag{A.13}$$

$$J_2(k,k',u,v) = \big\langle \langle \mathbf{1}_{[u,t_k]}, \beta_k\rangle_{\mathcal{H}}, \langle \mathbf{1}_{[v,t_{k'}]}, \beta_{k'}\rangle_{\mathcal{H}}\big\rangle_{\mathcal{H}}, \tag{A.14}$$

$$J_3(k,k',u,v) = \big\langle \langle \mathbf{1}_{[v,t_{k'}]}, \beta_k\rangle_{\mathcal{H}}, \langle \mathbf{1}_{[u,t_k]}, \beta_{k'}\rangle_{\mathcal{H}}\big\rangle_{\mathcal{H}}, \tag{A.15}$$

and where the function β is defined by $\beta_k(u,v) = \mathbf{1}_{t_k<u<v<t_{k+1}}$.

Next, observe that we have $\langle \beta_k, \beta_{k'}\rangle_{\mathcal{H}^{\otimes 2}} \geq 0$ for all k and k'. Indeed, when $k = k'$, this stems from the fact that $\langle \beta_k, \beta_{k'}\rangle_{\mathcal{H}^{\otimes 2}}$ can be identified with $\mathbb{E}[|\delta F^{jj'}_{t_kt_{k+1}}|^2]$, while for $k \neq k'$ the expression for $\langle \beta_k, \beta_{k'}\rangle_{\mathcal{H}^{\otimes 2}}$ is given by (3.6), and the product of the measures μ therein gives a positive contribution. So the Cauchy–Schwarz inequality implies that

$$\big|J_1(k,k',u,v)\big| \leq K|t_k-u|^H|t_{k'}-v|^H \langle \beta_k, \beta_{k'}\rangle_{\mathcal{H}^{\otimes 2}}, \tag{A.16}$$

and we easily get the following bound:

$$\begin{aligned}\Bigg|\sum_{l,l'=0}^{\frac{\nu r}{T}-1}\sum_{t_k\in\tilde{D}_l}\sum_{t_{k'}\in\tilde{D}_{l'}} J_1(k,k',t_{k_l},t_{k_{l'}})\Bigg| &\leq K\nu^{-2H}\sum_{k,k'=0}^{\lfloor\frac{nr}{T}\rfloor-1}\langle \beta_k, \beta_{k'}\rangle_{\mathcal{H}^{\otimes 2}} \\ &= K\nu^{-2H}\sum_{k,k'=0}^{\lfloor\frac{nr}{T}\rfloor-1}\mathbb{E}\big[\delta F^{jj'}_{t_kt_{k+1}}\delta F^{jj'}_{t_{k'}t_{k'+1}}\big] \\ &= K\nu^{-2H}\mathbb{E}\big[\big|F^{jj'}_{\eta(r)}\big|^2\big].\end{aligned}$$

It then follows from Lemma 3.4 that

$$\begin{aligned}&\Bigg|\sum_{l,l'=0}^{\frac{\nu r}{T}-1}\sum_{t_k\in\tilde{D}_l}\sum_{t_{k'}\in\tilde{D}_{l'}} J_1(k,k',t_{k_l},t_{k_{l'}})\Bigg| \\ &\qquad \leq K\nu^{-2H}n^{1-4H}\eta(r) \leq K\nu^{-2H}n^{1-4H}r.\end{aligned} \tag{A.17}$$

Let us now handle the terms J_2 and J_3 in (A.12). First, by self-similarity of B we obtain

$$J_2(k,k',u,v) = n^{-6H}T^{6H}\big\langle\big\langle \mathbf{1}_{[\frac{nu}{T},k]}(a),\phi_k(b,a)\big\rangle_{\mathcal{H}},\big\langle \mathbf{1}_{[\frac{nv}{T},k']}(c),\phi_{k'}(b,c)\big\rangle_{\mathcal{H}}\big\rangle_{\mathcal{H}}, \tag{A.18}$$

where we have denoted $\phi_k(u,v)=\mathbf{1}_{k<u<v<k+1}$, and the letters a, b, c designate the pairing for our inner product in $\mathcal{H}$. In order to estimate the quantity (A.18), we assume first that k, k' satisfies $|k-k'|>2$. In this case, we can approximate the functions $\mathbf{1}_{k<u<v<k+1}$ in the definition of ϕ_k by sums of indicators of rectangles. Namely, for $k\le\lfloor\frac{nt}{T}\rfloor$ we set

$$\phi_k^{\ell}(u,v)=\sum_{i=0}^{\ell-1}\mathbf{1}_{[k+\frac{i}{\ell},k+\frac{i+1}{\ell}]}(u)\times\mathbf{1}_{[k+\frac{i+1}{\ell},k+1]}(v), \tag{A.19}$$

then the convergence $\lim_{\ell\to\infty}\|\phi_k-\phi_k^{\ell}\|_{\mathcal{H}^{\otimes 2}}=0$ holds true whenever $H>\frac{1}{4}$. Applying the convergence of ϕ_k^{ℓ} to (A.18) and taking into account expression (A.19), we obtain

$$\begin{aligned}
&J_2(k,k',u,v)\\
&\quad = n^{-6H}T^{6H}\lim_{\ell\to\infty}\big\langle\big\langle \mathbf{1}_{[\frac{nu}{T},k]}(a),\phi_k^{\ell}(b,a)\big\rangle_{\mathcal{H}},\big\langle \mathbf{1}_{[\frac{nv}{T},k']}(c),\phi_{k'}^{\ell}(b,c)\big\rangle_{\mathcal{H}}\big\rangle_{\mathcal{H}}\\
&\quad \le n^{-6H}T^{6H}\lim_{\ell\to\infty}\sum_{i,j=1}^{\ell-1}d_{ij}\tilde{d}_{ij},
\end{aligned}$$

where we denote

$$\begin{aligned}
d_{ij}&=\langle\mathbf{1}_{[k+\frac{i}{\ell},k+\frac{i+1}{\ell}]},\mathbf{1}_{[k'+\frac{j}{\ell},k'+\frac{j+1}{\ell}]}\rangle_{\mathcal{H}},\\
\tilde{d}_{ij}&=\langle\mathbf{1}_{[\frac{nu}{T},k]},\mathbf{1}_{[k+\frac{i+1}{\ell},k+1]}\rangle_{\mathcal{H}}\langle\mathbf{1}_{[\frac{nv}{T},k']},\mathbf{1}_{[k'+\frac{j+1}{\ell},k'+1]}\rangle_{\mathcal{H}}.
\end{aligned}$$

It is easy to see that, for all $i,j\le\ell-1$, we have $|\tilde{d}_{ij}|\le K$. Therefore, taking into account the fact that $d_{ij}<0$ for $k,k':|k-k'|>2$, we obtain

$$\begin{aligned}
J_2(k,k',u,v)&\le\frac{K}{n^{6H}}\lim_{\ell\to\infty}\sum_{i,j=1}^{\ell-1}|d_{ij}|=\frac{K}{n^{6H}}\lim_{\ell\to\infty}\sum_{i,j=1}^{\ell-1}d_{ij}\\
&=\frac{K}{n^{6H}}\langle\mathbf{1}_{[k,k+1]},\mathbf{1}_{[k',k'+1]}\rangle_{\mathcal{H}}\le Kn^{-6H}|k-k'|^{2H-2}.
\end{aligned} \tag{A.20}$$

The estimate (A.20) also holds true for J_3, and the proof is similar. In addition, for $|k-k'|\le 2$ the relation (A.18) shows that

$$\big|J_e(k,k',u,v)\big|\le Kn^{-6H}. \tag{A.21}$$

We now invoke relations (A.20) and (A.21), together with the fact that the summation $\sum_{k>1} |k|^{2H-2}$ is finite and

$$\#\left(\bigcup_{l=0}^{\frac{\nu r}{T}-1} \tilde{D}_l\right) = \#\{k : t_k \le t \text{ and } t_k < r\} \le \frac{nr}{T},$$

which yields for $e = 2, 3$:

$$\left|\sum_{l,l'=0}^{\frac{\nu r}{T}-1} \sum_{t_k \in \tilde{D}_l} \sum_{t_{k'} \in \tilde{D}_{l'}} J_e\big(k, k', t_{k_l}, t_{k_{l'}}\big)\right| \le K n^{1-6H} r. \tag{A.22}$$

Gathering our bounds (A.17) and (A.22) for J_1, J_2 and J_3, it is readily checked from our decompositions (A.10) and (A.12) that for $i = j \neq j'$ we have

$$\mathbb{E}\left(\left|\sum_{l=0}^{\frac{\nu r}{T}-1} \zeta_l^n\right|^2\right) \le K \nu^{-2H} n^{1-4H} r. \tag{A.23}$$

Furthermore, using the stationarity of the increments of F and B, plus the equivalence of L_p-norms in finite chaos, we obtain from (A.23) the desired estimate (A.9). Moreover, the estimate (A.23) holds true for other i, j, j'. The proof is similar and is omitted. $\square$

LEMMA A.5. *Let F be defined in* (3.14). *Then the following estimate holds true for $H > \frac{1}{4}$*:

$$\mathbb{E}\left(\left|\sum_{t_k=s}^{t-} \delta B_{st_k} \otimes F_{t_k t_{k+1}}\right|^2\right) \le K n^{1-4H} (t-s)^{2H+1}, \qquad (s,t) \in \mathcal{S}_2(\llbracket 0, T \rrbracket).$$

PROOF. Since B has stationary increment, it suffices to prove the lemma for $s = 0$. As in the proof of Proposition 4.10, we consider the sum

$$M_t^{ijj'} = \sum_{t_k=0}^{t-} B_{t_k}^i F_{t_k t_{k+1}}^{jj'},$$

for $i = j \neq j'$. The other cases can be considered similarly. Let J_e, $e = 1, 2, 3$ be the quantities defined in (A.13), (A.14), (A.15). By (A.11), we have

$$\mathbb{E}\left(\left|\sum_{t_k=0}^{t-} B_{t_k}^i F_{t_k t_{k+1}}^{jj'}\right|^2\right) = \sum_{t_{k_1}, t_{k_2}=0}^{t-} \sum_{e=1}^{3} J_e(k_1, k_2, 0, 0). \tag{A.24}$$

Applying (A.16) and taking into account Lemma 3.4 yields

$$\sum_{t_{k_1}, t_{k_2}=0}^{t-} J_1(k_1, k_2, 0, 0) \le n^{1-4H} t^{1+2H}. \tag{A.25}$$

On the other hand, it follows from (A.20) and (A.21) that for $e = 2, 3$ we have

$$\sum_{t_{k_1}, t_{k_2}=0}^{t-} J_e(k_1, k_2, 0, 0) \le n^{1-6H} t \le n^{1-4H} t^{1+2H}. \tag{A.26}$$

The lemma then follows by applying (A.25) and (A.26) to (A.24). □

A.3. Inverse of a linear equation. Let us consider two linear rough differential equations:

$$P_t = \mathrm{Id} + \sum_{l=1}^{m} \int_0^t M_s^l P_s \, dB_s^l, \tag{A.27}$$

$$Q_t = \mathrm{Id} - \sum_{l=1}^{m} \int_0^t Q_s M_s^l \, dB_s^l, \tag{A.28}$$

where for $l = 1, \dots, m$, Id is the identity matrix, and $M^l = (M^l(i, j))_{i,j=1,\dots,d}$ is a $\mathbb{R}^{d\times d}$-valued process on $[0, T]$. Note that the solutions $P = (P(i, j))$ and $Q = (Q(i, j))$ are also $\mathbb{R}^{d\times d}$-valued. The following lemma shows that for fixed $t \in [0, T]$ the matrices P_t and Q_t are inverse of each other.

LEMMA A.6. *Suppose that P and Q are respectively the unique solutions of* (A.27) *and* (A.28). *Then $PQ = QP \equiv I$.*

PROOF. Denote $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$. Rewrite (A.27) and (A.28) as

$$P_t(r, j) = \delta_{rj} + \sum_{l=1}^{m} \sum_{v=1}^{d} \int_0^t M_s^l(r, v) P_s(v, j) \, dB_s^l,$$

$$Q_t(i, r) = \delta_{ir} - \sum_{l=1}^{m} \sum_{v=1}^{d} \int_0^t Q_s(i, v) M_s^l(v, r) \, dB_s^l.$$

By the Itô–Stratonovich formula for rough paths integrals, we obtain

$$\begin{aligned} Q_t(i, r) P_t(r, j) &= \delta_{ir}\delta_{rj} + \int_0^t Q_s(i, r) \, dP_s(r, j) + \int_0^t P_s(r, j) \, dQ_s(i, r) \\ &= \delta_{ir}\delta_{rj} + \int_0^t Q_s(i, r) \sum_{l=1}^{m} \sum_{v=1}^{d} M_s^l(r, v) P_s(v, j) \, dB_s^l \\ &\quad - \int_0^t P_s(r, j) \sum_{l=1}^{m} \sum_{v=1}^{d} Q_s(i, v) M_s^l(v, r) \, dB_s^l. \end{aligned} \tag{A.29}$$

Now summing up the two sides of (A.29) in r we obtain

$$\sum_{r=1}^{d} Q_t(i,r)P_t(r,j) = \sum_{r=1}^{d} \delta_{ir}\delta_{rj} = \delta_{ij}.$$

This completes the proof. $\square$

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