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Rough differential equations with power type nonlinearities

Prakash Chakraborty[a,*], Samy Tindel[b]

[a] *Department of Statistics, Purdue University, 150 N. University Street, W. Lafayette, IN 47907, USA*
[b] *Department of Mathematics, Purdue University, 150 N. University Street, W. Lafayette, IN 47907, USA*

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Abstract

In this note we consider differential equations driven by a signal x which is γ-Hölder with $\gamma > \frac{1}{3}$, and is assumed to possess a lift as a rough path. Our main point is to obtain existence of solutions when the coefficients of the equation behave like power functions of the form $|\xi|^\kappa$ with $\kappa \in (0, 1)$. Two different methods are used in order to construct solutions: (i) In a 1-d setting, we resort to a rough version of Lamperti's transform. (ii) For multidimensional situations, we quantify some improved regularity estimates when the solution approaches the origin.

Keywords: Rough path; Rough differential equation

1. Introduction

This article is concerned with the following $\mathbb{R}^m$-valued integral equation:

$$y_t = a + \sum_{j=1}^{d} \int_0^t \sigma^j(y_s) dx_s^j, \quad t \in [0, T] \tag{1}$$

where $x : [0, T] \to \mathbb{R}^d$ is a noisy function in the Hölder space $\mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$ with $\gamma > \frac{1}{3}$, $a \in \mathbb{R}^m$ is the initial value and σ^j are vector fields on $\mathbb{R}^m$. We shall resort to rough path

* Corresponding author.
E-mail addresses: chakra15@purdue.edu (P. Chakraborty), stindel@purdue.edu (S. Tindel).

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techniques in order to make sense of the noisy integral in Eq. (1), and we refer to [3–5,8] for further details on the rough path theory. Our main goal is to understand how to define solutions to (1) when the coefficients σ^j behave like power functions.

Indeed, the rough path theory allows to consider very general noisy signals x as drivers of Eq. (1), but it requires heavy regularity assumptions on the coefficients σ^j in order to get existence and uniqueness of solutions. More specifically, given the regularity of the coefficient σ, a minimal sufficient regularity of the driving signal that guarantees existence and uniqueness of the solution is provided in [4]. However, for differential equations driven by Brownian motion (which means in particular that $x \in \mathcal{C}^{\frac{1}{2}-}$) the condition amounts to the coefficient being twice differentiable. This is obviously far from being optimal with respect to the classical stochastic calculus approach for Brownian motion.

One of the current challenges in rough path analysis is thus to improve the regularity conditions on the coefficients of (1), and still get solutions to the differential system at stake. Among the irregular coefficients which can be thought of, power type functions of the form $\sigma^j(\xi) = |\xi|^\kappa$ with $\kappa \in (0, 1)$ play a special role. On the one hand these coefficients are related to classical population dynamics models (see e.g [2] for a review), which make them interesting in their own right. On the other hand, the fact that these coefficients vanish at the origin grant them some special properties which can be exploited in order to construct Hölder-continuous solutions. Roughly speaking, Eq. (1) behaves like a noiseless equation when y approaches 0, and one expects existence of a γ-Hölder solution whenever $\gamma + \kappa > 1$. This heuristic argument is explained at length in the introduction of [6], and the current contribution can be seen as the first implementation of such an idea in a genuinely rough context.

Let us now recall some of the results obtained for equations driven by a Brownian motion B. For power type coefficients, most of the results concern one dimensional cases of the form:

$$y_t = a + \int_0^t \sigma(y_s) dB_s, \quad t \in [0, T]. \tag{2}$$

The classical result [11, Theorem 1] involves stochastic integrals in the Itô sense, and gives existence and uniqueness for $\sigma(\xi) = |\xi|^\kappa$ with $\kappa \geq \frac{1}{2}$. However, the rough path setting is more related to Stratonovich type integrals in the Brownian case. We thus refer the interested reader to the comprehensive study performed in [1], which studies singular stochastic differential equations and classifies them according to the nature of their solution. Comparing Eq. (2) interpreted in the Stratonovich sense with the systems analyzed in [1], their results can be read as follows: if $\sigma(\xi) = |\xi|^\kappa$ with $\kappa \geq \frac{1}{2}$ and the solution of (2) starts at a non-negative location, then it reaches zero almost surely. In addition, among solutions with vanishing local time at 0, there is a non-negative solution which is unique in law. However, in general we do not have uniqueness. The results we will obtain for a general rough path are not as sharp, but are at least compatible with the Brownian case. Let us also mention the works [9,10], where the authors study existence and uniqueness of solutions in the context of stochastic heat equations with space time white noise and power type coefficients.

As far as power type equations driven by general noisy signals x are concerned, we are only aware of the article [6] exploring Eq. (1) in the Young case $\gamma > 1/2$. The current contribution has thus to be seen as a generalization of [6], allowing to cope with γ-Hölder signals x with $\gamma \in (1/3, 1/2]$. Notice that we have restricted our analysis to $\gamma > 1/3$ in order to keep our computations to a reasonable size. However, we believe that our techniques can be adopted to obtain similar results when $\gamma < 1/3$, at the price of higher order rough path type expansions. As we will see, it turns out that when $\kappa + \gamma > 1$ Eq. (1) is well defined and yields a solution. More specifically, we shall obtain the following theorem in the 1-dimensional case (see Theorem 3.9 for a more precise and general formulation).

Theorem 1.1. *Consider a 1-dimensional signal* $x \in \mathcal{C}^\gamma$*, with* $\gamma \in (1/3, 1/2]$*. Let* σ *be the power function given by* $\sigma(\xi) = |\xi|^\kappa$ *and* ϕ *be the function defined by* $\phi(\xi) = \int_0^\xi \frac{ds}{\sigma(s)}$*. Assume* $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right]$ *and* $\kappa + \gamma > 1$*. Then the function* $y = \phi^{-1}(x + \phi(a))$ *is a solution of the equation*

$$y_t = a + \int_0^t \sigma(y_s) dx_s, \quad t \geq 0.$$

In the multidimensional case under a slightly increased regularity assumption on x, namely $x \in \mathcal{C}^{\gamma+}([0, T])$ as well as a roughness assumption (see Hypothesis 4.10 for precise statement), the following theorem holds under a few power type hypotheses on σ and its derivatives.

Theorem 1.2. *Consider a* d*-dimensional signal* $x \in \mathcal{C}^{\gamma+}$ *with* $\gamma \in (1/3, 1/2]$*, giving raise to a rough path. Assume* $\kappa + \gamma > 1$*, and that* $\sigma(\xi)$ *behaves like a power coefficient* $|\xi|^\kappa$ *near the origin. Then there exist a continuous function* y *defined on* $[0, T]$ *and an instant* $\tau \leq T$*, such that one of the following two possibilities holds:*

(A) $\tau = T$*:* y *is non-zero on* $[0, T]$*,* $y \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^m)$ *and* y *solves Eq.* (22) *on* $[0, T]$*.*
(B) $\tau < T$*: the path* y *sits in* $\mathcal{C}^\gamma([0, T]; \mathbb{R}^m)$ *and* y *solves Eq.* (22) *on* $[0, T]$*. Furthermore,* $y_s \neq 0$ *on* $[0, \tau)$*,* $\lim_{t\to\tau} y_t = 0$ *and* $y_t = 0$ *on the interval* $[\tau, T]$*.*

As mentioned above, Theorems 1.1 and 1.2 are the first existence results for power type coefficients in a truly rough context. As in [6], their proofs mainly hinge on a quantification of the regularity gain of the solution y when it approaches the origin. We should mention however that this quantification requires a significant amount of effort in the rough case. Indeed we resort to some discrete type expansions, whose analysis is based on precise estimates inspired by the numerical analysis of rough differential equations (see e.g. [7]).

Having stated the key results, we now describe the outline of this article. In Section 2, a short account of the necessary notions of rough path theory is provided. Section 3.1 deals with a few hypotheses we assume on the coefficient σ, all of which are satisfied by the power type coefficient $|\xi|^\kappa$. Section 3.2 proves the existence of a solution in the one-dimensional case. In Section 4 we proceed by considering a few stopping times and quantify the regularity gain mentioned above of the solution when it hits 0. We achieve this through discretization techniques as employed in Proposition 4.5. Finally we show Hölder continuity of our solution.

Notations. The following notations are used in this article:

1. For an arbitrary real $T > 0$, let $\mathcal{S}_k([0, T])$ be the kth order simplex defined by $\mathcal{S}_k([0, T]) = \{(s_1, \ldots, s_k) : 0 \leq s_1 \leq \cdots \leq s_k \leq T\}$.
2. For quantities a and b, let $a \lesssim b$ denote the existence of a constant c such that $a \leq cb$.
3. For an element z in the functional space $\mathcal{R}$, let $\mathcal{N}[z; \mathcal{R}]$ denote the corresponding norm of z in $\mathcal{R}$.

2. Rough path notions

The following is a short account of the rough path notions used in this article, mostly taken from [5]. We review the notion of controlled process as well as their integrals with respect to a rough path. We shall also give a version of an Itô-Stratonovich change of variable formula under reduced regularity condition.

2.1. Increments

For a vector space V and an integer $k \geq 1$, let $\mathcal{C}_k(V)$ be the set of functions $g : \mathcal{S}_k([0,T]) \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-increment, and we set $\mathcal{C}_*(V) = \cup_{k \geq 1} \mathcal{C}_k(V)$. Then the operator $\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V)$ is defined as follows

$$\delta g_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}} \tag{3}$$

where $\hat{t}_i$ means that this particular argument is omitted. It is easily verified that $\delta\delta = 0$ when considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$.

The sizes of these k-increments are measured by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ and $\mu > 0$ let

$$\|f\|_\mu = \sup_{(s,t) \in \mathcal{S}_2([0,T])} \frac{\|f_{st}\|}{|t-s|^\mu} \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V);\ \|f\|_\mu < \infty\}. \tag{4}$$

The usual Hölder space $\mathcal{C}_1^\mu(V)$ will be determined in the following way: for a continuous function $g \in \mathcal{C}_1(V)$, we simply set

$$\|g\|_\mu = \|\delta g\|_\mu$$

and we will say that $g \in \mathcal{C}_1^\mu(V)$ iff $\|g\|_\mu$ is finite.

Remark 2.1. Notice that $\|\cdot\|_\mu$ is only a semi-norm on $\mathcal{C}_1(V)$, but we will generally work on spaces for which the initial value of the function is fixed.

We shall also need to measure the regularity of increments in $\mathcal{C}_3(V)$. To this aim, similarly to (4), we introduce the following norm for $h \in \mathcal{C}_3(V)$:

$$\|h\|_\mu = \sup_{(s,u,t) \in \mathcal{S}_3([0,T])} \frac{|h_{sut}|}{|t-s|^\mu}. \tag{5}$$

Then the μ-Hölder continuous increments in $\mathcal{C}_3(V)$ are defined as:

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V);\ \|h\|_\mu < \infty\}.$$

Notice that the ratio in (5) could have been written as $\frac{|h_{sut}|}{|t-u|^{\mu_1}|u-s|^{\mu_2}}$ with $\mu_1 + \mu_2 = \mu$, in order to stress the dependence on u of our increment h. However, expression (5) is simpler and captures the regularities we need, since we are working on the simplex $\mathcal{S}_3$.

The building block of the rough path theory is the so-called sewing map lemma. We recall this fundamental result here for further use.

Proposition 2.2. *Let $h \in \mathcal{C}_3^\mu(V)$ for $\mu > 1$ be such that $\delta h = 0$. Then there exists a unique $g = \Lambda(h) \in \mathcal{C}_2^\mu(V)$ such that $\delta g = h$. Furthermore for such an h, the following relations hold true:*

$$\delta\Lambda(h) = h \ \textit{ and } \ \|\Lambda h\|_\mu \leq \frac{1}{2^\mu - 2}\|h\|_\mu.$$

2.2. Elementary computations in $\mathcal{C}_2$ and $\mathcal{C}_3$

Consider $V = \mathbb{R}$, and let $\mathcal{C}_k^\gamma$ for $\mathcal{C}_k^\gamma(\mathbb{R})$. Then $(\mathcal{C}_*, \delta)$ can be endowed with the following product: for $g \in \mathcal{C}_n$ and $h \in \mathcal{C}_m$ we let gh be the element of $\mathcal{C}_{m+n-1}$ defined by

$$(gh)_{t_1,\dots,t_{m+n-1}} = g_{t_1,\dots,t_n} h_{t_n,\dots,t_{m+n-1}}, \quad (t_1, \dots, t_{m+n-1}) \in \mathcal{S}_{m+n-1}([0,T]).$$

We now label a rule for discrete differentiation of products for further use throughout the article. Its proof is an elementary application of the definition (3), and is omitted for sake of conciseness.

Proposition 2.3. *The following rule holds true: Let $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then $gh \in \mathcal{C}_2$ and*

$$\delta(gh) = \delta g\, h - g\, \delta h.$$

The iterated integrals of smooth functions on $[0, T]$ are particular cases of elements of $\mathcal{C}_2$, which will be of interest. Specifically, for smooth real-valued functions f and g, let us denote $\int f dg$ by $\mathcal{I}(f dg)$ and consider it as an element of $\mathcal{C}_2$: for $(s, t) \in \mathcal{S}_2\,([0, T])$ we set

$$\mathcal{I}_{st}(f dg) = \left(\int f dg \right)_{st} = \int_s^t f_u dg_u.$$

2.3. Weakly controlled processes

One of our basic assumptions on the driving process x of Eq. (1) is that it gives raise to a geometric rough path. This assumption can be summarized as follows.

Hypothesis 2.4. The path $x : [0, T] \to \mathbb{R}^d$ belongs to the Hölder space $\mathcal{C}^\gamma([0, T]; \mathbb{R}^d)$ with $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $x_0 = 0$. In addition x admits a Lévy area above itself, that is, there exists a two index map $\mathbf{x}^2 : \mathcal{S}_2\,([0, T]) \to \mathbb{R}^{d,d}$ which belongs to $\mathcal{C}_2^{2\gamma}(\mathbb{R}^{d,d})$ and such that

$$\delta \mathbf{x}^{2;ij}_{sut} = \delta x^i_{su} \otimes \delta x^j_{ut}, \quad \text{and} \quad \mathbf{x}^{2;ij}_{st} + \mathbf{x}^{2;ji}_{st} = \delta x^i_{st} \otimes \delta x^j_{st}.$$

The γ-Hölder norm of x is denoted by:

$$\|\mathbf{x}\|_\gamma = \mathcal{N}(x; \mathcal{C}_1^\gamma([0, T], \mathbb{R}^d)) + \mathcal{N}(\mathbf{x}^2; \mathcal{C}_2^{2\gamma}([0, T], \mathbb{R}^{d,d})).$$

Preparing the ground for the upcoming change of variable formula in Proposition 2.9, we now define the notion weakly controlled process as a slight variation of the usual one.

Definition 2.5. Let z be a process in $\mathcal{C}_1^\gamma(\mathbb{R}^n)$ with $1/3 < \gamma \le 1/2$ and consider $\eta > \gamma$. We say that z is weakly controlled by x with a remainder of order η if $\delta z \in \mathcal{C}_2^\gamma(\mathbb{R}^n)$ can be decomposed into

$$\delta z^i = \zeta^{i i_1} \delta x^{i_1} + r^i, \quad \text{i.e.} \quad \delta z^i_{st} = \zeta^{i i_1}_s \delta x^{i_1}_{st} + r^i_{st}$$

for all $(s, t) \in \mathcal{S}_2\,([0, T])$. In the previous formula we assume $\zeta \in \mathcal{C}_1^{\eta-\gamma}(\mathbb{R}^{n,d})$ and r is a more regular remainder such that $r \in \mathcal{C}_2^\eta(\mathbb{R}^n)$. The space of weakly controlled paths will be denoted by $\mathcal{Q}_{\gamma,\eta}(\mathbb{R}^n)$ and a process $z \in \mathcal{Q}_{\gamma,\eta}(\mathbb{R}^n)$ can be considered as a couple (z, ζ). The natural semi-norm on $\mathcal{Q}_{\gamma,\eta}(\mathbb{R}^n)$ is given by

$$\begin{aligned}\mathcal{N}[z; \mathcal{Q}_{\gamma,\eta}(\mathbb{R}^n)] = \mathcal{N}[z; \mathcal{C}_1^\gamma(\mathbb{R}^n)] + \mathcal{N}[\zeta; \mathcal{C}_1^\infty(\mathbb{R}^{n,d})] \\ + \mathcal{N}[\zeta; \mathcal{C}_1^{\eta-\gamma}(\mathbb{R}^{n,d})] + \mathcal{N}[r; \mathcal{C}_2^\eta(\mathbb{R}^n)].\end{aligned}$$

Let $\mathrm{Lip}^{n+\lambda}$ denote the space of n-times differential functions with λ-Hölder nth derivative, endowed with the norm:

$$\|f\|_{n,\lambda} = \|f\|_\infty + \sum_{k=1}^n \|\partial^k f\|_\infty + \|\partial^n f\|_\lambda.$$

The following gives a composition rule which asserts that our rough path x composed with a $\text{Lip}^{1+\lambda}$ function is weakly controlled.

Proposition 2.6. *Let $f : \mathbb{R}^d \to \mathbb{R}^n$ be a $\text{Lip}^{1+\lambda}$ function and set $z = f(x)$. Then $z \in \mathcal{Q}_{\gamma,\sigma}(\mathbb{R}^n)$ with $\sigma = \gamma(\lambda + 1)$, where $\mathcal{Q}_{\gamma,\sigma}(\mathbb{R}^n)$ is introduced in Definition 2.5, and it can be decomposed into $\delta z = \zeta \delta x + r$, with*

$$\zeta^{ii_1} = \partial_{i_1} f_i(x) \quad \text{and} \quad r^i = \delta f_i(x) - \partial_{i_1} f_i(x) \delta x^{i_1}_{st}.$$

Furthermore, the norm of z as a controlled process can be bounded as follows:

$$\mathcal{N}[z; \mathcal{Q}_{\gamma,\sigma}] \le K \|f\|_{1,\lambda} (1 + \mathcal{N}^{1+\lambda}[x; \mathcal{C}_1^{\gamma}(\mathbb{R}^d)]),$$

where K is a positive constant.

Proof. The algebraic part of the assertion is straightforward. Just write

$$\delta z_{st} = f(x_t) - f(x_s) = \partial_{i_1} f(x_s) \delta x^{i_1}_{st} + r_{st}.$$

The estimate of $\mathcal{N}[z; \mathcal{Q}_{\gamma,\sigma}]$ is obtained from the estimates of $\mathcal{N}[z; \mathcal{C}_1^{\gamma}(\mathbb{R}^n)]$, $\mathcal{N}[\zeta; \mathcal{C}_1^{\infty}(\mathbb{R}^{n,d})]$, $\mathcal{N}[\zeta; \mathcal{C}_1^{\sigma-\gamma}(\mathbb{R}^{n,d})]$ and $\mathcal{N}[r; \mathcal{C}_2^{\sigma}(\mathbb{R}^n)]$. The details are similar to [5, Appendix] and left to the patient reader. □

More generally, we also need to specify the composition of a controlled process with a $\text{Lip}^{1+\lambda}$ function. The proof of this proposition is similar to Proposition 2.6 and omitted for sake of conciseness.

Proposition 2.7. *Let $z \in \mathcal{Q}_{\gamma,\sigma}(\mathbb{R}^n)$ with decomposition $\delta z = \tilde{\zeta} \delta x + \tilde{r}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be a $\text{Lip}^{1+\lambda}$ function. Set $w = g(x)$. Then $w \in \mathcal{Q}_{\gamma,\sigma}(\mathbb{R}^m)$ with $\sigma = \gamma(\lambda+1)$ and it can be decomposed into $\delta w = \zeta \delta x + r$, with*

$$\zeta^{ii_1} = \partial_{i_2} f_i(x) \tilde{\zeta}^{i_2,i_1}.$$

The class of weakly controlled paths provides a natural and basic set of functions which can be integrated with respect to a rough path. The basic proposition in this direction, whose proof can be found in [5], is summarized below.

Theorem 2.8. *For $1/3 < \gamma \le 1/2$, let x be a process satisfying Hypothesis 2.4. Furthermore let $m \in \mathcal{Q}_{\gamma,\eta}(\mathbb{R}^d)$ with $\eta + \gamma > 1$, whose decomposition is given by $m_0 = b \in \mathbb{R}^d$ and*

$$\delta m^i = \mu^{ii_1} \delta x^{i_1} + r^i \quad \text{where} \quad \mu \in \mathcal{C}_1^{\eta-\gamma}(\mathbb{R}^{d,d}), r \in \mathcal{C}_2^{\eta}(\mathbb{R}^n).$$

Define z by $z_0 = a \in \mathbb{R}^d$ and

$$\delta z = m^i \delta x^i + \mu^{ii_1} \mathbf{x}^{2;i_1 i} - \Lambda(r^i \delta x^i + \delta \mu^{ii_1} \mathbf{x}^{2;i_1 i}).$$

Finally, set

$$\mathcal{I}_{st}(m dx) = \int_s^t \langle m_u, dx_u \rangle_{\mathbb{R}^d} := \delta z_{st}.$$

Then this integral extends Young integration and coincides with the Riemann–Stieltjes integral of m with respect to x whenever these two functions are smooth. Furthermore, $\mathcal{I}_{st}(m dx)$ is the limit of modified Riemann sums:

$$\mathcal{I}_{st}(m dx) = \lim_{|\Pi_{st}| \to 0} \sum_{q=0}^{n-1} [m^i_{t_q} \delta x^i_{t_q t_{q+1}} + \mu^{ii_1}_{t_q} \mathbf{x}^{2;i_1 i}_{t_q t_{q+1}}],$$

for any $0 \le s < t \le T$*, where the limit is taken over all partitions* $\Pi_{st} = \{s = t_0, \dots, t_n = t\}$ *of* $[s,t]$*, as the mesh of the partition goes to zero.*

2.4. Itô-Stratonovich formula

We now state a change of variable formula for a function $g(x)$ of a rough path, under minimal assumptions on the regularity of g. To the best of our knowledge, this proposition cannot be found in literature, and therefore a short and elementary proof is included. The techniques of this proof will prove to be useful for the study of our system (1) in the one-dimensional case.

Proposition 2.9. *Let x satisfy Hypothesis* 2.4*. Let g be a* Lip$^{2+\lambda}$ *function such that* $(\lambda+2)\gamma > 1$*. Then*

$$[\delta(g(x))]_{st} = \mathcal{I}_{st}(\nabla g(x)dx) = \int_s^t \langle \nabla g(x_u), dx_u\rangle_{\mathbb{R}^d}, \tag{6}$$

where the integral above has to be understood in the sense of Theorem 2.8.

Proof. Consider a partition $\Pi_{st} = \{s = t_0 < \cdots t_n = t\}$ of $[s,t]$. The following identity holds trivially:

$$\begin{aligned} g(x_t) - g(x_s) &= \sum_{q=0}^{n-1}\left[g(x_{t_{q+1}}) - g(x_{t_q})\right] \\ &= \sum_{q=0}^{n-1}\left[\sum_i \partial_i g(x_{t_q})\delta x^i_{t_q t_{q+1}} + \frac{1}{2}\sum_{i_1,i_2}\partial^2_{i_1 i_2} g(x_{t_q})\delta x^{i_1}_{t_q t_{q+1}}\delta x^{i_2}_{t_q t_{q+1}} + r_{t_q t_{q+1}}\right] \end{aligned} \tag{7}$$

where

$$r_{t_q t_{q+1}} = g(t_{q+1}) - g(t_q) - \sum_i \partial_i g(x_{t_q})\delta x^i_{t_q t_{q+1}} - \frac{1}{2}\sum_{i_1,i_2}\partial^2_{i_1 i_2} g(x_{t_q})\delta x^{i_1}_{t_q t_{q+1}}\delta x^{i_2}_{t_q t_{q+1}}.$$

Furthermore, an elementary Taylor type argument shows that for all i_1, i_2 there exists an element $\xi^q_{i_1 i_2}$ of $[x_{t_q}, x_{t_{q+1}}]$ such that

$$\begin{aligned} r_{t_q t_{q+1}} &= \frac{1}{2}\sum_{i_1,i_2}\partial^2_{i_1 i_2} g(\xi^q_{i_1 i_2})\delta x^{i_1}_{t_q t_{q+1}}\delta x^{i_2}_{t_q t_{q+1}} - \frac{1}{2}\sum_{i_1,i_2}\partial^2_{i_1 i_2} g(x_{t_q})\delta x^{i_1}_{t_q t_{q+1}}\delta x^{i_2}_{t_q t_{q+1}} \\ &= \frac{1}{2}\sum_{i_1,i_2}\left(\partial^2_{i_1 i_2} g(\xi^q_{i_1 i_2}) - \partial^2_{i_1 i_2} g(x_{t_q})\right)\delta x^{i_1}_{t_q t_{q+1}}\delta x^{i_2}_{t_q t_{q+1}}. \end{aligned}$$

We now invoke the fact that $g \in \text{Lip}^{2+\lambda}$ in order to get

$$\left|r_{t_q t_{q+1}}\right| \le C|t_q - t_{q+1}|^{(2+\lambda)\gamma},$$

where C is a constant depending on g and x. Thus, since $(\lambda+2)\gamma > 1$, it is easily seen that

$$\lim_{|\Pi_{st}|\to 0}\sum_{q=0}^{n-1} r_{t_q t_{q+1}} = 0. \tag{8}$$

In addition, using Hypothesis 2.4 and continuity of the partial derivatives, we can write

$$\frac{1}{2}\sum_{i_1,i_2}\partial^2_{i_1 i_2} g(x_{t_q})\delta x^{i_1}_{t_q t_{q+1}}\delta x^{i_2}_{t_q t_{q+1}} = \sum_{i_1,i_2}\partial^2_{i_1 i_2} g(x_{t_q})\mathbf{x}^{\mathbf{2};i_1 i_2}_{t_q t_{q+1}}. \tag{9}$$

Plugging (8) and (9) into (7) we get

$$g(x_t) - g(x_s) = \lim_{|\Pi_{st}|\to 0} \sum_{q=0}^{n-1} \partial_i g(x_{t_q})\delta x^i_{t_q t_{q+1}} + \sum_{q=0}^{n-1} \partial^2_{i_1 i_2} g(x_{t_q}) \mathbf{x}^{\mathbf{2};i_1 i_2}_{t_q t_{q+1}}, \tag{10}$$

for all $(s,t) \in \mathcal{S}_2([0,T])$.
On the other hand looking at the decomposition of $\nabla g(x)$ as a weakly controlled process and using Proposition 2.6 we obtain:

$$\delta\,[\nabla g(x)]^i_{st} = \delta \partial_i g(x)_{st} = \partial^2_{i_1 i} g(x_s)\delta x^{i_1}_{st} + R^i_{st},$$

where R lies in $\mathcal{C}_2^{(1+\lambda)\gamma}$. Then using the Riemann sum representation Theorem 2.8 of rough integrals, we have

$$\mathcal{I}_{st}(\nabla g(x)dx) = \lim_{|\Pi_{st}|\to 0} \left[\sum_{q=0}^{n-1} \partial_i g(x_{t_q})\delta x^i_{t_q t_{q+1}} + \sum_{q=0}^{n-1} \partial^2_{i_1 i_2} g(x_{t_q}) \mathbf{x}^{\mathbf{2};i_1 i_2}_{t_q t_{q+1}} \right].$$

Comparing the above formula with (10) proves the result. □

3. Differential equations: setting and one-dimensional case

In this section we will give the general formulation and assumptions for Eq. (1). Then we state an existence result in dimension 1, which follows quickly from our preliminary considerations in Section 2.

3.1. Setting

Recall that we are considering the following rough differential equation:

$$y_t = a + \sum_{j=1}^{d} \int_0^t \sigma^j(y_s)dx^j_s, \tag{11}$$

where x satisfies Hypothesis 2.4 and $\sigma^1, \ldots, \sigma^d$ are vector fields on $\mathbb{R}^m$. In this section we will specify some general assumptions on the coefficient σ, which will prevail for the remainder of the article.
Let us start with a regularity assumption on σ:

Hypothesis 3.1. Let $\kappa > 0$ be a constant such that $\gamma + \kappa > 1$, where γ is introduced in Hypothesis 2.4. We assume that $\sigma(0) = 0$, and that the following two conditions are valid:

(i) For all $\xi_1, \xi_2 \in \mathbb{R}^m$ we have the following:

$$|\sigma(\xi_1) - \sigma(\xi_2)| \lesssim |\xi_1 - \xi_2|^\kappa, \tag{12}$$

(ii) Consider the function $\Psi = D\sigma \cdot \sigma$ defined on $\mathbb{R}^m$. For all $\xi_1, \xi_2 \in \mathbb{R}^m$ such that $\frac{1}{r} \le \frac{|\xi_1|}{|\xi_2|} \le r$ for a fixed $r > 1$, there exists a constant $\mathcal{N}_\Psi$ (depending on r, m and κ) satisfying:

$$|\Psi(\xi_1) - \Psi(\xi_2)| \le \mathcal{N}_\Psi \left| \frac{1}{|\xi_1|^{2(1-\kappa)}} + \frac{1}{|\xi_2|^{2(1-\kappa)}} \right| |\xi_1 - \xi_2|\,. \tag{13}$$

In addition to above, we assume that outside of a neighborhood of 0, σ behaves like a Lip^p_{loc} function with $p > \frac{1}{\gamma}$. In other words, σ is bounded with bounded two derivatives and the second derivative is locally Hölder continuous with order larger than $(\frac{1}{\gamma} - 2)$.

We also need a more specific assumption in dimension 1:

Hypothesis 3.2. Whenever $m = d = 1$, assume σ is positive on $\mathbb{R}_+$ and that ϕ defined by $\phi(\xi) = \int_0^\xi \frac{ds}{\sigma(s)}$ exists. Also consider $\kappa > 0$ as in Hypothesis 3.1. Then we assume for all $\xi_1, \xi_2 \in \mathbb{R}$ we have

$$|F(\xi_1) - F(\xi_2)| \lesssim |\xi_1 - \xi_2|^\lambda,$$

where F stands for the function $(D\sigma \cdot \sigma) \circ \phi^{-1}$ and $\lambda = \frac{2\kappa-1}{1-\kappa} \wedge 1$.

We now give a typical example of a coefficient σ satisfying our standing assumptions.

Proposition 3.3. *Let $\chi : \mathbb{R} \to \mathbb{R}^+$ be a smooth cutoff function such that $\chi(z) = 1$ if $|z| \le \frac{M}{2}$ and $\chi(z) = 0$ if $|z| \ge M$, for a given $M > 0$. Assume that $\sigma = (\sigma^1, \dots, \sigma^m)$ where each $\sigma^i : \mathbb{R}^m \to \mathbb{R}$ is defined by the κth power of the Euclidean norm: $\sigma^i(\xi) = (\sum_j (\xi^j)^2)^{\kappa/2} \chi(|\xi|)$. Then inequality (13) holds true for all $\xi_1, \xi_2 \in \mathbb{R}^m$ such that $\frac{1}{r} \le \frac{|\xi_1|}{|\xi_2|} \le r$.*

Proof. We only handle inequality (13) when ξ_1, ξ_2 are close to 0, which is the relevant case in our situation. We can thus assume that each σ^i is of the form $\sigma^i(\xi) = |\xi|^\kappa$ in the sequel. For notational sake we will set $\tilde{\sigma}(\xi) = |\xi|^\kappa$ in the remainder of the proof.

Observe that $\Psi : \mathbb{R}^m \to \mathbb{R}^m$ defined by $\Psi(\xi) = (D\sigma \cdot \sigma)(\xi)$ satisfies $\Psi^i(\xi) = \sum_k \sigma^k(\xi) \partial_k \sigma^i(\xi)$. Consequently,

$$\nabla \Psi^{ij}(\xi) = \partial_j \Psi^i(\xi) = \partial_j \left[\sum_{k=1}^m \partial_k \sigma^i(\xi) \sigma^k(\xi) \right]$$
$$= \sum_{k=1}^m \left[\left(\partial_j \partial_k \sigma^i(\xi)\right) \sigma^k(\xi) + \left(\partial_k \sigma^i(\xi)\right) \left(\partial_j \sigma^k(\xi)\right) \right]. \tag{14}$$

The partial derivatives above, when evaluated for $\sigma^k(\xi) = \tilde{\sigma}(\xi) = |\xi|^\kappa = (\sum {\xi^i}^2)^{\kappa/2}$, turn out to be as follows:

$$\partial_k \tilde{\sigma}(\xi) = \kappa |\xi|^{\kappa-2} \xi_k \quad \text{and} \quad \partial_j \partial_k \tilde{\sigma}(\xi) = \kappa(\kappa-2)|\xi|^{\kappa-4} \xi_j \xi_k + \kappa |\xi|^{\kappa-2} \mathbf{1}_{(j=k)}.$$

Plugging these partial derivatives in the formula obtained in (14), we get

$$\nabla \Psi^{ij}(\xi) = 2\kappa(\kappa-1)|\xi|^{2\kappa-4} \xi^j (\xi \cdot \mathbf{1}) + \kappa |\xi|^{2(\kappa-1)}, \tag{15}$$

where $\xi \cdot \mathbf{1}$ denotes the inner product of ξ and the vector $\mathbf{1} \in \mathbb{R}^m$. Now we use the multivariate mean value theorem in integral form given by:

$$\Psi(\xi_1) - \Psi(\xi_2) = \int_0^1 \nabla \Psi(\xi_t) \cdot (\xi_1 - \xi_2) \, dt,$$

where we have set $\xi_t = (1-t)\xi_2 + t\xi_1$ for $t \in [0, 1]$. From (15) we thus obtain

$$\Psi^i(\xi_1) - \Psi^i(\xi_2) = \sum_{j=1}^m \int_0^1 \left(2\kappa(\kappa-1)|\xi_t|^{2\kappa-4} \xi_t^j (\xi_t \cdot \mathbf{1}) + \kappa |\xi_t|^{2(\kappa-1)} \right) \left(\xi_1^j - \xi_2^j \right) dt.$$

Assume wlog that $|\xi_1| \le |\xi_2|$, which implies by our assumption on ξ_1, ξ_2 that $1 \le \frac{|\xi_2|}{|\xi_1|} \le r$. Now observe

$$|\xi_1|^{2(1-\kappa)} \left| \Psi^i(\xi_1) - \Psi^i(\xi_2) \right|$$

$$= |\xi_1|^{2(1-\kappa)} \left| \sum_{j=1}^{m} \int_0^1 \left(2\kappa(\kappa-1)|\xi_t|^{2\kappa-4} \xi_t^j \, (\xi_t \cdot \mathbf{1}) + \kappa |\xi_t|^{2(\kappa-1)} \right) \left(\xi_1^j - \xi_2^j \right) dt \right|$$

$$\leq \sum_{j=1}^{m} \int_0^1 \left(2\kappa(\kappa-1) \left| \frac{\xi_t}{|\xi_1|} \right|^{2\kappa-4} \left| \frac{\xi_t^j}{|\xi_1|} \right| \left| \frac{(\xi_t \cdot \mathbf{1})}{|\xi_1|} \right| + \kappa \left| \frac{\xi_t}{|\xi_1|} \right|^{2(\kappa-1)} \right) \left| \xi_1^j - \xi_2^j \right| dt. \tag{16}$$

Since $\frac{\xi_t}{|\xi_1|} = (1-t)\frac{\xi_2}{|\xi_1|} + t\frac{\xi_1}{|\xi_1|}$ and $1 \leq \frac{|\xi_2|}{|\xi_1|} \leq r$ we must have $1 \leq |\frac{\xi_t}{|\xi_1|}| \leq r$. Using this information in (16) we get

$$|\xi_1|^{2(1-\kappa)} \left| \Psi^i(\xi_1) - \Psi^i(\xi_2) \right| \lesssim \sum_{j=1}^{m} |\xi_1 - \xi_2| \lesssim |\xi_1 - \xi_2|.$$

This yields (13). □

Remark 3.4. A sufficient condition for σ to satisfy Hypothesis 3.1 is the boundedness of $|\xi_1|^{2(1-\kappa)} |\nabla \Psi(\tilde{\xi})|$ for any $\tilde{\xi}$ such that $1 \leq \frac{|\tilde{\xi}|}{|\xi_1|} \leq r$.

Remark 3.5. Let χ be defined as in Proposition 3.3. It can be easily shown that perturbations of the power function, e.g. $\sigma(\xi) = (\sigma^1(\xi), \ldots, \sigma^m(\xi))$ where each σ^j is of the form $\sigma^j(\xi) = (|\xi|^\kappa + \sin(|\xi|^\kappa))\chi(\xi)$, also fall under the purview of Hypothesis 3.1.

Finally we add some assumptions on the first and second order derivatives of σ, which will be mainly invoked in the proof of Proposition 4.5.

Hypothesis 3.6. The derivatives of σ satisfy the following: there exists a $\ell_0 > 0$ such that for all ξ with $0 < |\xi| \leq \ell_0$ we have

$$|D\sigma(\xi)| \lesssim |\xi|^{\kappa-1} \quad \text{and} \quad |D^2\sigma(\xi)| \lesssim |\xi|^{\kappa-2}. \tag{17}$$

Remark 3.7. Observe that Hypotheses 3.1 and 3.6 imply: there exists a $\ell_0 > 0$ such that for all ξ with $0 < |\xi| \leq \ell_0$ we have

$$|D\sigma \cdot \sigma(\xi)| \lesssim |\xi|^{2\kappa-1}. \tag{18}$$

In addition, the reader can check that (17) and (18) are satisfied for σ as in Proposition 3.3.

Definition 3.8. Let $\mathcal{N}_{\alpha,F}$ be defined as:

$$\mathcal{N}_{\alpha,F} := \sup \left\{ \frac{|F(\xi)|}{|\xi|^\alpha}; |\xi| \neq 0 \right\}, \tag{19}$$

where $\alpha = \kappa$ if $F = \sigma$ and $\alpha = 2\kappa - 1$ if $F = \Psi = (D\sigma \cdot \sigma)$.

3.2. One-dimensional differential equations

In the one-dimensional case, similarly to what is done for more regular coefficients (See [12]), one can prove that a suitable function of x solves Eq. (11). This stems from an application of our extension of Itô's formula (see Proposition 2.9) and is obtained in the following theorem.

Theorem 3.9. *Consider Eq. (11) with $m = d = 1$, let $\sigma : \mathbb{R} \to \mathbb{R}$ and assume Hypothesis 3.2 to hold true. Assume $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $\kappa + \gamma > 1$. Let ϕ be the function defined in Hypothesis 3.2.*

Then the function $y = \phi^{-1}(x + \phi(a))$ *is a solution of the equation*

$$y_t = a + \int_0^t \sigma(y_s) dx_s, \quad t \geq 0. \tag{20}$$

Proof. Let $\psi(\xi) = \phi^{-1}(\xi + \phi(a))$. Due to the definition of ϕ, some elementary computations show that $\psi'(\xi) = \frac{1}{\phi'(\phi^{-1}(\xi+\phi(a)))} = \sigma(\psi(\xi))$ and thus we are reduced to show

$$\delta\psi(x)_{st} = \int_s^t \psi'(x_u) dx_u. \tag{21}$$

To this aim, observe that the second derivative of ψ satisfies

$$\psi''(\xi) = D\sigma(\psi(\xi))\psi'(\xi) = (D\sigma \cdot \sigma)(\psi(\xi)).$$

Using Hypothesis 3.2, ψ'' is thus $\lambda-$Hölder continuous where $\lambda = \frac{2\kappa-1}{1-\kappa} \wedge 1$, that is, ψ is a $\mathrm{Lip}^{2+\lambda}$ function. Moreover, since $\kappa + \gamma > 1$ and $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right]$ we find $(\lambda + 2)\gamma > 1$. Consequently we can invoke Proposition 2.9 and hence we obtain directly (21). The result is now proved. □

Remark 3.10. It is readily checked that the power coefficient $\sigma(\xi) = |\xi|^\kappa$ satisfies the conditions of Theorem 3.9, with a function F defined by $F(\xi) = c_\kappa |\xi|^\lambda \mathrm{sgn}(\xi)$ and where the exponent λ is given by $\lambda = \frac{2\kappa-1}{1-\kappa}$.

Remark 3.11. If $a = 0$, we do not have uniqueness of solution since in addition to the solution defined above, $y \equiv 0$ solves Eq. (20). This is not in contradiction to the results stated in [1] where the authors deal with equations with non-vanishing coefficients. In our case, $\sigma(0) = 0$.

Remark 3.12. As the reader might see, Theorem 3.9 is an easy consequence of the change of variable formula (6). This is in contrast with the corresponding proof in [6], which relied on a negative moment estimate and non trivial extensions of Young's integral in the fractional calculus framework.

4. Multidimensional differential equations

In the multidimensional case, our strategy in order to construct a solution is based (as in [6]) on quantifying an additional smoothness of the solution y as it approaches the origin. However, our computations here are more involved than in [6], due to the fact that we are handling a rough process x.

4.1. Prelude

In this section, we will introduce a sequence of stopping times, similarly to [6]. We assume that each component $\sigma^j : \mathbb{R}^m \to \mathbb{R}^m$ satisfies Hypothesis 3.1 and we consider the following equation for a fixed $a \in \mathbb{R}^m \setminus \{0\}$:

$$y_t = a + \sum_{j=1}^{d} \int_0^t \sigma^j(y_u)\, dx_u^j, \quad t \in [0, T], \tag{22}$$

where $T > 0$ is a fixed arbitrary horizon and $\mathbf{x} = (x, \mathbf{x^2})$ is a γ-rough path above x, as given in Hypothesis 2.4.

Our considerations start from the fact that, as long as we are away from 0, we can solve Eq. (22) as a rough path equation with regular coefficients. In particular the following can be shown under the above set-up. See [4].

Theorem 4.1. *Assume Hypothesis 3.1 is fulfilled. Then there exist a continuous function y defined on $[0, T]$ and an instant $\tau \le T$, such that one of the following two possibilities holds:*

(A) *$\tau = T$, y is non-zero on $[0, T]$, $y \in \mathcal{C}^\gamma([0, T]; \mathbb{R}^m)$ and y solves Eq. (22) on $[0, T]$, where the integrals $\int \sigma^j(y_u)\, dx_u^j$ are understood in the rough path sense.*

(B) *We have $\tau < T$. Then for any $t < \tau$, the path y sits in $\mathcal{C}^\gamma([0, t]; \mathbb{R}^m)$ and y solves Eq. (22) on $[0, t]$. Furthermore, $y_s \neq 0$ on $[0, \tau)$, $\lim_{t\to\tau} y_t = 0$ and $y_t = 0$ on the interval $[\tau, T]$.*

Option (A) above leads to classical solutions of Eq. (22). In the rest of this section, we will assume (B), that is the function y given by Theorem 4.1 vanishes in the interval $[\tau, T]$. The aim of this section is to prove the following:

- The path y is globally γ-Hölder continuous on $[0, T]$.

To achieve this we will require some additional hypotheses on x (See Hypothesis 4.6).

Quantification of the increased smoothness of the solution as it approaches the origin would require a partition of the interval $(0, \tau]$ as follows. Let $a_j = 2^{-j}$ and consider the following decomposition of $\mathbb{R}_+$:

$$\mathbb{R}_+ = \bigcup_{j=-1}^{\infty} I_j,$$

where

$$I_{-1} = [1, \infty), \quad \text{and} \quad I_q = [a_{q+1}, a_q), \quad q \ge 0.$$

Also consider:

$$J_{-1} = [3/4, \infty), \quad \text{and} \quad J_q = \left[\frac{a_{q+2} + a_{q+1}}{2}, \frac{a_{q+1} + a_q}{2}\right) =: \left[\hat{a}_{q+1}, \hat{a}_q\right), \quad q \ge 0.$$

Observe that owing to the definition of a_q, we have $\hat{a}_q = \frac{3}{2^{q+2}}$. Let q_0 be such that $a \in I_{q_0}$. Define $\lambda_0 = 0$ and

$$\tau_0 = \inf\{t \ge 0 : |y_t| \notin I_{q_0}\}.$$

By definition, $y_{\tau_0} \in J_{\hat{q}_0}$ with $\hat{q}_0 \in \{q_0, q_0 - 1\}$. Now define

$$\lambda_1 = \inf\{t \ge \tau_0 : |y_t| \notin J_{\hat{q}_0}\}.$$

Thus we get a sequence of stopping times $\lambda_0 < \tau_0 < \cdots < \lambda_k < \tau_k$, such that

$$y_t \in \left[\frac{b_1}{2^{q_k}}, \frac{b_2}{2^{q_k}}\right], \quad \text{for} \quad t \in [\lambda_k, \tau_k] \cup [\tau_k, \lambda_{k+1}], \tag{23}$$

where $b_1 = \frac{3}{8}$, $b_2 = \frac{3}{4}$ and $q_{k+1} = q_k + \ell$, with $\ell \in \{-1, 0, 1\}$, for $q_k \ge 1$. If $q_k = 0$ or $q_k = 1$, then we can choose the upper bound b_2 as $b_2 = \infty$.

Remark 4.2. Since this problem relies heavily on radial variables in $\mathbb{R}^m$, we alleviate vectorial notations and carry out the computations below for $m = d = 1$. Generalizations to higher dimensions are straight forward.

4.2. Regularity estimates

Let $\pi = \{0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T\}$ be a partition of the interval $[0, T]$ for $n \in \mathbb{N}$. Denote by $\mathcal{C}_2(\pi)$ the collection of functions R on π such that $R_{t_k t_{k+1}} = 0$ for $k = 0, 1, \dots n-1$. We now introduce some operators on discrete time increments, which are similar to those in Section 2. First, we define the operator $\delta : \mathcal{C}_2(\pi) \to \mathcal{C}_3(\pi)$ by

$$\delta R_{sut} = R_{st} - R_{su} - R_{ut} \quad \text{for } s, u, t \in \pi. \tag{24}$$

The Hölder seminorms we will consider are similar to those introduced in (4) and (5). Namely, for $R \in \mathcal{C}_2(\pi)$ we set

$$\|R\|_\mu = \sup_{u,v\in\pi} \frac{R_{uv}}{|u-v|^\mu} \quad \text{and} \quad \|\delta R\|_\mu = \sup_{s,u,t\in\pi} \frac{|\delta R_{sut}|}{|t-s|^\mu}.$$

We now state a sewing lemma for discrete increments which is similar to [7, Lemma 2.5]. Its proof is included here for completeness.

Lemma 4.3. *For $\mu > 1$ and $R \in \mathcal{C}_2(\pi)$, we have*

$$\|R\|_\mu \le K_\mu \|\delta R\|_\mu,$$

where $K_\mu = 2^\mu \sum_{l=1}^{\infty} \frac{1}{l^\mu}$.

Proof. Consider some fixed $t_i, t_j \in \pi$. Since $R \in \mathcal{C}_2(\pi)$ we have $\sum_{k=i}^{j-1} R_{t_k t_{k+1}} = 0$. Hence, for an arbitrary sequence of partitions $\{\pi_l;\, 1 \le l \le j-i-1\}$, where each π_l is a subset of $\pi \cap [t_i, t_j]$ with $l+1$ elements, we can write (thanks to a trivial telescoping sum argument):

$$R_{t_i t_j} = R_{t_i t_j} - \sum_{k=i}^{j-1} R_{t_k t_{k+1}} = \sum_{l=1}^{j-i-1} (R^{\pi_l} - R^{\pi_{l+1}}), \tag{25}$$

where we have set $R^{\pi_l} = \sum_{k=0}^{l-1} R_{t^l_k t^l_{k+1}}$. We now specify the choice of partitions π_l recursively: Define $\pi_{j-i} = \pi \cap [t_i, t_j]$. Given a partition π_l with $l+1$ elements, $l = 2, \dots, j-i$, we can find $t^l_{k_l} \in \pi_l \setminus \{t_i, t_j\}$ such that

$$t^l_{k_l+1} - t^l_{k_l-1} \le \frac{2(t_j - t_i)}{l}. \tag{26}$$

Denote by π_{l-1} the partition $\pi_l \setminus \{t^l_{k_l}\}$. Owing to (24), we obtain:

$$|R^{\pi_{l-1}} - R^{\pi_l}| = \left|\delta R_{t^l_{k_l-1} t^l_{k_l} t^l_{k_l+1}}\right| \le \|\delta R\|_\mu (t^l_{k_l+1} - t^l_{k_l-1})^\mu \le \|\delta R\|_\mu \frac{2^\mu (t_j - t_i)^\mu}{l^\mu},$$

where the second inequality follows from (26). Now plugging the above estimate in (25) we get

$$\left|R_{t_i t_j}\right| \le 2^\mu (t_j - t_i)^\mu \|\delta R\|_\mu \sum_{l=1}^{j-i-1} \frac{1}{(l+1)^\mu} \le K_\mu (t_j - t_i)^\mu \|\delta R\|_\mu.$$

By dividing both sides by $(t_i - t_j)^\mu$ and taking supremum over $t_i, t_j \in \pi$, we obtain the desired estimate. □

Next we define an increment R which is obtained as a remainder in rough path type expansions.

Definition 4.4. Let y and τ be defined as in Theorem 4.1. For $(s,t) \in \mathcal{S}_2([0,\tau])$, let R_{st} be defined by the following decomposition:

$$\delta y_{st} = \sigma(y_s)\delta x_{st} + (D\sigma \cdot \sigma)(y_s)\mathbf{x}^{\mathbf{2}}_{st} + R_{st}. \tag{27}$$

The theorem below quantifies the regularity improvement for the solution y of Eq. (22) as it gets closer to 0.

Proposition 4.5. *Consider a rough path x satisfying Hypothesis 2.4. Assume σ and $(D\sigma \cdot \sigma)$ follow Hypothesis 3.1. Also assume Hypothesis 3.6 holds. Then there exist constants $c_{0,x}$, $c_{1,x}$ and $c_{2,x}$ such that for $s,t \in [\lambda_k, \lambda_{k+1})$ satisfying $|t-s| \le c_{0,x}2^{-\alpha q_k}$, with $\alpha := \frac{1-\kappa}{\gamma}$, we have the following bounds:*

$$\mathcal{N}\left[y; \mathcal{C}_1^{\gamma}([s,t])\right] \le c_{1,x}2^{-\kappa q_k} \tag{28}$$

and

$$\mathcal{N}\left[R; \mathcal{C}_2^{3\gamma}([s,t])\right] \le c_{2,x}2^{(2-3\kappa)q_k}. \tag{29}$$

Proof. We divide this proof in several steps.

Step 1: Setting. Consider the dyadic partition on $[s,t]$. Specifically, we set

$$[\![s,t]\!] = \left\{ t_i : t_i = s + \frac{i(t-s)}{2^n}; i = 0, \ldots, 2^n \right\}$$

for all $n \in \mathbb{N}$. Define y^n on $[\![s,t]\!]$ by setting $y^n_s = y_s$, and

$$\delta y^n_{t_i t_{i+1}} = \sigma(y^n_{t_i})\delta x_{t_i t_{i+1}} + (D\sigma \cdot \sigma)(y^n_{t_i})\mathbf{x}^{\mathbf{2}}_{t_i t_{i+1}}.$$

We also introduce a discrete type remainder R^n, defined for all $(u,v) \in \mathcal{S}_2([\![s,t]\!])$, as follows:

$$R^n_{uv} = \delta y^n_{uv} - \sigma(y^n_{su})\delta x_{uv} - (D\sigma \cdot \sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv}.$$

Since $\gamma > 1/3$ and σ is sufficiently smooth away from zero, a second order expansion argument (see [4, Section 10.3]) shows that δy^n_{st} converges to δy_{st}.

Step 2: Induction hypothesis. Recall that we are working in $[\lambda_k, \lambda_{k+1})$. Hence, using (23) we can choose n large enough so that

$$y^n_u \in \left[\frac{a_1}{2^{q_k}}, \frac{a_2}{2^{q_k}}\right] \quad \text{for} \quad u \in [\![s,t]\!], \tag{30}$$

where $a_1 = \frac{2}{8}$ and $a_2 = \frac{7}{8}$. In addition, using Hypothesis 3.1, (19) and (30), we also have

$$|\sigma(y^n_u)| \le \mathcal{N}_{\kappa,\sigma}|y^n_u|^{\kappa} \le \mathcal{N}_{\kappa,\sigma}\left(\frac{a_2}{2^{q_k}}\right)^{\kappa} \tag{31}$$

as well as:

$$|(D\sigma \cdot \sigma)(y^n_u)| \le \mathcal{N}_{2\kappa-1,\Psi}|y^n_u|^{2\kappa-1} \le \mathcal{N}_{2\kappa-1,\Psi}\left(\frac{a_2}{2^{q_k}}\right)^{2\kappa-1}. \tag{32}$$

We now assume that s and t are close enough, namely for a given constant $c_0 > 0$, we have

$$|t-s| \le c_0 2^{-\alpha q_k} = T_0. \tag{33}$$

We will proceed by induction on the points of the partition t_i. That is, for $q \le 2^n - 1$ we assume that R^n satisfies the following relation:

$$\mathcal{N}[R^n; \mathcal{C}_2^{3\gamma}[\![s,t_q]\!]] \le c_2 2^{(2-3\kappa)q_k} \tag{34}$$

where c_2 is a constant to be fixed later. We will try to propagate this induction assumption to $[\![s, t_{q+1}]\!]$.

Step 3: A priori bounds on y^n. For $(u, v) \in \mathcal{S}_2\left([\![s, t_q]\!]\right)$ we have:

$$\delta y^n_{uv} = \sigma(y^n_u)\delta x_{uv} + (D\sigma \cdot \sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv} + R^n_{uv}. \tag{35}$$

Hence, using (31), (32) and our induction assumption (34) we get:

$$\begin{aligned}\mathcal{N}[y^n; \mathcal{C}^\gamma_1[\![s, t_q]\!]] \le \mathcal{N}_{\kappa,\sigma}\left(\frac{a_2}{2^{q_k}}\right)^\kappa \|\mathbf{x}\|_\gamma + \mathcal{N}_{2\kappa-1,\Psi}\left(\frac{a_2}{2^{q_k}}\right)^{2\kappa-1} \|\mathbf{x}\|_\gamma |t_q - s|^\gamma \\ + \mathcal{N}[R^n; \mathcal{C}^{3\gamma}_2[\![s, t]\!]]|t_q - s|^{2\gamma}.\end{aligned}$$

Since $|t_q - s| \le T_0 = c_0 2^{-\alpha q_k}$, we thus have

$$\begin{aligned}\mathcal{N}[y^n; \mathcal{C}^\gamma_1[\![s, t_q]\!]] \le \mathcal{N}_{\kappa,\sigma}\left(\frac{a_2}{2^{q_k}}\right)^\kappa \|\mathbf{x}\|_\gamma + \mathcal{N}_{2\kappa-1,\Psi}\left(\frac{a_2}{2^{q_k}}\right)^{2\kappa-1} \|\mathbf{x}\|_\gamma \left(c_0 2^{-\alpha q_k}\right)^\gamma \\ + \mathcal{N}[R^n; \mathcal{C}^{3\gamma}_2[\![s, t]\!]]\left(c_0 2^{-\alpha q_k}\right)^{2\gamma}.\end{aligned}$$

Therefore taking into account the fact that $\alpha = \frac{1-\kappa}{\gamma}$ and our assumption (34), we obtain:

$$\mathcal{N}[y^n; \mathcal{C}^\gamma_1[\![s, t_q]\!]] \le \tilde{c}\, 2^{-\kappa q_k} \tag{36}$$

where the constant $\tilde{c}$ is given by:

$$\tilde{c} = \mathcal{N}_{\kappa,\sigma} a_2^\kappa \|\mathbf{x}\|_\gamma + \mathcal{N}_{2\kappa-1,\Psi} a_2^{2\kappa-1} c_0^\gamma \|\mathbf{x}\|_\gamma + c_2 c_0^{2\gamma}. \tag{37}$$

Step 4: Induction propagation. Recall that $R^n_{uv} = \delta y^n_{uv} - \sigma(y^n_{su})\delta x_{uv} - (D\sigma \cdot \sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv}$. Hence invoking Proposition 2.3 we have:

$$\delta R^n_{uvw} = \mathcal{A}^{n,1}_{uvw} + \mathcal{A}^{n,2}_{uvw} + \mathcal{A}^{n,3}_{uvw}, \tag{38}$$

with

$$\mathcal{A}^{n,1}_{uvw} = -\delta\sigma(y^n)_{uv}\delta x_{vw}, \qquad \mathcal{A}^{n,2}_{uvw} = -\delta((D\sigma \cdot \sigma)(y^n))_{uv}\mathbf{x}^{\mathbf{2}}_{vw}$$

and

$$\mathcal{A}^{n,3}_{uvw} = (D\sigma \cdot \sigma)(y^n_u)\delta\mathbf{x}^{\mathbf{2}}_{uvw}.$$

We now treat those terms separately. The term $\mathcal{A}^{n,1}_{uvw}$ in (38) can be expressed using Taylor expansion, which yields

$$\mathcal{A}^{n,1}_{uvw} = -\left(D\sigma(y^n_u)\delta y^n_{uv} + \frac{1}{2}D^2\sigma(\xi^n)\left(\delta y^n_{uv}\right)^2\right)\delta x_{vw},$$

for some $\xi^n \in [y^n_u, y^n_v]$. Now, using (35) the above becomes

$$\begin{aligned}\mathcal{A}^{n,1}_{uvw} &= -\, D\sigma(y^n_u)\left(\sigma(y^n_u)\delta x_{uv} + (D\sigma \cdot \sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv} + R^n_{uv}\right)\delta x_{vw} \\ &\quad - \frac{1}{2}D^2\sigma(\xi^n)\left(\delta y^n_{uv}\right)^2\delta x_{vw} \\ &= -\,(D\sigma \cdot \sigma)(y^n_u)\delta x_{uv}\delta x_{vw} - D\sigma(y^n_u)(D\sigma \cdot \sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv}\delta x_{vw} \\ &\qquad - D\sigma(y^n_u)R^n_{uv}\delta x_{vw} - \frac{1}{2}D^2\sigma(\xi^n)(\delta y^n_{uv})^2\delta x_{vw}.\end{aligned} \tag{39}$$

Due to Hypothesis 2.4, the first term of (39) cancels $\mathcal{A}^{n,3}_{uvw}$ in (38). Therefore we end up with:

$$\mathcal{A}^{n,1}_{uvw} + \mathcal{A}^{n,3}_{uvw} = -D\sigma(y^n_u)(D\sigma\cdot\sigma)(y^n_u)\mathbf{x}^{\mathbf{2}}_{uv}\delta x_{vw} - D\sigma(y^n_u)R^n_{uv}\delta x_{vw} - \frac{1}{2}D^2\sigma(\xi^n_w)(\delta y^n_{uv})^2\delta x_{vw}.$$

Taking into account (12), (17) and (18) (similarly to what we did for (31)–(32)), as well as Hypothesis 2.4 and relation (33) for $|t-s|$, plus the induction (34) on R^n, we easily get:

$$\mathcal{A}^{n,1}_{uvw} + \mathcal{A}^{n,3}_{uvw} \le \Bigg\{\left(\frac{\tilde a_1}{2^{q_k}}\right)^{\kappa-1}\left(\frac{\tilde a_2}{2^{q_k}}\right)^{2\kappa-1}\|\mathbf{x}\|^2_\gamma + \left(\frac{\tilde a_1}{2^{q_k}}\right)^{\kappa-1}\|\mathbf{x}\|_\gamma T_0^\gamma \mathcal{N}[R^n;\mathcal{C}^{3\gamma}_2[\![s,t_q]\!]] + \frac{1}{2}\left(\frac{\tilde a_1}{2^{q_k}}\right)^{\kappa-2}\|\mathbf{x}\|_\gamma \mathcal{N}[y^n;\mathcal{C}^\gamma_1[\![s,t_q]\!]]^2\Bigg\}|w-u|^{3\gamma}, \tag{40}$$

where we have incorporated the constants on the right hand side of inequalities (17) inside $\tilde a_1$ and that of inequality (18) inside $\tilde a_2$.

We are now left with the estimation of $\mathcal{A}^{n,2}$. To bound this last term we first use inequality (13) with $r=\frac{7}{2}$. Taking into account (30), we get

$$\left|\mathcal{A}^{n,2}_{uvw}\right| \le \mathcal{N}_\Psi(|y^n_u|^{-2(1-\kappa)} + |y^n_v|^{-2(1-\kappa)})|y^n_v - y^n_u|\|\mathbf{x}\|_\gamma |w-v|^{2\gamma}.$$

Invoking (30) again and the definition of $\mathcal{N}[y^n;\mathcal{C}^\gamma_1[\![s,t_q]\!]]$, this yields:

$$\left|\mathcal{A}^{n,2}_{uvw}\right| \le \mathcal{N}_\Psi\left(\frac{2^{q_k}}{a_1}\right)^{2(1-\kappa)}\mathcal{N}[y^n;\mathcal{C}^\gamma_1[\![s,t_q]\!]]|v-u|^\gamma\|\mathbf{x}\|_\gamma|w-v|^{2\gamma}.$$

Finally using the a priori bound on y^n stated in (36) we obtain:

$$\left|\mathcal{A}^{n,2}_{uvw}\right| \le \mathcal{N}_\Psi\left(\frac{2^{q_k}}{a_1}\right)^{2(1-\kappa)}\tilde c 2^{-\kappa q_k}\|\mathbf{x}\|_\gamma|w-u|^{3\gamma}, \tag{41}$$

which can be recast as:

$$\left|\mathcal{A}^{n,2}_{uvw}\right| \le \frac{\mathcal{N}_\Psi}{a_1^{2(1-\kappa)}}\,\tilde c\,\|\mathbf{x}\|_\gamma 2^{(2-3\kappa)q_k}|w-u|^{3\gamma}. \tag{42}$$

We can now plug (40) and (42) back into (38) in order to get:

$$\begin{aligned}\mathcal{N}[\delta R^n;\mathcal{C}^{3\gamma}_3[\![s,t_{q+1}]\!]] \le& \left(\frac{\tilde a_1}{2^{q_k}}\right)^{\kappa-1}\left(\frac{\tilde a_2}{2^{q_k}}\right)^{2\kappa-1}\|\mathbf{x}\|^2_\gamma \\ &+\left(\frac{\tilde a_1}{2^{q_k}}\right)^{\kappa-1}\|\mathbf{x}\|_\gamma T_0^\gamma\mathcal{N}[R^n;\mathcal{C}^{3\gamma}_2[\![s,t_q]\!]] \\ &+\frac{1}{2}\left(\frac{\tilde a_1}{2^{q_k}}\right)^{\kappa-2}\|\mathbf{x}\|_\gamma\mathcal{N}[y^n;\mathcal{C}^\gamma_1[\![s,t_q]\!]]^2 \\ &+\frac{1}{a_1^{2(1-\kappa)}}\mathcal{N}_\Psi\tilde c\|\mathbf{x}\|_\gamma 2^{(2-3\kappa)q_k}.\end{aligned}$$

Therefore, thanks to our induction assumption (34) and the a priori bound (36), the above becomes

$$\mathcal{N}[\delta R^n;\mathcal{C}^{3\gamma}_3[\![s,t_{q+1}]\!]] \le d2^{(2-3\kappa)q_k}$$

with

$$d = \left(\tilde a_1^{\kappa-1}\tilde a_2^{2\kappa-1}\|\mathbf{x}\|^2_\gamma + \tilde a_1^{\kappa-1}\|\mathbf{x}\|_\gamma c_0^\gamma c_2 + \frac{1}{2}\tilde a_1^{\kappa-2}\tilde c^2\|\mathbf{x}\|_\gamma + \frac{1}{\tilde a_1^{2(1-\kappa)}}\mathcal{N}_\Psi\tilde c\|\mathbf{x}\|_\gamma\right). \tag{43}$$

Then using the discrete sewing Lemma 4.3, we obtain

$$\mathcal{N}[R^n; \mathcal{C}_2^{3\gamma}[\![s, t_{q+1}]\!]] \leq K_{3\gamma}\mathcal{N}[\delta R^n; \mathcal{C}_3^{3\gamma}[\![s, t_{q+1}]\!]] \leq \hat{c}2^{(2-3\kappa)q_k}, \tag{44}$$

where $K_{3\gamma} = \sum_{l=1}^{\infty} \frac{1}{l^{3\gamma}}$ and $\hat{c} = dK_{3\gamma}$.

Plugging in the value of $\tilde{c}$ from (37) in the expression for d in (43) we find that $\hat{c}$ can be decomposed as

$$\hat{c} = dK_{3\gamma} = (d_{1,x} + d_{2,x})K_{3\gamma},$$

where

$$d_{1,x} = \left(\tilde{a}_1^{\kappa-1}\tilde{a}_2^{2\kappa-1}\|\mathbf{x}\|_\gamma^2 + \frac{1}{2}\tilde{a}_1^{\kappa-2}\mathcal{N}_{\kappa,\sigma}^2 a_2^{2\kappa}\|\mathbf{x}\|_\gamma^3 + \frac{1}{a_1^{2(1-\kappa)}}\mathcal{N}_\Psi\mathcal{N}_{\kappa,\sigma}a_2^\kappa\|\mathbf{x}\|_\gamma^2 \right)$$

and $d_{2,x}$ consist of terms containing positive powers of c_0, where we recall that c_0 is defined by (33).

Looking at inequality (44), we need $\hat{c}$ to be less than c_2 in order to complete the induction propagation. Let us now fix $c_2 = \frac{3}{2}d_{1,x}K_{3\gamma} = c_{2,x}$ and choose $c_0 = c_{0,x}$ small enough so that $d_{2,x} < \frac{d_{1,x}}{2}$. This implies $\hat{c} = dK_{3\gamma} = (d_{1,x} + d_{2,x})K_{3\gamma} < \frac{3}{2}d_{1,x}K_{3\gamma} = c_{2,x}$, which is what we required. Our propagation is hence established.

Step 5: Conclusion. Completing the iterations over t_q in $[\![s, t]\!]$ we get that relation (34) is valid for $\mathcal{N}[R^n; \mathcal{C}_3^{3\gamma}[\![s, t]\!]]$. Next, put the values of $c_{0,x}$ and $c_{2,x}$ in $\tilde{c}$ as defined in (36) and call this new value $c_{1,x}$. We thus get the following uniform bound over n:

$$\mathcal{N}[y^n; \mathcal{C}_1^\gamma[\![s, t]\!]] \leq c_{1,x}2^{-\kappa q_k}.$$

Our claims (29) and (28) are now achieved by taking limits over n. □

In order to further analyze the increments of y^n, we need to increase slightly the regularity assumptions on x. This is summarized in the following hypothesis:

Hypothesis 4.6. There exists $\varepsilon_1 > 0$ such that for $\gamma_1 = \gamma + \varepsilon_1$, we have $\|\mathbf{x}\|_{\gamma_1} < \infty$.

The extra regularity imposed on $\mathbf{x}$ allows us to improve our estimates on remainders (in rough path expansions) in the following way.

Proposition 4.7. *Let us assume that Hypothesis 4.6 holds, as well as Hypotheses 3.1 and 3.6. For $k \geq 0$, consider $(s, t) \in \mathcal{S}_2([\lambda_k, \lambda_{k+1}))$ such that $|t - s| \leq c_{0,x}2^{-\alpha q_k}$, where $c_{0,x}$ is defined in Proposition 4.5. Then the following second order decomposition for δy is satisfied:*

$$\delta y_{st} = \sigma(y_s)\,\delta x_{st} + r_{st}, \quad \textit{with} \quad |r_{st}| \leq c_{3,x}\,2^{-\kappa_{\varepsilon_1}q_k}|t - s|^\gamma, \tag{45}$$

where we have set $\kappa_{\varepsilon_1} = \kappa + 2\varepsilon_1\alpha$.

Proof. From (27) we have

$$|r_{st}| = |(D\sigma \cdot \sigma)(y_s)\mathbf{x}_{st}^{\mathbf{2}} + R_{st}| \leq |(D\sigma \cdot \sigma)(y_s)|\,|\mathbf{x}_{st}^{\mathbf{2}}| + |R_{st}|. \tag{46}$$

Under the constraints we have imposed on s, t, namely $s, t \in [\lambda_k, \lambda_{k+1})$ such that $|t - s| \leq c_{0,x}2^{-\alpha q_k}$, and recalling that we have set $\gamma_1 = \gamma + \varepsilon_1$, we have

$$\begin{aligned}\sup_{s,t} \frac{|\mathbf{x}_{st}^{\mathbf{2}}|}{|t - s|^\gamma} &= \sup_{s,t} \frac{|\mathbf{x}_{st}^{\mathbf{2}}|}{|t - s|^{2\gamma+2\varepsilon_1}}|t - s|^{\gamma+2\varepsilon_1} \leq \sup_{s,t} \frac{|\mathbf{x}_{st}^{\mathbf{2}}|}{|t - s|^{2\gamma_1}} \sup_{s,t}|t - s|^{\gamma+2\varepsilon_1} \\ &\leq \mathcal{N}\left[\mathbf{x}^{\mathbf{2}}; \mathcal{C}_2^{2\gamma_1}\right](c_{0,x}2^{-\alpha q_k})^{\gamma+2\varepsilon_1} \end{aligned} \tag{47}$$

where we have used $\sup_{s,t}$ to stand for supremum over the set $\{(s,t) : s,t \in [\lambda_k, \lambda_{k+1})$ and $|t - s| \le c_{0,x} 2^{-\alpha q_k}\}$.

Note that under Hypothesis 4.6, the quantity $\|\mathbf{x}\|_{\gamma_1}$ is finite and hence (47) can be read as:

$$\sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{\gamma}} \le \|\mathbf{x}\|_{\gamma_1} c_{0,x}^{\gamma+2\varepsilon_1} 2^{-\alpha(\gamma+2\varepsilon_1)q_k}. \tag{48}$$

Moreover, owing to (29) applied to $\gamma := \gamma + \varepsilon_1$, and κ as in Hypothesis 3.1, we get

$$\begin{aligned}\sup_{s,t} \frac{|R_{st}|}{|t-s|^{\gamma}} &= \sup_{s,t} \frac{|R_{st}|}{|t-s|^{3(\gamma+\varepsilon_1)}} |t-s|^{2\gamma+3\varepsilon_1} \le \sup_{s,t} \frac{|R_{st}|}{|t-s|^{3\gamma_1}} \sup_{s,t} |t-s|^{2\gamma+3\varepsilon_1} \\ &\le \tilde{c}_{2,x} 2^{(2-3\kappa)q_k} (c_{0,x} 2^{-\alpha q_k})^{2\gamma+3\varepsilon_1}. \end{aligned} \tag{49}$$

Here we have used the notation $\tilde{c}_{2,x}$ to stand for the coefficient $c_{2,x}$ in (29), with $\|\mathbf{x}\|_{\gamma}$ replaced by $\|\mathbf{x}\|_{\gamma_1}$. Thus we have

$$\sup_{s,t} \frac{|R_{st}|}{|t-s|^{\gamma}} \le \tilde{c}_{2,x} c_{0,x}^{2\gamma+3\varepsilon_1} 2^{-(\alpha(2\gamma+3\varepsilon_1)+3\kappa-2)q_k}. \tag{50}$$

Now incorporating (48) and (50) in (46), and recalling that $\alpha = \frac{1-\kappa}{\gamma}$, we easily get:

$$\begin{aligned}\sup_{s,t} \frac{|r_{st}|}{|t-s|^{\gamma}} &\le \mathcal{N}_{2\kappa-1,\Psi} \left(\frac{b_2}{2^{q_k}}\right)^{2\kappa-1} \|\mathbf{x}\|_{\gamma_1} c_{0,x}^{\gamma+2\varepsilon_1} 2^{-\alpha(\gamma+2\varepsilon_1)q_k} \\ &\quad + \tilde{c}_{2,x} c_{0,x}^{2\gamma+3\varepsilon_1} 2^{-(\alpha(2\gamma+3\varepsilon_1)+3\kappa-2)q_k} \\ &= \mathcal{N}_{2\kappa-1,\Psi} b_2^{2\kappa-1} \|\mathbf{x}\|_{\gamma_1} c_{0,x}^{\gamma+2\varepsilon_1} 2^{-(\kappa+2\varepsilon_1\alpha)q_k} + \tilde{c}_{2,x} c_{0,x}^{2\gamma+3\varepsilon_1} 2^{-(\kappa+3\varepsilon_1\alpha)q_k}.\end{aligned}$$

Collecting terms and recalling that we have set $\kappa_{\varepsilon_1} = \kappa + 2\varepsilon_1\alpha$, we end up with:

$$\sup_{s,t} \frac{|r_{st}|}{|t-s|^{\gamma}} \le c_{3,x} 2^{-(\kappa+2\varepsilon_1\alpha)q_k} = c_{3,x} 2^{-\kappa_{\varepsilon_1} q_k},$$

which is our claim (45). □

Thanks to our previous efforts, we can now slightly enlarge the interval on which our improved regularity estimates hold true:

Corollary 4.8. *Let the assumptions of Proposition 4.7 prevail, and consider $0 < \varepsilon_1 < 1 - \gamma$ as in Hypothesis 4.6. Then with $\alpha = \gamma^{-1}(1-\kappa)$, there exist $0 < \varepsilon_2 < \alpha$ and a constant $c_{4,x}$ such that for all $(s,t) \in \mathcal{S}_2([\lambda_k, \lambda_{k+1}))$ satisfying $|t-s| \le c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k}$ we have*

$$|\delta y_{st}| \le c_{5,x} 2^{-q_k \kappa_{\varepsilon_2}^-} |t-s|^{\gamma}, \quad \text{where} \quad \kappa_{\varepsilon_2}^- = \kappa - (1-\gamma)\varepsilon_2. \tag{51}$$

Moreover, under the same conditions on (s,t), decomposition (45) still holds true, with

$$|r_{st}| \le c_{6,x} 2^{-q_k \kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^{\gamma}, \quad \text{where} \quad \kappa_{\varepsilon_1,\varepsilon_2} = \kappa + 2\alpha\varepsilon_1 - \gamma\varepsilon_2 - 2\varepsilon_1\varepsilon_2. \tag{52}$$

Proof. We split our computations in 2 steps.

Step 1: Proof of (51). Start from inequality (28), which is valid for $|t-s| \le c_{0,x} 2^{-\alpha q_k}$. Now let $m \in \mathbb{N}$ and consider $s,t \in [\lambda_k, \lambda_{k+1})$ such that $c_{0,x}(m-1)2^{-\alpha q_k} < |t-s| \le c_{0,x} m 2^{-\alpha q_k}$. We partition the interval $[s,t]$ by setting $t_j = s + c_{0,x} j 2^{-\alpha q_k}$ for $j = 0, \ldots, m-1$ and $t_m = t$. Then we simply write

$$|\delta y_{st}| \le \sum_{j=0}^{m-1} |\delta y_{t_j t_{j+1}}| \le c_{1,x} 2^{-q_k \kappa} \sum_{j=0}^{m-1} (t_{j+1} - t_j)^{\gamma} \le c_{1,x} 2^{-q_k \kappa} m^{1-\gamma} |t-s|^{\gamma},$$

where the last inequality stems from the fact that $t_{j+1} - t_j \le (t-s)/m$. Now the upper bound (51) is easily deduced by applying the above inequality to a generic $m \le [2^{\varepsilon_2 q_k}] + 1$, where $0 < \varepsilon_2 < \frac{\kappa}{1-\gamma}$. This ensures $\kappa^-_{\varepsilon_2} = \kappa - (1-\gamma)\varepsilon_2 > 0$.

Step 2: Proof of (52). We proceed as in the proof of Proposition 4.7, but now with a relaxed constraint on (s,t), namely $|t-s| \le c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k}$ where $\varepsilon_2 > 0$ satisfies:

$$\varepsilon_2 < \min\left(\frac{\kappa}{1-\gamma}, \frac{\varepsilon_1 \alpha}{\gamma + \varepsilon_1}\right). \tag{53}$$

The equivalent of relation (49) is thus

$$\begin{aligned}\sup_{s,t} \frac{|R_{st}|}{|t-s|^{\gamma}} &= \sup_{s,t} \frac{|R_{st}|}{|t-s|^{3(\gamma+\varepsilon_1)}} |t-s|^{2\gamma+3\varepsilon_1} \le \sup_{s,t} \frac{|R_{s,t}|}{|t-s|^{3\gamma_1}} \sup_{s,t} |t-s|^{2\gamma+3\varepsilon_1} \\ &\le \tilde{c}_{2,x} 2^{(2-3\kappa)q_k} (c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k})^{2\gamma+3\varepsilon_1}.\end{aligned} \tag{54}$$

As in Proposition 4.7 we have used the notation $\tilde{c}_{2,x}$ to stand for the coefficient $c_{2,x}$ with $\|\mathbf{x}\|_{\gamma}$ replaced by $\|\mathbf{x}\|_{\gamma_1}$ and $\sup_{s,t}$ to stand for supremum over the set $\{(s,t) : s,t \in [\lambda_k, \lambda_{k+1})$ and $|t-s| \le c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k}\}$. Collecting the exponents in (54) we thus end up with:

$$\sup_{s,t} \frac{|R_{st}|}{|t-s|^{\gamma}} \le \tilde{c}_{2,x} c_{4,x} 2^{-(\kappa+3\varepsilon_1\alpha-2\varepsilon_2\gamma-3\varepsilon_1\varepsilon_2)q_k}. \tag{55}$$

Similarly to (47), we also get:

$$\begin{aligned}\sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{\gamma}} &= \sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{2\gamma+2\varepsilon_1}} |t-s|^{\gamma+2\varepsilon_1} \le \sup_{s,t} \frac{|\mathbf{x}^{\mathbf{2}}_{st}|}{|t-s|^{2\gamma_1}} \sup_{s,t} |t-s|^{\gamma+2\varepsilon_1} \\ &\le \|\mathbf{x}\|_{\gamma_1} (c_{4,x} 2^{-(\alpha-\varepsilon_2)q_k})^{\gamma+2\varepsilon_1}.\end{aligned} \tag{56}$$

Consequently, owing to Hypothesis 3.6, we get the following relation:

$$\begin{aligned}|(D\sigma\cdot\sigma)(y_s)\mathbf{x}^{\mathbf{2}}_{st}| &\le \mathcal{N}_{2\kappa-1,\Psi} \left(\frac{b_2}{2^{q_k}}\right)^{2\kappa-1} \|\mathbf{x}\|_{\gamma_1} c_{4,x}^{\gamma+2\varepsilon_1} 2^{-(\alpha-\varepsilon_2)(\gamma+2\varepsilon_1)q_k} \\ &= \mathcal{N}_{2\kappa-1, D\sigma\cdot\sigma} b_2^{2\kappa-1} \|\mathbf{x}\|_{\gamma_1} c_{4,x}^{\gamma+2\varepsilon_1} 2^{-(\kappa+2\varepsilon_1\alpha-\varepsilon_2\gamma-2\varepsilon_1\varepsilon_2)q_k}.\end{aligned} \tag{57}$$

Notice that under the conditions on ε_2 in (53), we have $\kappa + 2\varepsilon_1\alpha - \varepsilon_2\gamma - 2\varepsilon_1\varepsilon_2 < \kappa + 3\varepsilon_1\alpha - 2\varepsilon_2\gamma - 3\varepsilon_1\varepsilon_2$. Therefore incorporating (55) and (57) we have:

$$|r_{st}| \le |(D\sigma\cdot\sigma)(y_s)\mathbf{x}^{\mathbf{2}}_{st}| + |R_{st}| \lesssim 2^{-q_k \kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^{\gamma}$$

which is our claim (52). □

4.3. Estimates for stopping times

Thanks to the previous estimates on improved regularity for the solution y to Eq. (22), we will now get a sharp control on the difference $\lambda_{k+1} - \lambda_k$. Otherwise stated we shall control the speed at which y might converge to 0, which is the key step in order to control the global Hölder continuity of y. This section is similar to what has been done in [6], and proofs are included for sake of completeness. We start with a lower bound on the difference $\lambda_{k+1} - \lambda_k$.

Proposition 4.9. *Assume σ and $(D\sigma\cdot\sigma)$ follow Hypothesis 3.1. Also assume Hypothesis 3.6 holds. Then the sequence of stopping times $\{\lambda_k,\ k \ge 1\}$ defined by (23) satisfies*

$$\lambda_{k+1} - \lambda_k \ge c_{5,x}\, 2^{-\alpha q_k}, \tag{58}$$

where we recall that $\alpha = (1-\kappa)/\gamma$.

Proof. We show that the difference $\tau_k - \lambda_k$ satisfies a lower bound of the form

$$\tau_k - \lambda_k \geq c_{6,x}\, 2^{-\alpha q_k}. \tag{59}$$

There exists a similar bound for $\lambda_{k+1} - \tau_k$, and consequently we get our claim (58).

To arrive at inequality (59) we observe that in order to leave the interval $[\lambda_k, \tau_k)$, an increment of size at least $2^{-(q_k+1)}$ must occur. This is because at λ_k the solution lies at the mid point of I_{q_k}, an interval of size 2^{-q_k}. Thus, if $|\delta y_{st}| \geq 2^{-(q_k+1)}$ and $|t-s| \leq c_{0,x} 2^{-\alpha q_k}$, relation (28) provides us with:

$$c_{1,x} \frac{|t-s|^\gamma}{2^{\kappa q_k}} \geq \frac{1}{2^{q_k+1}}, \tag{60}$$

which implies

$$|t-s| \geq \left(2c_{1,x}\right)^{-\frac{1}{\gamma}} 2^{-\frac{(1-\kappa)q_k}{\gamma}} = \left(2c_{1,x}\right)^{-\frac{1}{\gamma}} 2^{-\alpha q_k}.$$

This completes the proof. □

In order to sharpen Proposition 4.9, we introduce a roughness hypothesis on x, again as in [6]. This assumption is satisfied when x is a fractional Brownian motion.

Hypothesis 4.10. We assume that for $\hat{\varepsilon}$ arbitrarily small there exists a constant $c > 0$ such that for every s in $[0, T]$, every ϵ in $(0, T/2]$, and every ϕ in $\mathbb{R}^d$ with $|\phi| = 1$, there exists t in $[0, T]$ such that $\epsilon/2 < |t-s| < \epsilon$ and

$$|\langle \phi, \delta x_{st} \rangle| > c\, \epsilon^{\gamma+\hat{\varepsilon}}.$$

The largest such constant is called the modulus of $(\gamma+\hat{\varepsilon})$-Hölder roughness of x, and is denoted by $L_{\gamma,\hat{\varepsilon}}(x)$.

Under this hypothesis, we are also able to upper bound the difference $\lambda_{k+1} - \lambda_k$.

Proposition 4.11. *Assume σ and $(D\sigma \cdot \sigma)$ follow Hypothesis 3.1. Also assume Hypothesis 3.6 holds and $\sigma(\xi) \gtrsim |\xi|^\kappa$. Then for all $\varepsilon_2 < \frac{\alpha\varepsilon_1}{\gamma+\varepsilon_1} \wedge \frac{\kappa}{1-\gamma}$ and q_k large enough (that is for k large enough, since $\lim_{k\to\infty} q_k = \infty$ under Assumption (B) of Theorem 4.1), the sequence of stopping times $\{\lambda_k,\ k \geq 1\}$ defined by (23) satisfies*

$$\lambda_{k+1} - \lambda_k \leq c_{x,\varepsilon_2} 2^{-q_k(\alpha-\varepsilon_2)}, \tag{61}$$

where we recall that $\alpha = (1-\kappa)/\gamma$. Furthermore, inequality (51) can be extended as follows: there exists a constant c_x such that for $s, t \in [\lambda_k, \lambda_{k+1})$ we have

$$|\delta y_{st}| \leq c_x 2^{-\kappa\bar{\varepsilon}_2 q_k} |t-s|^\gamma. \tag{62}$$

Proof. We prove by contradiction. Assume the contrary, that is, (61) does not hold. This implies that for some $\varepsilon_2 < \frac{\alpha\varepsilon_1}{\gamma+\varepsilon_1} \wedge \frac{\kappa}{1-\gamma}$

$$\lambda_{k+1} - \lambda_k \geq C 2^{-q_k(\alpha-\varepsilon_2)} \tag{63}$$

holds for infinitely many values of k, for any constant C. Consequently

$$\lambda_{k+1} - \lambda_k \geq C\, 2^{-q_k(1-\kappa)/(\gamma+\hat{\varepsilon})}, \tag{64}$$

for an $\hat{\varepsilon}$ small enough so that $(1-\kappa)/(\gamma+\hat{\varepsilon}) \geq \alpha - \varepsilon_2$. We now show that there exists $s, t \in [\lambda_k, \lambda_{k+1}]$ such that $|\delta y_{st}| > |J_{q_k}|$ providing us with our contradiction. Here $|J_{q_k}|$ denotes the size of the interval J_{q_k}.

To achieve this we now use Hypothesis 4.10. Taking into account we are in the one-dimensional case let us choose

$$\varepsilon := \frac{c_1\, 2^{-\frac{q_k(1-\kappa)}{\gamma+\hat{\varepsilon}}}}{\left[L_{\gamma,\hat{\varepsilon}}(x)\right]^{\frac{1}{\gamma+\hat{\varepsilon}}}} \leq C\, 2^{-\frac{q_k(1-\kappa)}{\gamma+\hat{\varepsilon}}},$$

where the inequality is true for a fixed constant c_1 and a large enough constant C. Due to (63) and Hypothesis 4.10 there now exist $s, t \in [\lambda_k, \lambda_{k+1}]$ such that

$$\frac{\varepsilon}{2} \leq |t-s| \leq \varepsilon, \quad \text{and} \quad |\delta x_{st}| \geq c_1^{\gamma+\hat{\varepsilon}}\, 2^{-q_k(1-\kappa)}. \tag{65}$$

Moreover, due to our assumptions on σ and because $y_s \geq b_1 2^{-q_k} \geq 2^{-q_k-2}$, we have $|\sigma(y_s)| \geq c 2^{-q_k\kappa}$ for $s \in [\lambda_k, \lambda_{k+1}]$. Consequently, for s, t as in (65)

$$|\sigma(y_s)\delta x_{st}| \geq c c_1^{\gamma+\hat{\varepsilon}}\, 2^{-q_k}.$$

For fixed ε, c_1 can be chosen arbitrarily large (by increasing k or decreasing $\hat{\varepsilon}$) such that $c c_1^{\gamma+\hat{\varepsilon}} \geq 6$. We thus have

$$|\sigma(y_s)\delta x_{st}| \geq 6 \cdot 2^{-q_k} = 2|J_{q_k}|.$$

In particular the size of this increment is larger than twice the size of J_{q_k} (see relation (23)).

Recall, $\hat{\varepsilon}$ is small enough so that $(1-\kappa)/(\gamma+\hat{\varepsilon}) \geq \alpha - \varepsilon_2$, so that from the bound on $|t-s|$ in (65) we have $|t-s| \leq c_{7,x} 2^{-q_k(\alpha-\varepsilon_2)}$. With s, t as in relation (65) we use the fact that $\delta y_{st} = \sigma(y_s)\delta x_{st} + r_{st}$ and the bound (52) to get

$$|\delta y_{st}| \gtrsim A^1_{st} - A^2_{st}, \quad \text{with} \quad A^1_{st} = 6 \cdot 2^{-q_k}, \quad A^2_{st} \leq c_{6,x} 2^{-q_k \kappa_{\varepsilon_1,\varepsilon_2}} |t-s|^{\gamma} \leq c_{9,x} 2^{-q_k \mu_{\varepsilon_2}},$$

where we recall that $\kappa_{\varepsilon_1,\varepsilon_2} = \kappa + 2\alpha\varepsilon_1 - \gamma\varepsilon_2 - 2\varepsilon_1\varepsilon_2$ to obtain

$$\mu_{\varepsilon_2} = \kappa_{\varepsilon_1,\varepsilon_2} + (\alpha - \varepsilon_2)\gamma = 1 + 2\alpha\varepsilon_1 - 2(\gamma+\varepsilon_1)\varepsilon_2.$$

Compared to 2^{-q_k}, A^2_{st} can be made negligible for large enough q_k by making sure that $\mu_{\varepsilon_2} > 1$. One can ensure $\mu_{\varepsilon_2} > 1$ by choosing ε_1 large enough and ε_2 small enough. As a consequence $|\delta y_{st}| \gtrsim A^1_{st} - A^2_{st}$, where A^1_{st} is larger than twice $|J_{q_k}| = 3 \cdot 2^{-q_k}$ and A^2_{st} is negligible compared to A^1_{st} as q_k gets large. That is, $|\delta y_{st}| > |J_{q_k}|$ for k large enough. We now have our contradiction and this proves (61). □

4.4. Hölder continuity

Eventually the control of the stopping times λ_k leads to the main result of this section, that is the existence of a $\mathcal{C}^\gamma$ solution to Eq. (22). The crucial step in this direction is detailed in the proposition below. It is achieved under the additional assumption $\gamma + \kappa > 1$, and yields directly the proof of Theorem 1.2.

Proposition 4.12. *Suppose that our noise x satisfies Hypotheses 4.6* and *4.10. Assume σ and $(D\sigma \cdot \sigma)$ follow Hypotheses 3.1* and *3.6 holds as well. Also assume $\sigma(\xi) \gtrsim |\xi|^\kappa$ and that $\gamma + \kappa > 1$. Then, the function y given in Theorem 4.1 belongs to $\mathcal{C}^\gamma([0,T];\mathbb{R}^m)$.*

Proof. We start with the assumption that y satisfies condition (B) in Theorem 4.1. We first consider $s = \lambda_k$ and $t = \lambda_l$ with $k < l$ and decompose the increments $|\delta y_{st}|$ as:

$$|\delta y_{st}| \leq \sum_{j=k}^{l-1} \left|\delta y_{\lambda_j \lambda_{j+1}}\right|.$$

Due to Proposition 4.11 we have $\lambda_{k+1} - \lambda_k \leq c_{x,\varepsilon_2} 2^{-q_k(\alpha-\varepsilon_2)}$ for a large enough k. An application of Corollary 4.8 yields

$$|\delta y_{st}| \leq \sum_{j=k}^{l-1} \left|\delta y_{\lambda_j \lambda_{j+1}}\right| \leq c_{5,x} \sum_{j=k}^{l-1} 2^{-q_j \kappa_{\varepsilon_2}^-} |\lambda_{j+1} - \lambda_j|^{\gamma}. \tag{66}$$

Rewriting inequality (58),

$$2^{-\frac{q_j(1-\kappa)}{\gamma}} \leq c_{7,x}^{-1} \left(\lambda_{j+1} - \lambda_j\right)$$

which implies

$$2^{-q_j \kappa_{\varepsilon_2}^-} \leq (c_{7,x})^{-\frac{\gamma \kappa_{\varepsilon_2}^-}{1-\kappa}} \left(\lambda_{j+1} - \lambda_j\right)^{\frac{\gamma \kappa_{\varepsilon_2}^-}{1-\kappa}}.$$

Using this inequality in (66) and defining $c_{8,x} = c_{5,x}(c_{7,x})^{-\frac{\gamma \kappa_{\varepsilon_2}^-}{1-\kappa}}$, we get:

$$|\delta y_{st}| \leq c_{8,x} \sum_{j=k}^{l-1} |\lambda_{j+1} - \lambda_j|^{\tilde{\mu}_{\varepsilon_2}}, \quad \text{where} \quad \tilde{\mu}_{\varepsilon_2} = \gamma \left(1 + \frac{\kappa_{\varepsilon_2}^-}{1-\kappa}\right).$$

Recall $\kappa_{\varepsilon_2}^- = \kappa - (1-\gamma)\varepsilon_2$, which can be made arbitrarily close to κ. Hence under the assumption $\gamma + \kappa > 1$, $\tilde{\mu}_{\varepsilon_2}$ is of the form $\tilde{\mu}_{\varepsilon_2} = 1 + \varepsilon_3$. We thus obtain

$$|\delta y_{st}| \leq c_{8,x} \sum_{j=k}^{l-1} |\lambda_{j+1} - \lambda_j|^{1+\varepsilon_3} \leq c_{8,x} |\lambda_l - \lambda_k|^{1+\varepsilon_3} \leq c_{8,x}\, \tau^{1+\varepsilon_3-\gamma} |t-s|^{\gamma},$$

where we recall $s = \lambda_k$ and $t = \lambda_l$. Having proved our claim for this special case, the general case for $s < \lambda_k \leq \lambda_l < t$ is obtained by the following decomposition

$$\delta y_{st} = \delta y_{s\lambda_k} + \delta y_{\lambda_k \lambda_l} + \delta y_{\lambda_l t}.$$

Finally, we make use of (62) in order to bound $\delta y_{s\lambda_k}$ and $\delta y_{\lambda_l t}$. □

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