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One-dimensional reflected rough differential equations

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Abstract

We prove existence and uniqueness of the solution of a one-dimensional rough differential equation driven by a step-2 rough path and reflected at zero. The whole difficulty of the problem (at least as far as uniqueness is concerned) lies in the non-continuity of the Skorohod map with respect to the topologies under consideration in the rough case. Our argument to overcome this obstacle is inspired by some ideas we introduced in a previous work dealing with rough kinetic PDEs arXiv:1604.00437.

1. Introduction

In its original formulation [19], Lyons' rough paths theory aimed at the study of the standard differential model

$$dy_t = f(y_t)\,dx_t \quad , \quad y_0 = a \in \mathbb{R}^d \ , \ t \in [0,T] \, , \tag{1.1}$$

where $f : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^N;\mathbb{R}^d)$ is a smooth enough application and $x : [0,T] \to \mathbb{R}^N$, $y : [0,T] \to \mathbb{R}^d$ are (typically non-differentiable) continuous paths. In order to deal with

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the lack of regularity one has to drop both the classical differential or integral formulation of the problem and turn to a description of the motion on arbitrarily small, but finite scales. Eq. (1.1) can be interpreted as the requirement that increments of y should behave locally as some "germ" given by a Taylor-like polynomial approximation of the right hand side. A rough path $\mathbb{X}$ constructed above the irregular signal x is the given appropriate monomials with which such a local approximation is constructed. One of the key results of the rough paths theory is that, under appropriate conditions, only one continuous function y can satisfy all these local constraints. In this case we say that the path y satisfies the *rough differential equations* (RDEs) (1.1).

While the approach of Lyons [19–21] stresses more the control theoretic sides of the theory, and in particular the mapping from rough paths over x to rough path over y, it has been Davie [4] who observed the usefulness of these local expansions. Following Davie's insight, one of the author of the present paper [12] introduced a suitable Banach space where these local expansions can be studied efficiently. The work of Friz and Victoir [11] showed also how to systematically generate and analyze the local expansions for (1.1) leading to a very complete theory for RDEs.

It later turned out that these principles, or at least some adaptation of them, remain valid for other – less standard – differential models, such as delay [22] or Volterra [6] rough equations and homogenization of fast/slow systems [17]. The basic idea of local coherent expansions as effective description of rough dynamical systems has been developed more recently in numerous PDE settings (see e.g. [13–15], to mention but a few spin-offs amongst a flourishing literature) leading to the development of the general framework of regularity structures by Hairer [16], which allows to handle local expansions of a large class of distributions. For a recent nice introduction to rough paths theory and some applications see [10].

This being said, in the vast majority of the situations so far covered by rough paths analysis, and especially in all the above quoted references, the success of the method lies in an essential way on fixed-point and contraction mapping methods to establish existence and more importantly uniqueness of the object under consideration. Unfortunately, the existence of such a contraction property is not known in the case of the reflected rough differential equation, which we propose to study in this paper. To be more precise, we will focus on the one-dimensional RDE reflected at 0, which can be described as follows: given a time $T > 0$, a smooth function $f : \mathbb{R} \to \mathcal{L}(\mathbb{R}^N; \mathbb{R})$ and a p-variation N-dimensional rough path $\mathbb{X}$ with $2 \leqslant p < 3$ (see Definition 1), find an $\mathbb{R}_{\geqslant 0}$-valued path $y \in V_1^p([0, T])$ and an $\mathbb{R}_{\geqslant 0}$-valued increasing function (or "reflection measure") $m \in V_1^1([0, T])$ that together satisfy

$$\mathrm{d}y(t) = f(y(t))\mathrm{d}\mathbb{X}_t + \mathrm{d}m_t\,, \qquad y_t \mathrm{d}m_t = 0\,. \tag{1.2}$$

Thus, the idea morally is to exhibit a path y that somehow follows the dynamics in (1.1), but is also forced to stay positive thanks to the intervention of some regular "local time" m at 0. Of course, at this point, it is not exactly clear how to understand the right hand side of (1.2), and we shall later give a more specific interpretation of the system, based on rough paths principles (see Definition 2).

The stochastic counterpart of (1.2), where $\mathbb{X}$ is a standard N-dimensional Brownian motion and the right hand side is interpreted as an Itô integral, has been receiving a lot of attention since the 60s (see e.g. [18,23–25]), with several successive generalizations regarding the (possibly multidimensional) containment domain of y. This Brownian reflected equation has also been investigated more recently through the exhibition of Wong–Zakai-type approximation algorithms [3].

When $1 \leqslant p < 2$, Problem (1.2) can be naturally interpreted and analyzed by means of Young integration techniques. This situation was first considered by Ferrante and Rovira in [9] for the

d-dimensional positive domain $\mathbb{R}^d_{\geq 0}$, with exhibition of an existence result therein. Using some sharp p-variation estimates for the Skorohod map, Falkowski and Slominski [7,8] have recently provided a full treatment of the Young case (at least when considering reflection on hyperplanes), by proving both existence and uniqueness of the solution.

The more complex rough (or step-2) version of (1.2), which somehow extends the Brownian model, has been first considered by Aida in [1], and further analyzed by the same author in [2] for more general multidimensional domains. Nevertheless, in these two references, only *existence* of a solution to (1.2) can be established and the *uniqueness* issue is left open. The lack of regularity of the Skorohod map clearly appears as the main obstacle towards a uniqueness result in the approach followed in [1,2] (see Section 3.1 for more details on the problem).

Our aim in this study is to complete the above picture in the one-dimensional situation, that is to prove uniqueness of a solution to the problem (1.2). Actually, for the reader's convenience, we will also provide a detailed proof of the existence of a solution in this setting, and simplify at the same time some of the arguments used by Aida in [1,2]. The subsequent analysis accordingly offers a thorough – and totally self-contained – proof of well-posedness of the problem (1.2).

The strategy is inspired by the recent results on rough conservation laws [5]. Indeed, there is an analogy between (1.2) and the kinetic formulation of conservation laws where the so-called kinetic measure appears. As for (1.2), this measure is unknown and becomes part of the solution which brings significant difficulties, especially in the proof of uniqueness. The latter is then based on a tensorization-type argument, also known as doubling of variables, and subsequent estimation of the difference of two solutions.

In the case of (1.2), we put forward a fairly simple proof of uniqueness based on a direct estimation of a difference of two solutions. In particular, in this finite dimensional setting no technical tensorization method is needed. The existence is then derived from a compactness result, starting from a smooth approximation of the rough path $\mathbb{X}$. In both cases, the key of the procedure consists in deriving sharp estimates for the remainder term which measures the difference between the (explicit) local expansion and the unknown of the problem. The strategy thus heavily relies on the so-called *sewing lemma* at the core of the rough paths machinery (see Lemma 1). The estimates on the remainder are then converted via a *rough Gronwall lemma* (see Lemma 2) into estimates for the unknown (resp. for some function thereof) in order to establish existence (resp. uniqueness).

The paper is organized along a very simple division. In Section 2, we start with a few reminders on the rough paths setting and topologies, which allows us to give a rigorous interpretation of the problem (1.2), as well as the statement of our well-posedness result (Theorem 4). We also introduce the two main technical ingredients of our analysis therein, namely the above-mentioned sewing and Gronwall lemmas, with statements borrowed from [5]. Section 3 is devoted to the uniqueness issue: we first elaborate on the difficulties raised by the rough case (with respect to the "Young" case) and then display our solution to the problem (Theorem 9). Section 4 closes the study with the proof of existence: we will first provide an exhaustive treatment of the problem in the one-dimensional situation (the main topic of the paper), and then give a few details on possible extensions of our arguments to more general multidimensional domains (Section 4.2).

2. Setting and main result

To settle our analysis, we will need the following notations and definitions taken from rough paths theory. First of all, let us recall the definition of the increment operator, denoted by δ. If g is a path defined on $[0, T]$ and $s, t \in [0, T]$ then $\delta g_{st} := g_t - g_s$, if g is a 2-index map defined on $[0,T]^2$ then $\delta g_{sut} := g_{st} - g_{su} - g_{ut}$. For $g : [0,T] \to E$ and $\varphi : E \to F$ (with E, F two Banach spaces), we will also use the convenient notations

$$\begin{aligned} [\![\varphi]\!](g)_{st} &:= \int_0^1 \mathrm{d}\tau\, \varphi(g_s + \tau(\delta g)_{st}) \quad , \\ [\![[\![\varphi]\!]]\!](g)_{st} &:= \int_0^1 \mathrm{d}\tau \int_0^\tau \mathrm{d}\sigma\, \varphi(g_s + \sigma(\delta g)_{st}) \,. \end{aligned} \tag{2.1}$$

Observe in particular that if φ is a smooth enough mapping, then

$$\delta\varphi(g)_{st} = [\![\nabla\varphi]\!](g)_{st}\delta g_{st} \quad \text{and} \quad [\![\varphi]\!](g)_{st} - \varphi(g_s) = [\![[\![\nabla\varphi]\!]]\!](g)_{st}\delta g_{st} \,. \tag{2.2}$$

In the sequel, given an interval I we call a *control on* I (and denote it by ω) any superadditive map on $\mathcal{S}_I := \{(s,t) \in I^2 : s \leqslant t\}$, that is, any map $\omega : \mathcal{S}_I \to [0,\infty[$ such that,

$$\omega(s,u) + \omega(u,t) \leqslant \omega(s,t), \qquad s \leqslant u \leqslant t.$$

We will say that a control ω is *regular* if $\lim_{|t-s|\to 0}\omega(s,t) = 0$. Also, given a control ω on a time interval $I = [a,b]$, we will use the notation $\omega(I) := \omega(a,b)$.

Now, given a time interval I, a parameter $p > 0$, a Banach space E and a function $g : \mathcal{S}_I \to E$, we define the p-variation seminorm of g as

$$\|g\|_{\bar{V}_2^p(I;E)} := \sup_{(t_i)\in\mathcal{P}(I)} \left(\sum_i |g_{t_i t_{i+1}}|^p \right)^{\frac{1}{p}},$$

where $\mathcal{P}(I)$ denotes the set of all partitions of the interval I, and we denote by $\bar{V}_2^p(I;E)$ the set of maps $g : \mathcal{S}_I \to E$ for which this quantity is finite. In this case,

$$\omega_g(s,t) := \|g\|^p_{\bar{V}_2^p([s,t];E)}$$

defines a control on I, and we denote by $V_2^p(I;E)$ the set of elements $g \in \bar{V}_2^p(I;E)$ for which ω_g is regular on I. We then denote by $\bar{V}_1^p(I;E)$, resp. $V_1^p(I;E)$, the set of paths $g : I \to E$ such that $\delta g \in \bar{V}_2^p(I;E)$, resp. $\delta g \in V_2^p(I;E)$. Finally, we define the space $\bar{V}^p_{2,\mathrm{loc}}(I;E)$ of maps $g : \mathcal{S}_I \to E$ such that there exists a countable covering $\{I_k\}_k$ of I satisfying $g \in \bar{V}_2^p(I_k;E)$ for every k. We write $g \in V^p_{2,\mathrm{loc}}(I;E)$ if the related controls can be chosen regular.

Definition 1. Fix a time $T > 0$ and let $N \geqslant 1$, $2 \leqslant p < 3$. Then we call a continuous N-dimensional p-variation rough path on $[0,T]$ any pair

$$\mathbb{X} = (\mathbb{X}^1, \mathbb{X}^2) \in V_2^p([0,T];\mathbb{R}^N) \times V_2^{p/2}([0,T];\mathbb{R}^{N,N}) \tag{2.3}$$

that satisfies the relation

$$\delta\mathbb{X}^{2;ij}_{sut} = \mathbb{X}^{1,i}_{su}\,\mathbb{X}^{1,j}_{ut} \,, \qquad s < u < t \in [0,T]\,,\ i,j \in \{1,\ldots,N\}\,. \tag{2.4}$$

Such a rough path $\mathbb{X}$ is said to be geometric if it can be obtained as the limit, for the p-variation topology involved in (2.3), of a sequence of smooth rough paths $(\mathbb{X}^\varepsilon)_{\varepsilon>0}$, that is with $\mathbb{X}^\varepsilon = (\mathbb{X}^{\varepsilon,1}, \mathbb{X}^{\varepsilon,2})$ explicitly defined as

$$\mathbb{X}^{\varepsilon,1,i}_{st} := \delta x^{\varepsilon,i}_{st} \,, \qquad \mathbb{X}^{\varepsilon,2,ij}_{st} := \int_s^t \delta x^{\varepsilon,i}_{su}\, \mathrm{d}x^{\varepsilon,j}_u \,,$$

for some smooth path $x^\varepsilon : [0,T] \to \mathbb{R}^N$.

We are now in a position to provide a clear interpretation of the problem (1.2).

Definition 2. Given a time $T > 0$, a real $a \geq 0$, a differentiable function $f : \mathbb{R} \to \mathcal{L}(\mathbb{R}^N; \mathbb{R})$ and a p-variation N-dimensional rough path $\mathbb{X}$ with $2 \leqslant p < 3$, a pair $(y, m) \in V_1^p([0,T]; \mathbb{R}_{\geqslant 0}) \times V_1^1([0,T]; \mathbb{R}_{\geqslant 0})$ is said to solve the problem (1.2) on $[0,T]$ with initial condition a if there exists a 2-index map $y^\natural \in V_{2,\mathrm{loc}}^{p/3}([0,T]; \mathbb{R})$ such that for all $s < t \in [0,T]$, we have

$$\begin{cases} \delta y_{st} = f_i(y_s)\mathbb{X}_{st}^{1,i} + f_{2,ij}(y_s)\mathbb{X}_{st}^{2,ij} + \delta m_{st} + y_{st}^\natural \\ \quad y_0 = a \quad \text{and} \quad m_t = \displaystyle\int_0^t \mathbf{1}_{\{y_u=0\}} \mathrm{d}m_u \end{cases}, \tag{2.5}$$

where we have set $f_{2,ij}(\xi) := f_i'(\xi) f_j(\xi)$ and $m([0,t]) := m_t$.

Remark 3. Eq. (2.5) should be read as the given local expansion of the function y: it says that around each time point s the function can be approximated by the germ

$$t \mapsto y_s + f(y_s)\mathbb{X}_{st}^1 + f_2(y_s)\mathbb{X}_{st}^2 + \delta m_{st}$$

up to terms of order $\omega(s,t)^{p/3}$ where ω is a control. The term δm_{st} is characteristic for this reflected problem: the measure m increases only at times u where $y_u = 0$ effectively "kicking" the path y away from the negative axis. In some sense it can be considered as a Lagrange multiplier enforcing the constraint $y_u \geq 0$ for all $u \in [0,T]$.

With this interpretation in hand, our well-posedness result reads as follows:

Theorem 4. *Let $T > 0$ and $a > 0$. If $f \in \mathcal{C}_b^3(\mathbb{R}; \mathcal{L}(\mathbb{R}^N; \mathbb{R}))$, that is if f is 3-time differentiable, bounded with bounded derivatives, and if $\mathbb{X}$ is a continuous geometric N-dimensional p-variation rough path, then Problem (1.2) admits a unique solution (y, m) on $[0,T]$ with initial condition a.*

Let us conclude this preliminary section with a presentation of the two main technical results that will be used in our analysis, and the proofs of which are elementary and can be found e.g. in [5] (Lemma 2.1 and Lemma 2.7, respectively).

Lemma 1 (*Sewing Lemma*). *Fix an interval I, a Banach space E and a parameter $\zeta > 1$. Consider a map $G : I^3 \to E$ such that $G \in \{\delta H;\ H : I^2 \to E\}$ and for every $s < u < t \in I$,*

$$|G_{sut}| \leqslant \omega(s,t)^\zeta,$$

for some regular control ω on I. Then there exists a unique element $\Lambda G \in V_2^{1/\zeta}(I; E)$ such that $\delta(\Lambda G) = G$ and for every $s < t \in I$,

$$|(\Lambda G)_{st}| \leqslant C_\zeta \omega(s,t)^\zeta, \tag{2.6}$$

for some universal constant C_ζ.

Lemma 2 (*Rough Gronwall Lemma*). *Fix a time horizon $T > 0$ and let $g : [0,T] \to [0,\infty)$ be a path such that for some constants $C, L > 0$, $\kappa \geqslant 1$ and some controls ω_1, ω_2 on $[0,T]$ with ω_1 being regular, one has*

$$\delta g_{st} \leqslant C\big(\sup_{0 \leqslant r \leqslant t} g_r\big)\,\omega_1(s,t)^{\frac{1}{\kappa}} + \omega_2(s,t), \tag{2.7}$$

for every $s < t \in [0, T]$ satisfying $\omega_1(s,t) \leqslant L$. Then it holds

$$\sup_{0\leqslant t\leqslant T} g_t \leqslant 2e^{c_{L,\kappa}\,\omega_1(0,T)} \left\{ g_0 + \sup_{0\leqslant t\leqslant T} (\omega_2(0,t)e^{-c_{L,\kappa}\,\omega_1(0,t)}) \right\},$$

where $c_{L,\kappa}$ is defined as

$$c_{L,\kappa} = \sup\left(\frac{1}{L},\ (2Ce^2)^\kappa\right). \tag{2.8}$$

3. Uniqueness

In this section we shall first briefly review (essentially following [7]) the contraction method which allows to get uniqueness for reflected equations when Eq. (1.2) can be interpreted in the Young sense. Then we will show why one cannot expect to extend directly this simple contraction principle to the rough case. Eventually our main result is shown in Section 3.2, thanks to the new ingredients alluded to in the introduction.

3.1. Illustration of the difficulty

As we already pointed it out in the introduction, proving uniqueness of a solution to the rough reflected Eq. (2.5) is the most intricate part of the study. In order to illustrate this difficulty, let us briefly go back here to the strategy in the so-called Young situation, i.e. we assume for the moment that the driving path x is of finite p-variation for $1 \leq p < 2$ (we set $\mathbb{X}^{1,i}_{st} := \delta x^i_{st}$ in the sequel). In this case, the corresponding notion of a solution to (1.2) consists of a pair $(y, m) \in V_1^p([0,T]; \mathbb{R}_{\geqslant 0}) \times V_1^1([0,T]; \mathbb{R}_{\geqslant 0})$ such that $y_0 = a$ and for all $s < t$,

$$\delta y_{st} = f_i(y_s)\mathbb{X}^{1,i}_{st} + \delta m_{st} + y^\natural_{st} \tag{3.1}$$

for some path $y^\natural \in V^{p/2}_{2,\mathrm{loc}}([0,T]; \mathbb{R})$ and with m satisfying the additional constraint

$$m_t = \int_0^t \mathbf{1}_{\{y_u=0\}} \mathrm{d}m_u\ . \tag{3.2}$$

Let us also recall the general definition of the Skorohod map.

Definition 5. Let g be a continuous $\mathbb{R}$-valued path defined on some interval $I = [\ell_1, \ell_2]$. The Skorohod problem in the domain $\mathbb{R}_{\geqslant 0}$ associated with g consists in finding a pair $(y, m) \in \mathcal{C}(I; \mathbb{R}_{\geqslant 0}) \times V_1^1(I; \mathbb{R}_{\geqslant 0})$ such that for all $s < t$ with $s, t \in I$ we have:

$$\begin{cases} \delta y_{st} = \delta g_{st} + \delta m_{st}\ , \\ y_{\ell_1} = g_{\ell_1}\ ,\ m_t = \displaystyle\int_0^t \mathbf{1}_{\{y_u=0\}} \mathrm{d}m_u \end{cases}.$$

The application $g \mapsto (y, m)$ is called Skorohod map.

It turns out that the uniqueness issue associated with (3.1) can be readily handled by using the regularity property of the (one-dimensional) Skorohod map, as developed in [7]. Namely, it is well known that for every $g \in \bar{V}_1^p(I)$, there exists a unique pair $(y, m) \in \bar{V}_1^p(I) \times \bar{V}_1^1(I)$ satisfying Definition 5.

Now the key point towards uniqueness for (3.1) lies in the fact that for $1 \leqslant p < 2$, according to [7, Theorem 2.2],

$$\text{the map} \quad \Phi : \bar{V}_1^p(I) \to \bar{V}_1^p(I)\ ,\ g \mapsto m \quad \text{is Lipschitz-continuous} \tag{3.3}$$

for any finite interval I. Based on this property, we can easily prove the following uniqueness result:

Proposition 6. *Consider a finite horizon $T > 0$ and an initial condition $a \geq 0$. We consider a function $f \in \mathcal{C}_b^3(\mathbb{R}; \mathcal{L}(\mathbb{R}^N; \mathbb{R}))$ and a N-dimensional p-variation path x with $1 \leq p < 2$. Then uniqueness holds for* Eq. (3.1) *on* $[0, T]$.

Proof. We shall implement the usual contraction argument for differential equations to our system (3.1). In order to ease notations throughout this proof, we use the convention $a \lesssim b$ if there exists a constant c such that $a \leqslant c\, b$.

Step 1: Bounding the remainders. Let (y, μ) and (z, ν) be two solutions of problem (3.1). We set $\Delta := y - z$, $\Delta^\natural := y^\natural - z^\natural$ and

$$\omega_y(s,t) := \|y\|^p_{\bar{V}_1^p([s,t])} \quad , \quad \omega_\Delta(s,t) := \|y - z\|^p_{\bar{V}_1^p([s,t])} \quad , \quad \omega_{\mathbb{X}}(s,t) := \|x\|^p_{\bar{V}_1^p([s,t])} .$$

Then writing decomposition (3.1) for y and z and invoking elementary inequalities for the rectangular increment $\delta(f_i(y) - f_i(z))_{su}$, it holds that

$$\begin{aligned} |\delta \Delta^\natural_{sut}| &= |\delta(f_i(y) - f_i(z))_{su} \mathbb{X}^{1,i}_{ut}| \\ &\lesssim \big[\|y - z\|_{\infty;[s,u]} \omega_y(s,u)^{1/p} + \omega_\Delta(s,u)^{1/p}\big] \omega_{\mathbb{X}}(u,t)^{1/p} , \end{aligned}$$

which, by Lemma 1 and since $1 \leq p < 2$, yields that

$$|\Delta^\natural_{st}| \lesssim \big[\|y - z\|_{\infty;[s,t]} \omega_y(s,t)^{1/p} + \omega_\Delta(s,t)^{1/p}\big] \omega_{\mathbb{X}}(s,t)^{1/p} . \tag{3.4}$$

Step 2: Bounding the measures. Since y (resp. z) can be seen as the solution of a Skorohod problem, let us call g (resp. h) the corresponding non reflected path. Then according to (3.1), the respective decompositions of the increments of g and h can be written for $0 \leq s < t \leq T$ as:

$$\delta g_{st} = f_i(y_s) \mathbb{X}^{1,i}_{st} + y^\natural_{st}, \quad \text{and} \quad \delta h_{st} = f_i(z_s) \mathbb{X}^{1,i}_{st} + z^\natural_{st}.$$

In particular, if the initial condition for both y and z is $a \geqslant 0$, we get:

$$g_r = f_i(a) \mathbb{X}^i_{0r} + y^\natural_{0r}, \quad \text{and} \quad h_r = f_i(a) \mathbb{X}^i_{0r} + z^\natural_{0r}, \tag{3.5}$$

for all $r \leqslant T$. Besides, due to (3.3), we can assert that for all $0 \leq s < t \leq T$:

$$\|\mu - \nu\|_{\bar{V}_1^p([s,t])} \lesssim \|g - h\|_{\bar{V}_1^p([s,t])} . \tag{3.6}$$

Step 3: Conclusion. Observe that owing to (3.5) we have:

$$\delta(g - h)_{ru} = \Delta^\natural_{0u} - \Delta^\natural_{0r} = \delta \Delta^\natural_{0ru} + \Delta^\natural_{ru} = \delta(f_i(y) - f_i(z))_{0r} \mathbb{X}^i_{ru} + \Delta^\natural_{ru} ,$$

so, by (3.4) and (3.6), for all $0 < T_0 \leq T$ and $0 \leq s < t \leq T_0$,

$$\begin{aligned} \|\mu - \nu\|_{\bar{V}_1^p([s,t])} \lesssim \big[&\|y - z\|_{\infty;[0,T_0]} + \|y - z\|_{\infty;[0,T_0]} \omega_y(s,t)^{1/p} + \omega_\Delta(s,t)^{1/p}\big] \\ &\times \omega_{\mathbb{X}}(s,t)^{1/p} . \end{aligned} \tag{3.7}$$

Going back to expansion (3.1) (for both y and z), we know that

$$|\delta(y - z)_{st}| \leq |f_i(y_s) - f_i(z_s)| |\mathbb{X}^i_{st}| + |\delta(\mu - \nu)_{st}| + |\Delta^\natural_{st}|$$

and thus, using (3.4)–(3.7), we get, for any $0 < T_0 < T$,

$$\omega_\Delta(0, T_0) \lesssim \big[\|\Delta\|^p_{\infty;[0,T_0]} + \omega_\Delta(0, T_0)\big] \omega_{\mathbb{X}}(0, T_0) ,$$

which easily leads us to the expected uniqueness result by a standard contraction and patching argument. □

Remark 7. Note that the arguments in the proof of Proposition 6 could also provide us with a regularity property for the corresponding "Ito" map $\mathbb{X} \mapsto y$.

Let us now turn to the rough situation $2 \leq p < 3$, with associated reflected problem (2.5). We use the same notations (y, μ), (z, ν) and $\Delta^\natural$ as in the proof of Proposition 6. Then thanks to (2.2) and (2.4) it can be shown (see (3.11) for similar computations) that

$$\delta\Delta^\natural_{sut} = A^i_{su}\mathbb{X}^{1,i}_{ut} + \delta(f_{2,ij}(y) - f_{2,ij}(z))_{su}\mathbb{X}^{2,ij}_{ut} , \tag{3.8}$$

where we recall that $f_{2,ij} = f_i' f_j$ and where we have set

$$\begin{aligned} A^i_{su} &:= (f_i'(y_s)f_{2,jk}(y_s) - f_i'(z_s)f_{2,jk}(z_s))\mathbb{X}^{2,jk}_{su} + (f_i'(y_s) - f_i'(z_s))(\delta\mu_{su} + y^\natural_{su}) \\ &\quad + f_i'(z_s)[\delta(\mu-\nu)_{su} + \Delta^\natural_{su}] \\ &\quad + \big([\![[\![f_i'']\!]]\!](y)_{su}(\delta y_{su})^2 - [\![[\![f_i'']\!]]\!](z)_{su}(\delta z_{su})^2\big) . \end{aligned}$$

With the conditions of Lemma 1 in mind, the latter expression clearly emphasizes the need for a control of $\|\mu-\nu\|_{\bar{V}^q([0,T])}$, with q such that $\frac{1}{p}+\frac{1}{q} > 1$, in terms of $\|y-z\|_{\bar{V}^p([0,T])}$. In light of the above strategy for the Young situation, we would expect this control to be (again) a consequence of some Lipschitz-continuity property for the Skorohod map.

This is where the whole difficulty of the rough case arises: when $p \geq 2$, the following result indeed annihilates any hope for such a regularity statement.

Proposition 8. *For all $p > q \geq 1$, the Skorohod map $\Phi : g \mapsto m$ defined by (3.3) is not Hölder-continuous when considered as an application from $\bar{V}_1^p([0,1];\mathbb{R})$ to $\bar{V}_1^q([0,1];\mathbb{R})$.*

Proof. Assume that there exist constants $\lambda \in (0,1]$ and $C_{p,q,\lambda} > 0$ such that for all $f, g \in \bar{V}_1^p([0,1];\mathbb{R})$,

$$\|\Phi(f) - \Phi(g)\|_{\bar{V}_1^q([0,1];\mathbb{R})} \leq C_{p,q,\lambda}\|f-g\|^\lambda_{\bar{V}_1^p([0,1];\mathbb{R})} . \tag{3.9}$$

In particular, we would have, for all increasing functions $f, g : [0,1] \to \mathbb{R}$,

$$\begin{aligned} \|f-g\|_{\bar{V}_1^q([0,1];\mathbb{R})} &= \|(f-f(0)) - (g-g(0))\|_{\bar{V}_1^q([0,1];\mathbb{R})} \\ &= \|\Phi(-f+f(0)) - \Phi(-g+g(0))\|_{\bar{V}_1^q([0,1];\mathbb{R})} \\ &\leq C_{p,q,\lambda}\|f-g\|^\lambda_{\bar{V}_1^p([0,1];\mathbb{R})} , \end{aligned}$$

and so, for every function $F : [0,1] \to \mathbb{R}$ with bounded variation,

$$\|F\|_{\bar{V}_1^q([0,1];\mathbb{R})} \leq C_{p,q,\lambda}\|F\|^\lambda_{\bar{V}_1^p([0,1];\mathbb{R})} . \tag{3.10}$$

We now show that relation (3.10) is impossible, by exhibiting a simple counter example. Indeed, consider the sequence (F_n) of step-functions given by the formula: for every $t \in [0,1]$,

$$F_n(t) = \sum_{i\geq 0}\left\{\frac{1}{2n^{\frac{1}{p}}}\mathbf{1}_{\{t^{2n}_{2i} \leq t < t^{2n}_{2i+1}\}} - \frac{1}{2n^{\frac{1}{p}}}\mathbf{1}_{\{t^{2n}_{2i+1} \leq t < t^{2n}_{2i+2}\}}\right\} ,$$

where we have set $t_i^n = \frac{i}{n}$. It is readily checked that

$$\|F_n\|_{\bar{V}_1^p([0,1];\mathbb{R})} = \left(\sum_{i=1}^{2n}\left(\frac{1}{n^{\frac{1}{p}}}\right)^p\right)^{\frac{1}{p}} = 2^{\frac{1}{p}}$$

and in the same way we get the following relation for $q < p$:

$$\|F_n\|_{\bar{V}_1^q([0,1];\mathbb{R})} = \left(\sum_{i=1}^{2n}\left(\frac{1}{n^{\frac{1}{p}}}\right)^q\right)^{\frac{1}{q}} = 2^{\frac{1}{q}} n^{\frac{1}{q}-\frac{1}{p}} \overset{n\to\infty}{\longrightarrow} \infty\,,$$

which of course contradicts (3.10). □

3.2. Main result

Proposition 8 shows that one cannot use Hölder continuity properties of the Skorohod map in order to get uniqueness for Eq. (1.2) in the rough case. The current section shows how to circumvent this problem thanks to the full implementation of rough paths methods and our rough Gronwall Lemma 2.

Theorem 9. *Let $T > 0$ and $a > 0$. If $f \in \mathcal{C}_b^3(\mathbb{R}; \mathcal{L}(\mathbb{R}^N; \mathbb{R}))$ and $\mathbb{X}$ is an N-dimensional p-variation rough path, then Problem (1.2) admits at most one solution (y, m) on $[0, T]$ with initial condition a.*

Proof. Let (y, μ) and (z, ν) be two solutions for (2.5). Set $Y := (y, z) \in V_1^p([0, T]; \mathbb{R}^2)$ and with decomposition (2.5) in mind, write

$$\delta Y_{st} = F_i(Y_s)\mathbb{X}_{st}^{1,i} + F_{2,ij}(Y_s)\mathbb{X}_{st}^{2,ij} + \delta M_{st} + Y_{st}^\natural\,, \qquad 0 \leqslant s \leqslant t \leqslant T. \tag{3.11}$$

where we use the shorthands $F_i(Y) := (f_i(y), f_i(z))$, $F_{2,ij}(Y) := (f_{2,ij}(y), f_{2,ij}(z))$, $M := (\mu, \nu) \in V_1^1([0, T]; \mathbb{R}^2)$ and $Y^\natural := (y^\natural, z^\natural) \in V_{2,\mathrm{loc}}^{p/3}([0, T]; \mathbb{R}^2)$. From now on and until the end of the proof, we fix an interval $I \subset [0, T]$ such that $Y^\natural \in V_2^{p/3}(I; \mathbb{R}^2)$ and consider the following controls on I:

$$\omega_Y(s,t) := \|Y\|_{\bar{V}_1^p([s,t])}^p\,, \qquad \omega_{Y,\natural}(s,t) := \|Y^\natural\|_{\bar{V}_2^{p/3}([s,t])}^{p/3}\,,$$

$$\omega_\Delta(s,t) := \|y-z\|_{\bar{V}_1^p([s,t])}^p\,, \qquad \omega_{\Delta,\natural}(s,t) := \|y^\natural - z^\natural\|_{\bar{V}_2^{p/3}([s,t])}^{p/3}\,,$$

$$\omega_M(s,t) := \|M\|_{\bar{V}_1^1([s,t])} = \|\mu\|_{\bar{V}_1^1([s,t])} + \|\nu\|_{\bar{V}_1^1([s,t])}\,.$$

Without loss of generality, we will assume that $\omega_{\mathbb{X}}(I) \leqslant 1$, where $\omega_{\mathbb{X}}$ is a fixed control such that

$$|\mathbb{X}_{s,t}^1| + |\mathbb{X}_{s,t}^2|^{1/2} \leqslant \omega_{\mathbb{X}}(s,t)^{1/p}, \quad 0 \leqslant s \leqslant t \leqslant T\,.$$

Now, consider a smooth function $\varphi : \mathbb{R} \to \mathbb{R}_{\geqslant 0}$ and set $h(x_1, x_2) := \varphi(x_1 - x_2)$ for all $x_1, x_2 \in \mathbb{R}$. A direct computation via Taylor expansion, combined with (3.11), shows that

$$\begin{aligned}\delta h(Y)_{st} = [\![\nabla h]\!](Y)_{st}\,\delta Y_{st} &= H_i(Y_s)\mathbb{X}_{st}^{1,i} + H_{2,ij}(Y_s)\mathbb{X}_{st}^{2,ij} \\ &\quad + \int_s^t \varphi'(y_u - z_u)(\mathrm{d}\mu_u - \mathrm{d}\nu_u) + h_{st}^\natural\end{aligned} \tag{3.12}$$

where $h^\natural$ is a map in $V_2^{p/3}(I;\mathbb{R})$, and where we have set, for all $Y=(y,z)\in\mathbb{R}^2$,

$$H_i(Y) := \nabla h(Y)F_i(Y) = \varphi'(y-z)(f_i(y)-f_i(z)) \tag{3.13}$$

$$\begin{aligned} H_{2,ij}(Y) &:= \nabla H_i(Y)F_j(Y) \\ &= \varphi'(y-z)(f_{2,ij}(y)-f_{2,ij}(z)) \\ &\quad +\varphi''(y-z)(f_i(y)-f_i(z))(f_j(y)-f_j(z)) \,. \end{aligned} \tag{3.14}$$

Step 1: A general estimate on $h^\natural$. Given that $h^\natural$ is a remainder term, we wish to use the sewing map to estimate it. Applying δ to Eq. (3.12) and using (2.4), we get, for $0\leqslant s\leqslant u\leqslant t\leqslant T$:

$$\begin{aligned} \delta h^\natural_{sut} &= \delta H_i(Y)_{su}\mathbb{X}^{1,i}_{ut} - H_{2,ij}(Y_s)\mathbb{X}^{1,j}_{su}\mathbb{X}^{1,i}_{ut} + \delta H_{2,ij}(Y)_{su}\mathbb{X}^{2,ij}_{ut} \\ &= \left(\delta H_i(Y)_{su} - H_{2,ij}(Y_s)\mathbb{X}^{1,j}_{su}\right)\mathbb{X}^{1,i}_{ut} + \delta H_{2,ij}(Y)_{su}\mathbb{X}^{2,ij}_{ut}\,. \end{aligned} \tag{3.15}$$

We need to expand the quantity $\delta H_i(Y)_{su} - H_{2,ij}(Y_s)\mathbb{X}^{1,j}_{su}$ in (3.15), in order to show that $h^\natural$ is suitably small and depends in a very precise way on φ and on the difference $\varDelta := y-z$. In fact, by Taylor expansion and using (3.11) we get

$$\begin{aligned} \delta H_i(Y)_{su} - H_{2,ij}(Y_s)\mathbb{X}^{1,j}_{su} &= [\![\nabla H_i]\!](Y)_{su}\delta Y_{su} - H_{2,ij}(Y_s)\mathbb{X}^{1,j}_{su} \\ &= [\![\nabla H_i]\!](Y)_{su}F_j(Y_s)\mathbb{X}^{1,j}_{su} + [\![\nabla H_i]\!](Y)_{su}F_{2,jk}(Y_s)\mathbb{X}^{2,jk}_{su} + [\![\nabla H_i]\!](Y)_{su}Y^\natural_{su} \\ &\quad + [\![\nabla H_i]\!](Y)_{su}\delta M_{su} - H_{2,ij}(Y_s)\mathbb{X}^{1,j}_{su} \\ &= ([\![\nabla H_i]\!](Y)_{su} - \nabla H_i(Y_s))F_j(Y_s)\mathbb{X}^{1,j}_{su} + [\![\nabla H_i]\!](Y)_{su}F_{2,jk}(Y_s)\mathbb{X}^{2,jk}_{su} \\ &\quad + [\![\nabla H_i]\!](Y)_{su}Y^\natural_{su} + [\![\nabla H_i]\!](Y)_{su}\delta M_{su}\,, \end{aligned}$$

since $H_{2,ij}(Y) = \nabla H_i(Y)F_j(Y)$. Plugging this identity back into Eq. (3.15) and neglecting to write down explicitly the time indexes, we end up with:

$$\begin{aligned} \delta h^\natural &= ([\![\nabla H_i]\!](Y) - \nabla H_i(Y))F_j(Y)\mathbb{X}^{1,j}\mathbb{X}^{1,i} + [\![\nabla H_i]\!](Y)F_{2,jk}(Y)\mathbb{X}^{2,jk}\mathbb{X}^{1,i} \\ &\quad + [\![\nabla H_i]\!](Y)Y^\natural\mathbb{X}^{1,i} + [\![\nabla H_i]\!](Y)\,\delta M\,\mathbb{X}^{1,i} + \delta H_{2,ij}(Y)\mathbb{X}^{2,ij}. \end{aligned} \tag{3.16}$$

Using elementary algebraic manipulations, as well as the relation $H_{2,ij}(Y) = \nabla H_i(Y)F_j(Y)$, we obtain:

$$\begin{aligned} &([\![\nabla H_i]\!](Y)_{su} - \nabla H_i(Y_s))F_j(Y_s) \\ &\quad = ([\![H_{2,ij}]\!](Y)_{su} - H_{2,ij}(Y_s)) + ([\![\nabla H_i]\!](Y)_{su}F_j(Y_s) - [\![H_{2,ij}]\!](Y)_{su}) \\ &\quad = [\![[\![\nabla H_{2,ij}]\!]]\!](Y)_{su}\delta Y_{su} - [\![\nabla H_i(\cdot)[\![\nabla F_j(\cdot)]\!]]\!](Y)_{su}\delta Y_{su}\,, \end{aligned}$$

where the identity $[\![H_{2,ij}]\!](Y)_{su} - H_{2,ij}(Y_s) = [\![[\![\nabla H_{2,ij}]\!]]\!](Y)_{su}\delta Y_{su}$ directly stems from (2.2), and where we define:

$$[\![\nabla H_i(\cdot)[\![\nabla F_j(\cdot)]\!]]\!](Y)_{su} := \int_0^1 \nabla H_i(Y_s + \tau\delta Y_{su})\int_0^\tau \nabla F_j(Y_s + \sigma\delta Y_{su})\,\mathrm{d}\sigma\,\mathrm{d}\tau\,.$$

Therefore, we can rewrite Eq. (3.16) as

$$\begin{aligned} \delta h^\natural &= [\![[\![\nabla H_{2,ij}]\!]]\!](Y)\delta Y\,\mathbb{X}^{1,j}\mathbb{X}^{1,i} - [\![\nabla H_i(\cdot)[\![\nabla F_j(\cdot)]\!]]\!](Y)\delta Y\,\mathbb{X}^{1,j}\mathbb{X}^{1,i} \\ &\quad + [\![\nabla H_i]\!](Y)F_{2,jk}(Y)\mathbb{X}^{2,jk}\mathbb{X}^{1,i} + [\![\nabla H_i]\!](Y)Y^\natural\mathbb{X}^{1,i} \\ &\quad + [\![\nabla H_i]\!](Y)\delta M\,\mathbb{X}^{1,i} + [\![\nabla H_{2,ij}]\!](Y)\delta Y\,\mathbb{X}^{2,ij}\,. \end{aligned} \tag{3.17}$$

In order to further evaluate the rhs of this relation in terms of the test function φ, let us write explicit expressions for the gradients $\nabla H_i(Y)$ and $\nabla H_{2,ij}(Y)$ computed at $(a,b) \in \mathbb{R}^2$:

$$\nabla H_i(Y)(a,b) = \varphi''(y-z)(f_i(y)-f_i(z))(a-b) + \varphi'(y-z)(f_i'(y)-f_i'(z))a \\ + \varphi'(y-z)f_i'(z)(a-b)$$

and

$$\nabla H_{2,ij}(Y)(a,b) = \varphi''(y-z)(f_{2,ij}(y)-f_{2,ij}(z))(a-b) + \varphi'(y-z) \\ \times (f_{2,ij}'(y)a - f_{2,ij}'(z)b) \\ + \varphi'''(y-z)(f_i(y)-f_i(z))(f_j(y)-f_j(z))(a-b) \\ + 2\varphi''(y-z)(f_i(y)-f_i(z))(f_j'(y)a - f_j'(z)b)\,.$$

At this point, consider the quantity

$$|||\varphi||| := \sup_{y,z\in\mathbb{R}} (|\varphi'(y-z)| + |y-z||\varphi''(y-z)| + |y-z|^2|\varphi'''(y-z)|)\,. \tag{3.18}$$

Then, denoting by C_f any quantity that only depends on f, we have for all $1 \le i, j, k \le N$ and $s < t \in I$,

$$|[\![\nabla H_i]\!](Y)_{st}| + |[\![\nabla H_{2,ij}]\!](Y)_{st}| \leqslant C_f\, |||\varphi|||$$
$$|[\![\nabla H_i]\!](Y)_{st} F_{2,jk}(Y_s)| \leqslant C_f\, |||\varphi|||\, \|y-z\|_{\infty;[s,t]}$$
$$|[\![\nabla H_i(\cdot)[\![\nabla F_j(\cdot)]\!]]\!](Y)_{st}\delta Y_{st}| \leqslant C_f\, |||\varphi|||\, (\omega_\Delta(s,t)^{1/p} + \|y-z\|_{\infty;[s,t]}\omega_Y(s,t)^{1/p})$$
$$|[\![[\![\nabla H_{2,ij}]\!]]\!](Y)_{st}\delta Y_{st}| \leqslant C_f\, |||\varphi|||\, (\omega_\Delta(s,t)^{1/p} + \|y-z\|_{\infty;[s,t]}\omega_Y(s,t)^{1/p})$$
$$|[\![\nabla H_i]\!](Y)_{st} Y^\natural_{st}| \leqslant C_f\, |||\varphi|||\, (\omega_{\Delta,\natural}(s,t)^{3/p} + \|y-z\|_{\infty;[s,t]}\omega_{Y,\natural}(s,t)^{3/p})\,.$$

Going back to (3.17), we get that for all $s < u < t \in I$,

$$|\delta h^\natural_{sut}| \leqslant C_f\, |||\varphi|||\, \big[\omega_\star(s,t) + \omega_{\mathbb{X}}(s,t)^{2/3}\omega_\Delta(s,t)^{1/3} + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{\Delta,\natural}(s,t)\big]^{3/p}$$

where $\omega_\star$ is the control on I given for every $s < t \in I$ by

$$\omega_\star(s,t) := \omega_M(s,t)^{p/3}\omega_{\mathbb{X}}(s,t)^{1/3} + \|y-z\|^{p/3}_{\infty;[s,t]}\omega_{\mathbb{X},Y}(s,t)\,,$$

with

$$\omega_{\mathbb{X},Y}(s,t) := \omega_{\mathbb{X}}(s,t) + \omega_Y(s,t)^{1/3}\omega_{\mathbb{X}}(s,t)^{2/3} + \omega_{Y,\natural}(s,t)\,.$$

We are therefore in a position to apply the sewing lemma and conclude that for all $s < t \in I$,

$$|h^\natural_{st}| \leqslant C_{f,p}\, |||\varphi|||\, \big[\omega_\star(s,t) + \omega_{\mathbb{X}}(s,t)^{2/3}\omega_\Delta(s,t)^{1/3} + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{\Delta,\natural}(s,t)\big]^{3/p} \tag{3.19}$$

for some quantity $C_{f,p}$ that only depends on f and p.

Step 2: A first application. Our aim now is to apply the previous bound to the non-smooth function $\varphi(\xi) = \varphi_0(\xi) := |\xi|$. To this end, we will rely on the smooth approximation φ_ε defined for $\varepsilon > 0$ as $\varphi_\varepsilon(\xi) = \sqrt{\varepsilon^2 + |\xi|^2}$ for all $\xi \in \mathbb{R}$. Let us denote the associated objects with $h_\varepsilon, h^\natural_\varepsilon, H_{\varepsilon,i}, H_{\varepsilon,2,ij}, \ldots$. In this case

$$|\varphi_\varepsilon'(\xi)| \leqslant 1\,, \qquad |\varphi_\varepsilon''(\xi)| \leqslant 1/\sqrt{\varepsilon^2 + |\xi|^2}\,, \qquad |\varphi_\varepsilon'''(\xi)| \leqslant 3/(\varepsilon^2 + |\xi|^2)$$

and so, with the notation (3.18), we have the uniform estimate $|||\varphi_\varepsilon||| \leqslant 3$. Plugging this estimate into (3.19), we get:

$$|h^\natural_{\varepsilon,st}| \leqslant C_{f,p}\, \big[\omega_\star(s,t) + \omega_{\mathbb{X}}(s,t)^{2/3}\omega_\Delta(s,t)^{1/3} + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{\Delta,\natural}(s,t)\big]^{3/p}. \tag{3.20}$$

Furthermore, explicit elementary computations show that

$$\lim_{\varepsilon\to 0}\varphi_\varepsilon = |\cdot|, \qquad \lim_{\varepsilon\to 0}\varphi'_\varepsilon = \text{sign}, \quad \text{and} \quad \lim_{\varepsilon\to 0}\varphi''_\varepsilon = \delta_0, \tag{3.21}$$

where the first two limits are simple limits of functions, and the last one is understood in the weak sense. Notice that we also use the convention sign(0) = 0 above.

With those preliminaries in mind, let us take limits in (3.12). To begin with, as $\varepsilon \to 0$, a standard dominated convergence argument and relation (3.21) yield:

$$\int_s^t \varphi'_\varepsilon(y_u - z_u)\mathrm{d}(\mu_u - \nu_u) \to \int_s^t \text{sign}(y_u - z_u)\mathrm{d}(\mu_u - \nu_u). \tag{3.22}$$

In addition, owing to the fact that $y_t \geqslant 0$, $z_t \geqslant 0$, we have

$$\int_s^t \text{sign}(y_u - z_u)\mathrm{d}(\mu_u - \nu_u)$$
$$= \int_s^t \mathbf{1}_{\{y_u > z_u \geqslant 0\}} d\mu_u - \int_s^t \mathbf{1}_{\{z_u > y_u \geqslant 0\}} d\mu_u - \int_s^t \mathbf{1}_{\{y_u > z_u \geqslant 0\}} d\nu_u + \int_s^t \mathbf{1}_{\{z_u > y_u \geqslant 0\}} d\nu_u .$$

Hence, using the conditions $\mu_t = \int_0^t \mathbf{1}_{\{y_u=0\}}\mathrm{d}\mu_u$, $\nu_t = \int_0^t \mathbf{1}_{\{z_u=0\}}\mathrm{d}\nu_u$, we end up with:

$$\int_s^t \text{sign}(y_u - z_u)\mathrm{d}(\mu_u - \nu_u) = -\Big[\int_s^t \mathbf{1}_{\{z_u > y_u \geqslant 0\}} d\mu_u + \int_s^t \mathbf{1}_{\{y_u > z_u \geqslant 0\}} d\nu_u\Big]$$
$$= -\Big[\int_s^t \mathbf{1}_{\{z_u > y_u \geqslant 0\}} d(\mu_u + \nu_u) + \int_s^t \mathbf{1}_{\{y_u > z_u \geqslant 0\}} d(\mu_u + \nu_u)\Big]$$
$$= -\int_s^t \mathbf{1}_{\{y_u \neq z_u\}}\mathrm{d}(\mu_u + \nu_u) \;=\; -\,\omega_M(s,t) + \int_s^t \mathbf{1}_{\{y_u = z_u\}}\mathrm{d}(\mu_u + \nu_u)\,. \tag{3.23}$$

Recall that H_i and $H_{2,ij}$ are defined respectively by (3.13) and (3.14). Thanks to (3.21), it thus clearly holds that

$$\lim_{\varepsilon\to 0} H_{\varepsilon,i}(Y) = \Psi_i(Y)\,, \quad \text{and} \quad \lim_{\varepsilon\to 0} H_{\varepsilon,2,ij}(Y) = \Psi_{2,ij}(Y)\,, \tag{3.24}$$

where the limits are simple limits of functions and where we have:

$$\Psi_i(Y) := \text{sign}(y - z)(f_i(y) - f_i(z))\,, \qquad \Psi_{2,ij}(Y) := \text{sign}(y - z)(f_{2,ij}(y) - f_{2,ij}(z))\,.$$

Taking relations (3.22)–(3.24) into account, we can now take limits as $\varepsilon \to 0$ in (3.12). This ensures the convergence of the quantity $h^\natural_{\varepsilon,st}$ to some limit $\Phi^\natural_{st}$ (for all $s < t \in I$), and using (3.19) we get that the path $\Phi(Y) := |y - z|$ satisfies the following equation:

$$\delta\Phi(Y)_{st} \;=\; \Psi_i(Y_s)\mathbb{X}^{1,i}_{st} + \Psi_{2,ij}(Y_s)\mathbb{X}^{2,ij}_{st} - \omega_M(s,t)$$
$$+ \int_s^t \mathbf{1}_{\{y_u = z_u\}}\mathrm{d}(\mu_u + \nu_u) + \Phi^\natural_{st}\,. \tag{3.25}$$

Moreover, invoking relation (3.20), we have for all $s < t \in I$:

$$|\Phi^\natural_{st}| \leqslant C_{f,p}\big[\omega_\star(s,t) + \omega_{\mathbb{X}}(s,t)^{2/3}\omega_\Delta(s,t)^{1/3} + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{\Delta,\natural}(s,t)\big]^{3/p}\,. \tag{3.26}$$

Here and in the sequel, we denote by $C_{f,p}$ any quantity that only depends on f and p.

Step 3: Bounds for ω_Δ and $\omega_{\Delta,\natural}$. Let us now estimate ω_Δ and $\omega_{\Delta,\natural}$ in terms of $\omega_\star$. To this end, we can first use the fact that the path $\Delta := y - z$ is (obviously) given by $h(Y)$ with the choice $\varphi(\xi) := \xi$. In this case $h^\natural = y^\natural - z^\natural$, $\|\varphi\| = 1$, so that (3.19) becomes

$$|y^\natural_{st} - z^\natural_{st}| \leqslant C_{f,p}\left[\omega_\star(s,t) + \omega_{\mathbb{X}}(s,t)^{2/3}\omega_\Delta(s,t)^{1/3} + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{\Delta,\natural}(s,t)\right]^{3/p}$$

for all $s < t \in I$, and accordingly we have

$$\omega_{\Delta,\natural}(s,t) \leqslant C^{(1)}_{f,p}\big[\omega_\star(s,t) + \omega_{\mathbb{X}}(s,t)^{2/3}\omega_\Delta(s,t)^{1/3} + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{\Delta,\natural}(s,t)\big]$$

for some fixed constant $C^{(1)}_{f,p}$. As a result, for any interval $I_0 \subset I$ satisfying

$$C^{(1)}_{f,p}\omega_{\mathbb{X}}(I_0)^{1/3} \leqslant 1/2\,, \tag{3.27}$$

and for all $s < t \in I_0$, we have

$$\omega_{\Delta,\natural}(s,t) \leqslant 2C^{(1)}_{f,p}\big[\omega_\star(s,t) + \omega_{\mathbb{X}}(s,t)^{2/3}\omega_\Delta(s,t)^{1/3}\big]\,. \tag{3.28}$$

Besides, going back to the equation satisfied by Δ (again, take $\varphi(\xi) = \xi$ in (3.12)), we easily obtain that for all $s < t \in I$,

$$|\delta\Delta_{st}| \leqslant C_{f,p}\big[\|y-z\|^p_{\infty;[s,t]}\omega_{\mathbb{X}}(s,t) + \omega_M(s,t)^p + \omega_{\Delta,\natural}(s,t)^3\big]^{1/p}\,,$$

so that the following inequality holds true:

$$\omega_\Delta(s,t) \leqslant C_{f,p}\big[\|y-z\|^p_{\infty;[s,t]}\omega_{\mathbb{X}}(s,t) + \omega_M(s,t)^p + \omega_{\Delta,\natural}(s,t)^3\big].$$

Therefore for any interval $I_0 \subset I$ satisfying (3.27), we get by (3.28)

$$\omega_\Delta(s,t) \leqslant C^{(2)}_{f,p}\big[\|y-z\|^p_{\infty;[s,t]}\omega_{\mathbb{X}}(s,t) + \omega_M(s,t)^p + \omega_\star(s,t)^3 + \omega_{\mathbb{X}}(s,t)^2\omega_\Delta(s,t)\big]\,,$$

for some constant $C^{(2)}_{f,p}$. Finally, for any interval $I_0 \subset I$ satisfying both (3.27) and

$$C^{(2)}_{f,p}\omega_{\mathbb{X}}(I_0)^2 \leqslant 1/2\,, \tag{3.29}$$

and for all $s < t \in I_0$, we have

$$\omega_\Delta(s,t) \leqslant 2C^{(2)}_{f,p}\big[\|y-z\|^p_{\infty;[s,t]}\omega_{\mathbb{X}}(s,t) + \omega_M(s,t)^p + \omega_\star(s,t)^3\big]\,. \tag{3.30}$$

Step 4: Conclusion. By injecting (3.28) and (3.30) into (3.26), we can derive the following assertion: for any interval $I_0 \subset I$ satisfying (3.27) and (3.29), and all $s < t \in I_0$, it holds that

$$|\Phi^\natural_{st}| \leqslant C_{f,p}\big[\omega_\star(s,t)^{3/p} + \|y-z\|_{\infty;[s,t]}\omega_{\mathbb{X}}(s,t)^{3/p} + \omega_M(s,t)\omega_{\mathbb{X}}(s,t)^{2/p}\big]\,,$$

which, by the definition of $\omega_\star$, gives

$$|\Phi^\natural_{st}| \leqslant C_{f,p}\big[\|y-z\|_{\infty;[s,t]}\omega_{\mathbb{X},Y}(s,t)^{3/p} + \omega_M(s,t)\omega_{\mathbb{X}}(s,t)^{1/p}\big]\,.$$

Going back to Eq. (3.25) and observing that $\|y-z\|_{\infty;[s,t]} = \sup_{[s,t]}\Phi(Y)$, we obtain that for any such interval I_0 and for all $s < t \in I_0$,

$$\delta\Phi(Y)_{st} + \omega_M(s,t) \leqslant C_{f,p}\Big(\sup_{[s,t]}\Phi(Y) + \omega_M(s,t)\Big)\big[\omega_{\mathbb{X}}(s,t) + \omega_{\mathbb{X},Y}(s,t)^3\big]^{1/p} + \int_s^t \mathbf{1}_{\{y_u=z_u\}}(\mathrm{d}\mu_u + \mathrm{d}\nu_u).$$

We are finally in a position to apply the Rough Gronwall Lemma 2 with $\omega_1 := \omega_{\mathbb{X}} + \omega^3_{\mathbb{X},Y}$ and $\omega_2(s,t) := \int_s^t \mathbf{1}_{\{y_u=z_u\}}(\mathrm{d}\mu_u + \mathrm{d}\nu_u)$, and assert that for every $s < t \in I_0$,

$$\sup_{[s,t]}\Phi(Y) + \omega_M(s,t) \leqslant C_{f,p,\mathbb{X},Y}\Big[\Phi(Y_s) + \int_s^t \mathbf{1}_{\{y_u=z_u\}}(\mathrm{d}\mu_u + \mathrm{d}\nu_u)\Big]\,,$$

that is

$$\sup_{r\in[s,t]} |y_r - z_r| + \omega_M(s,t) \leqslant C_{f,p,\mathbb{X},Y}\left[|y_s - z_s| + \int_s^t \mathbf{1}_{\{y_u=z_u\}}(\mathrm{d}\mu_u + \mathrm{d}\nu_u)\right], \tag{3.31}$$

for some constant $C_{f,p,\mathbb{X},Y}$.

Assume now that $[s,t]$ is an interval where $y \neq z$ in (s,t) but $y(s) = z(s)$. Then relation (3.31) yields

$$\sup_{r\in[s,t]} |y_r - z_r| + \omega_M(s,t) \leqslant 0,$$

which implies that $\sup_{r\in[s,t]}|y_r - z_r| = 0$ everywhere so we find a contradiction and such interval cannot exist. This concludes the proof of uniqueness. □

Remark 10. The key point of the above proof thus consists in the close follow-up of the "measure" control ω_M throughout the reasoning. The argument thus differs from the contraction strategy that usually prevails in rough analysis, and in particular regularity properties of the solution with respect to the driving rough path $\mathbb{X}$ can no longer be obtained as an almost-straightforward consequence of the procedure. Studying this regularity issue would actually mean facing the same non-continuity problems as those raised in Section 3.1 (recall the involvement of the difference $\mu - \nu$ in decomposition (3.8)), and accordingly this question seems to be out of reach for the moment.

Remark 11. As the reader can see, one of the cornerstones of our computations lies in the possibility to expand the integral $\int_s^t \mathrm{sign}(y_u - z_u)\mathrm{d}(\mu_u - \nu_u)$ as in (3.23), that is as the sum of $-\omega_M(s,t)$ and an integral from s to t that vanishes as soon as $y_r \neq z_r$ for every $r \in [s,t]$. Unfortunately, when turning to more general multidimensional reflection domain (see the forthcoming Definition 13 for a description of the rough equation in this context), a similar decomposition is certainly much more difficult to exhibit (if this exists), and therefore we currently fail to extend our arguments to multidimensional domains. To be more specific, using the notations of Definition 13, it is not hard to see that the d-dimensional analog of the above integral $\int_s^t \mathrm{sign}(y_u - z_u)\mathrm{d}(\mu_u - \nu_u)$ is given by

$$\int_s^t \frac{1}{|y_u - z_u|}\langle y_u - z_u, \mathbf{n}_y(u)\mathrm{d}|\mu|_u - \mathbf{n}_z(u)\mathrm{d}|\nu|_u\rangle,$$

and this expression happens to be much less flexible as soon as $d \geq 2$.

4. Existence

Although the existence issue for Eq. (1.2) in very general domains has been considered in [1,2], we wish to present here a self-contained and hopefully simpler treatment in our 1-dimensional setting. We then briefly sketch what is needed in order to extend our considerations to higher dimensional situations.

4.1. The one-dimensional case

Our existence result in the domain $\mathbb{R}_{\geqslant 0}$ can be read as follows.

Theorem 12. *Let $T > 0$, $2 \geqslant p < 3$ and $a > 0$. If $f \in \mathcal{C}_b^2(\mathbb{R}; \mathcal{L}(\mathbb{R}^N, \mathbb{R}))$ and $\mathbb{X}$ is a geometric N-dimensional p-variation rough path in the sense of Definition 1, then Problem (1.2) admits at least one solution (y, m) on $[0, T]$ with initial condition a.*

Just as in [1,2], our strategy towards existence will appeal to some a priori bound on the measure term of the (approximated) equation. The result more generally applies to the so-called Skorohod problem and it can be read as follows in the one-dimensional case.

Lemma 3. *Let g be a continuous $\mathbb{R}$-valued path defined on some interval $I = [\ell_1, \ell_2]$. Consider a solution $(y, m) \in \mathcal{C}(I; \mathbb{R}_{\geqslant 0}) \times V_1^1(I; \mathbb{R}_{\geqslant 0})$ of the Skorohod problem associated with g in the domain $\mathbb{R}_{\geqslant 0}$, as given in Definition 5. Then for all $s < t \in I$ it holds that*

$$\delta m_{st} \leqslant 8 \, \|g\|_{0,[s,t]} \, , \tag{4.1}$$

where $\|g\|_{0,[s,t]} := \sup_{s \leqslant u < v \leqslant t} |\delta g_{uv}|$.

The proof of (4.1) can be easily derived from the arguments of the proof of [3, Lemma 2.3] (namely, the same arguments as those leading to the forthcoming general Lemma 4). Let us provide some details though, not least to give the non-initiated reader an insight on how the specific constraints of the reflecting problem can be exploited.

Proof of Lemma 3. For all $s < t \in I$, one has

$$\begin{aligned} |\delta y_{st}|^2 \; = \; |\delta g_{st}|^2 + |\delta m_{st}|^2 + 2\delta g_{st} \delta m_{st} &= |\delta g_{st}|^2 + 2 \int_s^t \delta m_{su} \, \mathrm{d}m_u + 2 \int_s^t \delta g_{st} \, \mathrm{d}m_u \\ &= |\delta g_{st}|^2 + 2 \int_s^t \delta y_{su} \, \mathrm{d}m_u + 2 \int_s^t \delta g_{ut} \, \mathrm{d}m_u \, , \end{aligned}$$

where we have just used the fact that $\delta m_{su} = \delta y_{su} - \delta g_{su}$ for the last identity. Moreover, since $\int_s^t y_u \mathrm{d}m_u = \int_s^t y_u \mathbf{1}_{\{y_u = 0\}} \mathrm{d}m_u = 0$ and $y_s \geqslant 0$, we get:

$$|\delta y_{st}|^2 \leqslant |\delta g_{st}|^2 + 2 \int_s^t \delta g_{ut} \, \mathrm{d}m_u \, .$$

Therefore,

$$|\delta y_{st}|^2 \leqslant \|g\|_{0,[s,t]}^2 + 2 \, \|g\|_{0,[s,t]} \, \delta m_{st} \leqslant 5 \, \|g\|_{0,[s,t]}^2 + \frac{1}{4} \, |\delta m_{st}|^2 \, ,$$

and so $\|y\|_{0,[s,t]} \leqslant 3 \, \|g\|_{0,[s,t]} + \frac{1}{2} \, \delta m_{st}$. Finally,

$$\delta m_{st} \leqslant \|y\|_{0,[s,t]} + \|g\|_{0,[s,t]} \leqslant 4 \, \|g\|_{0,[s,t]} + \frac{1}{2} \, \delta m_{st} \, ,$$

and the result follows. □

Proof of Theorem 12. We start from a sequence of smooth rough paths $\mathbb{X}^\varepsilon$ converging to $\mathbb{X}$ as $\varepsilon \to 0$, in the space of continuous p-variation geometric rough paths. We can then find a regular control $\omega_{\mathbb{X}}$ such that, for all $s, t \in [0, T]$,

$$|\mathbb{X}_{st}^1| + |\mathbb{X}_{st}^2|^{1/2} \leqslant \omega_{\mathbb{X}}(s,t)^{1/p}, \qquad \sup_{\varepsilon > 0} (|\mathbb{X}_{st}^{\varepsilon,1}| + |\mathbb{X}_{st}^{\varepsilon,2}|^{1/2}) \leqslant \omega_{\mathbb{X}}(s,t)^{1/p} \, .$$

For every $\varepsilon > 0$, let X^ε be the path which corresponds to $\mathbb{X}^\varepsilon$ and consider the solution y^ε to reflected ODEs starting from y_0:

$$\begin{cases} \mathrm{d}y^\varepsilon_t = f(y^\varepsilon_t)\,\mathrm{d}X^\varepsilon_t + \mathrm{d}m^\varepsilon_t \\ y^\varepsilon_0 = y_0 \quad \text{and} \quad m^\varepsilon_t = \int_0^t \mathbf{1}_{\{y^\varepsilon_u = 0\}} \mathrm{d}m^\varepsilon_u \end{cases} .$$

Recall that the existence (and uniqueness) of such a solution is a standard result, based on the Lipschitz regularity of the Skorohod map with respect to the supremum norm. Then by Taylor expansion it is not difficult to show that these solutions correspond to rough solutions $(y^\varepsilon, m^\varepsilon)$ in the sense of (2.5), namely:

$$\delta y^\varepsilon_{st} = f_i(y^\varepsilon_s)\mathbb{X}^{\varepsilon,1,i}_{st} + f_{2,ij}(y^\varepsilon_s)\mathbb{X}^{\varepsilon,2,ij}_{st} + \delta m^\varepsilon_{st} + y^{\varepsilon,\natural}_{st} \qquad s,t \in [0,T] \tag{4.2}$$

where $y^{\varepsilon,\natural} \in V_2^{p/3}([0,T];\mathbb{R})$. Let us set from now on

$$\omega_{y^\varepsilon}(s,t) := \|y^\varepsilon\|^p_{\bar{V}_1^p([s,t];E)} \quad , \quad \omega_{\varepsilon,\natural}(s,t) := \|y^{\varepsilon,\natural}\|^{p/3}_{\bar{V}_2^{p/3}([s,t];E)} ,$$

$$\omega_{m^\varepsilon}(s,t) := \|m^\varepsilon\|_{\bar{V}_1^1([s,t];E)} = \delta m^\varepsilon_{st} = m^\varepsilon([s,t]) ,$$

and observe that from Eq. (4.2) we have

$$|\delta y^\varepsilon_{st}| \leqslant C_f(\omega_{\mathbb{X}}(s,t)^{1/p} + \omega_{\mathbb{X}}(s,t)^{2/p}) + \omega_{m^\varepsilon}(s,t) + \omega_{\varepsilon,\natural}(s,t)^{3/p} . \tag{4.3}$$

Here and in the sequel, we denote by C_f, resp. $C_{f,p}$, any quantity that only depends on f, resp. (f,p).

Step 1: Bounds on the approximate solutions. We would like to pass to the limit in ε and obtain solutions of the limiting problem. In order to do so we need uniform estimates for $y^{\varepsilon,\natural}_{st}$. They are obtained via an application of the sewing map.

To this end, one can proceed as in the proof of Theorem 9, Step 1. Specifically, we can just replace Y by y, H by f and H_2 by f_2 in relation (3.15). We then repeat all the steps up to relation (3.17), which yields the following relation for $\delta y^{\varepsilon,\natural}$ (for more simplicity, we neglect to write down the time indexes explicitly):

$$\begin{aligned} \delta y^{\varepsilon,\natural} = {} & [\![[\![\nabla f_{2,ij}]\!]]\!](y^\varepsilon)\delta y^\varepsilon\, \mathbb{X}^{\varepsilon,1,j}\mathbb{X}^{\varepsilon,1,i} - [\![\nabla f_i(\cdot)[\![\nabla f_j(\cdot)]\!]]\!](y^\varepsilon)\delta y^\varepsilon\, \mathbb{X}^{\varepsilon,1,j}\mathbb{X}^{\varepsilon,1,i} \\ & + [\![\nabla f_i]\!](y^\varepsilon) f_{2,jk}(y^\varepsilon)\mathbb{X}^{\varepsilon,2,jk}\mathbb{X}^{\varepsilon,1,i} + [\![\nabla f_i]\!](y^\varepsilon) y^{\varepsilon,\natural}\mathbb{X}^{\varepsilon,1,i} \\ & + [\![\nabla f_i]\!](y^\varepsilon)\delta m^\varepsilon\, \mathbb{X}^{\varepsilon,1,i} + [\![\nabla f_{2,ij}]\!](y^\varepsilon)\delta y^\varepsilon\, \mathbb{X}^{\varepsilon,2,ij} . \end{aligned} \tag{4.4}$$

Combining this expansion with (4.3), we get, for every interval $I \subset [0,T]$ such that $\omega_{\mathbb{X}}(I) \leqslant 1$ and all $s < u < t \in I$,

$$\begin{aligned} |\delta y^{\varepsilon,\natural}_{sut}| & \leqslant C_f\big[\omega_{y^\varepsilon}(s,t)^{1/p}\omega_{\mathbb{X}}(s,t)^{2/p} + (\omega_{\mathbb{X}}(s,t)^{2/p} + \omega_{m^\varepsilon}(s,t) \\ & \quad + \omega_{\varepsilon,\natural}(s,t)^{3/p})\omega_{\mathbb{X}}(s,t)^{1/p}\big] \\ & \leqslant C_{f,p}\big[\omega_{\mathbb{X}}(s,t) + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{m^\varepsilon}(s,t)^{p/3} + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{\varepsilon,\natural}(s,t)\big]^{3/p} . \end{aligned}$$

We are therefore in a position to apply the sewing lemma and assert that for every interval $I \subset [0,T]$ such that $\omega_{\mathbb{X}}(I) \leqslant 1$ and all $s < t \in I$, we have

$$|y^{\varepsilon,\natural}_{st}| \leqslant C_{f,p}\big[\omega_{\mathbb{X}}(s,t) + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{m^\varepsilon}(s,t)^{p/3} + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{\varepsilon,\natural}(s,t)\big]^{3/p} ,$$

which immediately entails that

$$\omega_{\varepsilon,\natural}(s,t) \leqslant C^{(1)}_{f,p}\big[\omega_{\mathbb{X}}(s,t) + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{m^\varepsilon}(s,t)^{p/3} + \omega_{\mathbb{X}}(I)^{1/3}\omega_{\varepsilon,\natural}(s,t)\big] ,$$

for some constant $C^{(1)}_{f,p}$. As a result, for every interval $I \subset [0, T]$ such that

$$\omega_{\mathbb{X}}(I) \leqslant 1 \quad \text{and} \quad C^{(1)}_{f,p}\omega_{\mathbb{X}}(I)^{1/3} \leqslant 1/2 , \tag{4.5}$$

one has

$$\omega_{\varepsilon,\natural}(s,t) \leqslant 2C^{(1)}_{f,p}\big[\omega_{\mathbb{X}}(s,t) + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_{m^\varepsilon}(s,t)^{p/3}\big] , \qquad s < t \in I. \tag{4.6}$$

Step 2: Control of the approximate measures. Consider the path $g^\varepsilon : [0, T] \to \mathbb{R}$ defined as $g^\varepsilon_t := y^\varepsilon_t - m^\varepsilon_t$, and observe that $(y^\varepsilon, m^\varepsilon)$ is then a solution of the Skorohod problem in $\mathbb{R}_{\geqslant 0}$ associated with g^ε, in the sense of Lemma 3. Therefore, by (4.1), it holds that

$$\omega_{m^\varepsilon}(s,t) \leqslant 8\, \|g^\varepsilon\|_{0,[s,t]} . \tag{4.7}$$

On the other hand, from Eq. (4.2), we have

$$\delta g^\varepsilon_{st} = f_i(y^\varepsilon_s)\mathbb{X}^{\varepsilon,1,i}_{st} + f_{2,ij}(y^\varepsilon_s)\mathbb{X}^{\varepsilon,2,ij}_{st} + y^{\varepsilon,\natural}_{st} , \qquad 0 \leqslant s \leqslant t \leqslant T ,$$

and so

$$\|g^\varepsilon\|_{0,[s,t]} \leqslant C_f\big[\omega_{\mathbb{X}}(s,t)^{1/p} + \omega_{\mathbb{X}}(s,t)^{2/p} + \omega_{\varepsilon,\natural}(s,t)^{3/p}\big] . \tag{4.8}$$

Injecting successively (4.8) and (4.6) into (4.7) yields that for every interval I satisfying the conditions in (4.5) and every $s < t \in I$,

$$\omega_{m^\varepsilon}(s,t) \leqslant C^{(2)}_{f,p}\big[\omega_{\mathbb{X}}(s,t)^{1/p} + \omega_{\mathbb{X}}(I)^{1/p}\omega_{m^\varepsilon}(s,t)\big] ,$$

for some constant $C^{(2)}_{f,p}$, and so, if we assume in addition that

$$C^{(2)}_{f,p}\omega_{\mathbb{X}}(I)^{1/p} \leqslant 1/2 , \tag{4.9}$$

we obtain

$$\omega_{m^\varepsilon}(s,t) \leqslant 2C^{(2)}_{f,p}\omega_{\mathbb{X}}(s,t)^{1/p} , \qquad s < t \in I . \tag{4.10}$$

From here we can easily conclude that

$$\omega_{m^\varepsilon}([0,T]) \leqslant C_{f,p,\mathbb{X}} \tag{4.11}$$

for some quantity $C_{f,p,\mathbb{X}}$ independent from ε.

Step 3: Passage to the limit for the measure. With all the bounds in place we can now pass to the limit as $\varepsilon \to 0$ via subsequences. We start with the measure. Using (4.11) we can assert that there exists a weakly convergent subsequence of measures $(m^{\varepsilon(k)})_{k\geqslant 1}$ on $[0, T]$, and we will denote by m their limit. Then it holds that

$$m([0,t]) = \lim_k m^{\varepsilon(k)}([0,t]) \qquad t \in \mathfrak{C} \tag{4.12}$$

where $\mathfrak{C} \subseteq [0, T]$ is the (dense) set of continuity points of the function $t \mapsto m([0, t])$. Now consider any interval I satisfying both the conditions in (4.5) and in (4.9), and for $s < t \in I$, introduce a sequence s_ℓ, resp. t_ℓ, of points in $\mathfrak{C}$ decreasing to s, resp. increasing to t, and such that $s_k < t_k$. Using (4.10), we have

$$m(]s_\ell, t_\ell]) = \lim_k m^{\varepsilon(k)}(]s_\ell, t_\ell]) \leqslant C_{f,p}\, \omega_{\mathbb{X}}(s,t)^{1/p} ,$$

and so $m([s,t]) \leqslant C_{f,p}\,\omega_{\mathbb{X}}(s,t)^{1/p}$, which proves that the function $m_t := m([0,t[)$ is continuous and accordingly that $m \in V_1^1([0,T];\mathbb{R}_{\geqslant 0})$, as expected.

Step 4: Passage to the limit for the path. Consider the subsequence $(y^{\varepsilon(k)}, m^{\varepsilon(k)})_k$ as defined in the previous step. Using (4.3) we have, for all $s,t \in [0,T]$,

$$\limsup_k |\delta y^{\varepsilon(k)}_{st}| \leqslant C_f(\omega_{\mathbb{X}}(s,t)^{1/p} + \omega_{\mathbb{X}}(s,t)^{2/p}) + \omega_m(s,t) + \limsup_k \omega_{\varepsilon(k),\natural}(s,t)^{3/p} ,$$

and for every interval I satisfying both the conditions in (4.5) and in (4.9) (we denote $\mathcal{J}$ the family of such intervals), we have

$$\limsup_k \omega_{\varepsilon(k),\natural}(s,t) \leqslant C_{f,p}\big[\omega_{\mathbb{X}}(s,t) + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_m(s,t)^{p/3}\big] , \quad s<t \in I .$$

From this bound we can choose a further subsequence, still called $(y^{\varepsilon(k)}, m^{\varepsilon(k)})_k$ so that $y^{\varepsilon(k)} \to y$ in $C([0,T];\mathbb{R}_{\geqslant 0})$. It is easy now to pass to the limit in Eq. (4.2) and conclude that there exists a map $y^\natural : \Delta_{[0,T]} \to \mathbb{R}$ such that

$$\delta y = f_i(y)\mathbb{X}^{1,i} + f_{2,ij}(y)\mathbb{X}^{2,ij} + \delta m + y^\natural ,$$

and

$$|y^\natural_{st}| \leqslant C_{f,p}\big[\omega_{\mathbb{X}}(s,t) + \omega_{\mathbb{X}}(s,t)^{1/3}\omega_m(s,t)^{p/3}\big]^{3/p}, \qquad s<t \in I \in \mathcal{J}.$$

The fact that $m_t = \int_0^t \mathbf{1}_{\{y_u=0\}}\,\mathrm{d}m_u$ (for all t) follows immediately from the relation $m^\varepsilon_t = \int_0^t \mathbf{1}_{\{y^\varepsilon_u=0\}}\,\mathrm{d}m^\varepsilon_u$, and finally the pair (y,m) does define a solution to the RRDE (2.5). □

4.2. *Generalization to multidimensional domains*

We conclude this study with a few details on possible extensions of the previous arguments (towards existence) to more general multidimensional domains. Together, these results will thus offer a simplification of some of the arguments and topologies used in [1,2].

Let us first extend Definition 2 of a reflected rough solution to more general settings, along the classical approach of the reflected problem. Let $D \subset \mathbb{R}^d$ be a connected domain and for every $x \in \partial D$, denote by $\mathcal{N}_x$ the set of inward unit normal vectors at x, that is

$$\mathcal{N}_x := \cup_{r>0}\mathcal{N}_{x,r} , \quad \mathcal{N}_{x,r} := \{n \in \mathbb{R}^d : |n| = 1,\ B(x-rn,r)\cap D = \emptyset\}$$

where $B(z,r) := \{y \in \mathbb{R}^d : |y-z| < r\}$, for $z \in \mathbb{R}^d$ and $r>0$.

Definition 13. Given a time $T>0$, an element $a \in D$, a differentiable function $f : \mathbb{R}^d \to \mathcal{L}(\mathbb{R}^N;\mathbb{R}^d)$ and a p-variation N-dimensional rough path $\mathbb{X}$ with $2 \leqslant p < 3$, a pair $(y,m) \in V_1^p([0,T];D) \times V_1^1([0,T];\mathbb{R}^d)$ is said to solve the reflected rough equation in D with initial condition a if there exists a 2-index map $y^\natural \in V_{2,\mathrm{loc}}^{p/3}([0,T];\mathbb{R}^d)$ such that for all $s,t \in [0,T]$, we have

$$\begin{cases} \delta y_{st} = f_i(y_s)\mathbb{X}^{1,i}_{st} + f_{2,ij}(y_s)\mathbb{X}^{2,ij}_{st} + \delta m_{st} + y^\natural_{st} \\ \quad y_0 = a \quad \text{and} \quad m_t = \displaystyle\int_0^t \mathbf{1}_{\{y_u\in\partial D\}} n_{y_u}\,\mathrm{d}|m|_u \end{cases} , \tag{4.13}$$

where we have set $f_{2,ij}(\xi) := \nabla f_i(\xi) f_j(\xi)$, $|m|_t := \|m\|_{\bar{V}_1^1([0,t];\mathbb{R}^d)}$ and for each $y \in \partial D$, $n_y \in \mathcal{N}_y$.

The existence of a solution for (4.13) can actually be derived from the same arguments as in the one-dimensional situation. The only step of the procedure needing for a revision is the so-called Step 2, since it involves the a priori bound (4.1) which is specific to the one-dimensional Skorohod problem. To this end, we shall exploit the following (sophisticated) substitute, borrowed from [1, Lemma 2.2].

Lemma 4. *Let $D \subset \mathbb{R}^d$ be connected domain that satisfies the two following assumptions:*

(A) *There exists a constant $r_0 > 0$ such that $\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset$ for any $x \in \partial D$;*

(B) *There exist constants $\delta_0 > 0$ and $\beta \geqslant 1$ satisfying: for every $x \in \partial D$, there exists a unit vector l_x such that $\langle l_x, n\rangle \geqslant 1/\beta$ for every $n \in \cup_{y\in B(x,\delta_0)\cap\partial D}\mathcal{N}_y$.*

Let $g \in V_1^p(I;\mathbb{R}^d)$, for some interval $I = [\ell_1, \ell_2]$, such that $g_{\ell_1} \in D$, and consider a solution $(y, m) \in \mathcal{C}(I; D) \times V_1^1(I;\mathbb{R}^d)$ of the Skorohod problem associated with g in the domain D, that is (y, m) satisfies for all $s < t \in I$

$$
\begin{cases} \delta y_{st} = \delta g_{st} + \delta m_{st} \ , \\ y_{\ell_1} = g_{\ell_1} \ , \ m_t = \displaystyle\int_0^t \mathbf{1}_{\{y_u=0\}} n_{y_u} \mathrm{d}|m|_u \end{cases} ,
$$

where $|m|_t := \|m\|_{\bar{V}_1^1([0,t];\mathbb{R}^d)}$ and for each $y \in \partial D$, $n_y \in \mathcal{N}_y$. Then for all $s < t \in I$ it holds that

$$
\|m\|_{V_1^1([s,t])} \leqslant C_1[e^{pC_2(1+\|g\|_{0,[s,t]})}\|g\|_{\bar{V}_1^p([s,t])} + 1](e^{C_2(1+\|g\|_{0,[s,t]})} + 1)\|g\|_{0,[s,t]} \ , \tag{4.14}
$$

where C_1, C_2 are constants depending only on the domain and $\|g\|_{0,[s,t]} := \sup_{s\leqslant u<v\leqslant t}|\delta g_{uv}|$.

Theorem 14. *Let $D \subset \mathbb{R}^d$ be a connected domain satisfying Conditions **(A)** and **(B)** of Lemma 4. Then there exists at least one solution (y, m) to the reflection problem (4.13) in D.*

Remark 15. Of course, Theorem 12 can retrospectively be obtained as a particular application of Theorem 14. Nevertheless, we have found it important, for didactic reasons, to first provide a full and self-contained treatment of the one-dimensional situation.

Proof of Theorem 14. As mentioned above, and apart from minor changes of notation due to the vectorial character of the equation, Steps 1, 3 and 4 of the proof of Theorem 12 can be readily transposed to this setting, and thus we only need to focus on the extension of Step 2.

In fact, with the same notations as in the one-dimensional proof and considering only those intervals $I = [s_0, t_0]$ satisfying the two conditions in (4.5), we have by (4.14), (4.8) and (4.6) that for all $s < t \in I$,

$$
\omega_{m^\varepsilon}(s,t) \leqslant \Psi(\omega_{g^\varepsilon,\varepsilon}(s,t)) \leqslant \Psi(C_{f,p}(\omega_{\mathbb{X}}(s,t) + \omega_{\mathbb{X}}(s,t)\omega_{m^\varepsilon}(s,t)^p)), \tag{4.15}
$$

where

$$
\Psi(\lambda) := C_1[e^{pC_2(1+\lambda^{1/p})}\lambda + 1](e^{C_2(1+\lambda^{1/p})} + 1)\lambda^{1/p}
$$

and $C_{f,p}$ is a fixed constant. Eq. (4.15) implies in particular that the control ω_{m^ε} is regular if $\omega_{\mathbb{X}}$ is regular, which is our case. Let G_I be the function

$$
G_I(\lambda) := \Psi(C_{f,p}(1 + \omega_{\mathbb{X}}(I)\lambda^p)).
$$

By choosing t_0 near to s_0 we can have both (4.5) and $G_I(3G_I(0)) \leqslant 2G_I(0)$, since $\omega_{\mathbb{X}}(I) \to 0$ as $t_0 \downarrow s_0$. This choice of t_0 depends only on $\omega_{\mathbb{X}}$ and $G_I(0)$ (which is actually independent of I).

Now Eq. (4.15) implies also that

$$\omega_{m^\varepsilon}(s_0, t) \leqslant G_I(\omega_{m^\varepsilon}(s_0, t)) , \qquad t \in I .$$

We want to establish that $\omega_{m^\varepsilon}(I) \leqslant 2G_I(0)$ and to this end we can apply the method of continuity. Let $\mathcal{A} \subseteq I$ be the set of $t \in I$ such that the property $\omega_{m^\varepsilon}(s_0, t) \leqslant 2G_I(0)$ is true. Note that $[s_0, s_0 + \delta] \subseteq \mathcal{A}$ for δ small enough by the continuity of the control ω_{m^ε}. Moreover $\mathcal{A}$ is closed in I since if $(t_n)_n \subseteq \mathcal{A}$ is a sequence converging to t_* then, again by regularity of ω_{m^ε} we have $\omega_{m^\varepsilon}(s_0, t_*) = \lim_n \omega_{m^\varepsilon}(s_0, t_n) \leqslant 2G_I(0)$. Finally $\mathcal{A}$ is also open in I since if $t_* \in \mathcal{A}$ then for δ small enough $\omega_{m^\varepsilon}(s_0, t) \leqslant 3G_I(0)$ for all $t \in (t_* - \delta, t_* + \delta) \cap I$. But then our choice of I guarantee that

$$\omega_{m^\varepsilon}(s_0, t) \leqslant G_I(\omega_{m^\varepsilon}(s_0, t)) \leqslant G_I(3G_I(0)) \leqslant 2G_I(0) , \qquad t \in (t_* - \delta, t_* + \delta) \cap I ,$$

from which we see that $(t_* - \delta, t_* + \delta) \cap I \subseteq \mathcal{A}$ and that $\mathcal{A}$ is open in I. We can then conclude that $\mathcal{A} = I$, namely that $\omega_{m^\varepsilon}(I) \leqslant 2G_I(0)$. Now we can reason in this way for any nonempty interval $I_{t,\delta} = (t - \delta, t + \delta) \cap [0, T]$ by choosing $\delta = \delta(t) > 0$ small enough to satisfy our conditions. In this way we construct an open covering $\cup_t I_{t,\delta(t)}$ of $[0, T]$ from which we can extract a finite covering $(I_k)_k$ independent of ε and such that

$$\omega_{m^\varepsilon}(I_k) \leqslant 2G_I(0)$$

for all I_k in the covering. This bound provides us with the expected substitute for (4.11), and we can then follow Steps 3 and 4 of the proof of Theorem 12 to get the conclusion. □

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References

[1] S. Aida, Reflected rough differential equations, Stochastic Process. Appl. 125 (9) (2015) 3570–3595.
[2] S. Aida, Rough differential equations containing path-dependent bounded variation terms. arXiv:1608.03083 (2016) http://arxiv.org/abs/1608.03083.
[3] S. Aida, K. Sasaki, Wong–Zakai approximation of solutions to reflecting stochastic differential equations on domains in Euclidean spaces, Stochastic Process. Appl. 123 (10) (2013) 3800–3827.
[4] A.M. Davie, Differential Equations Driven by Rough Paths: An Approach via Discrete Approximation, Appl. Math. Res. Express. AMRX 2 (2007) abm009, 40.
[5] A. Deya, M. Gubinelli, M. Hofmanová, S. Tindel, A priori estimates for rough PDEs with application to rough conservation laws. arXiv:1604.00437 (2016) http://arxiv.org/abs/1604.00437.
[6] A. Deya, S. Tindel, Rough Volterra equations 2: Convolutional generalized integrals, Stochastic Process. Appl. 121 (8) (2011) 1864–1899.
[7] A. Falkowski, L. Słomiński, Stochastic differential equations with constraints driven by processes with bounded p-variation, Probab. Math. Statist. 35 (2) (2015) 343–365 (2015).
[8] A. Falkowski, L. Slominski, Sweeping processes with stochastic perturbations generated by a fractional Brownian motion. Arxiv preprint.
[9] M. Ferrante, C. Rovira, Stochastic differential equations with non-negativity constraints driven by fractional Brownian motion, J. Evol. Equ. 13 (3) (2013) 617–632.
[10] P.K. Friz, M. Hairer, A Course on Rough Paths: With an Introduction to Regularity Structures, Springer, 2014.
[11] P.K. Friz, N.B. Victoir, Multidimensional Stochastic Processes as Rough Paths: Theory and Applications, Cambridge University Press, 2010.
[12] M. Gubinelli, Controlling rough paths, J. Funct. Anal. 216 (1) (2004) 86–140.
[13] M. Gubinelli, N. Perkowski, P. Imkeller, Paracontrolled distributions and singular PDEs, Forum Math., Pi 3 (6) (2015).
[14] M. Hairer, Rough stochastic PDEs, Commun. Pure Appl. Math. 64 (11) (2011) 1547–1585.

[15] M. Hairer, Solving the KPZ equation, Ann. of Math. (2) 178 (2) (2013) 559–664.
[16] M. Hairer, A theory of regularity structures, Invent. Math. 198 (2) (2014) 269–504.
[17] D. Kelly, I. Melbourne, Deterministic Homogenization for Fast-Slow Systems with Chaotic Noise. arXiv:1409.5748 (2014). http://arxiv.org/abs/1409.5748.
[18] P.L. Lions, A.S. Sznitman, Stochastic differential equations with reflecting boundary conditions, Comm. Pure Appl. Math. 37 (4) (1984) 511–537.
[19] T.J. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoam. 14 (2) (1998) 215–310.
[20] T.J. Lyons, Michael Caruana, Thierry Lévy, Differential Equations Driven By Rough Paths, in: Lecture Notes in Mathematics, vol. 1908, Springer, Berlin, 2007. Lectures from the 34th Summer School on Probability Theory held in Saint-Flour, July (2004) 6–24, With an introduction concerning the Summer School by Jean Picard.
[21] T. Lyons, Z. Qian, System Control and Rough Paths, in: Oxford Mathematical Monographs, Oxford Science Publications. Oxford University Press, Oxford, 2002, p. x+216.
[22] A. Neuenkirch, I. Nourdin, Samy Tindel, Delay equations driven by rough paths, Electron. J. Probab. 13 (67) (2008) 2031–2068.
[23] Y. Saisho, Stochastic differential equations for multi-dimensional domain with reflecting boundary, Probab. Theory Related Fields 74 (3) (1987) 455–477.
[24] A.V. Skorohod, Stochastic equations for diffusion processes in a bounded region 1, 2, Theor. Veroyatn. Primen. 6 (1961) 264–274, 7 (1962), 3-23.
[25] H. Tanaka, Stochastic differential equations with reflecting boundary condition in convex regions, Hiroshima Math. J. 9 (1979) 163–177.