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A CONSTRUCTION OF THE ROUGH PATH ABOVE FRACTIONAL BROWNIAN MOTION USING VOLTERRA'S REPRESENTATION

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This note is devoted to construct a rough path above a multidimensional fractional Brownian motion B with any Hurst parameter $H \in (0,1)$, by means of its representation as a Volterra Gaussian process. This approach yields some algebraic and computational simplifications with respect to [*Stochastic Process. Appl.* **120** (2010) 1444–1472], where the construction of a rough path over B was first introduced.

1. Introduction. Rough paths analysis is a theory introduced by Terry Lyons in the pioneering paper [13] which aims to solve differential equations driven by functions with finite p-variation with $p > 1$, or by Hölder continuous functions of order $\gamma \in (0,1)$. One possible shortcut to the rough path theory is the following summary (see [9, 10, 14] for a complete construction). Given a γ-Hölder d-dimensional process $X = (X(1), \dots, X(d))$ defined on an arbitrary interval $[0,T]$, assume that one can define some iterated integrals of the form

$$\mathbf{X}^{\mathbf{n}}_{st}(i_1, \dots, i_n) = \int_{s \le u_1 < \cdots < u_n \le t} dX_{u_1}(i_1)\, dX_{u_2}(i_2) \cdots dX_{u_n}(i_n), \tag{1}$$

for $0 \le s < t \le T$, $n \le \lfloor 1/\gamma \rfloor$ and $i_1, \dots, i_n \in \{1, \dots, d\}$. As long as X is a non-smooth function, the integral above cannot be defined rigorously in the Riemann sense (and not even in the Young sense if $\gamma \le 1/2$). However, it is reasonable to assume that some elements $\mathbf{X}^{\mathbf{n}}$ can be constructed, sharing the following three properties with usual iterated integrals (here and in the sequel, we denote by $\mathcal{S}_{k,T} = \{(u_1, \dots, u_k) : 0 \le u_1 < \cdots < u_k \le T\}$ the kth order simplex on $[0,T]$):

(1) *Regularity*: each component of $\mathbf{X}^{\mathbf{n}}$ is $n\gamma$-Hölder continuous [in the sense of the Hölder norm introduced in (11)] for all $n \le \lfloor 1/\gamma \rfloor$ and $\mathbf{X}^{\mathbf{1}}_{st} = X_t - X_s$.
(2) *Multiplicativity*: letting $(\delta \mathbf{X}^{\mathbf{n}})_{sut} := \mathbf{X}^{\mathbf{n}}_{st} - \mathbf{X}^{\mathbf{n}}_{su} - \mathbf{X}^{\mathbf{n}}_{ut}$ for $(s,u,t) \in \mathcal{S}_{3,T}$, one requires

$$(\delta \mathbf{X}^{\mathbf{n}})_{sut}(i_1, \dots, i_n) = \sum_{n_1=1}^{n-1} \mathbf{X}^{\mathbf{n_1}}_{su}(i_1, \dots, i_{n_1}) \mathbf{X}^{\mathbf{n-n_1}}_{ut}(i_{n_1+1}, \dots, i_n). \tag{2}$$

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(3) *Geometricity*: for any n, m such that $n+m \le \lfloor 1/\gamma \rfloor$and $(s,t) \in \mathcal{S}_{2,T}$, we have

$$\mathbf{X}^{\mathbf{n}}_{st}(i_1,\dots,i_n)\mathbf{X}^{\mathbf{m}}_{st}(j_1,\dots,j_m) = \sum_{\bar{k}\in \mathrm{Sh}(\bar{\imath},\bar{\jmath})} \mathbf{X}^{\mathbf{n+m}}_{st}(k_1,\dots,k_{n+m}), \tag{3}$$

where, for two tuples $\bar{\imath}$, $\bar{\jmath}$, $\Sigma_{(\bar{\imath},\bar{\jmath})}$ stands for the set of permutations of the indices contained in $(\bar{\imath},\bar{\jmath})$, and $\mathrm{Sh}(\bar{\imath},\bar{\jmath})$ is a subset of $\Sigma_{(\bar{\imath},\bar{\jmath})}$ defined by

$$\mathrm{Sh}(\bar{\imath},\bar{\jmath}) = \{\sigma \in \Sigma_{(\bar{\imath},\bar{\jmath})}; \sigma \text{ does not change the orderings of } \bar{\imath} \text{ and } \bar{\jmath}\}.$$

We shall call the family $\{\mathbf{X}^{\mathbf{n}}; n \le \lfloor 1/\gamma \rfloor\}$ a rough path over X (it is also referred to as the truncated signature of X in [9]).

Once a rough path over X is defined, the theory described in [9, 10, 14] can be seen as a procedure which allows us to construct, starting from the family $\{\mathbf{X}^{\mathbf{n}}; n \le \lfloor 1/\gamma \rfloor\}$, the complete stack $\{\mathbf{X}^{\mathbf{n}}; n \ge 1\}$. Furthermore, with the rough path over X in hand, one can also define rigorously and solve differential equations driven by X.

The above general framework leads thus naturally to the question of a rough path construction for standard stochastic processes. The first example one may have in mind concerning this issue is arguably the case of a d-dimensional fractional Brownian motion (fBm) $B = (B(1),\dots,B(d))$ with Hurst parameter $H \in (0,1)$. This is a Gaussian process with zero mean whose components are independent and with covariance function given by

$$\mathbf{E}(B_t(i)B_s(i)) = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}), \qquad s,t \in \mathbb{R}_+.$$

For $H = \frac{1}{2}$ this is just the usual Brownian motion. For any $H \in (0,1)$, the variance of the increments of B is then given by

$$\mathbf{E}\big[\big(B_t(i) - B_s(i)\big)^2\big] = (t-s)^{2H}, \qquad (s,t) \in \mathcal{S}_{2,T}, i = 1,\dots,d,$$

and this implies that almost surely the trajectories of the fBm are γ-Hölder continuous for any $\gamma < H$, which justifies the fact that the fBm is the canonical example for a rough path construction.

The first successful rough path analysis for B has been implemented in [5] by means of a linearization of the fBm path, and it leads to the construction of a family $\{\mathbf{B}^{\mathbf{1}}, \mathbf{B}^{\mathbf{2}}, \mathbf{B}^{\mathbf{3}}\}$ satisfying (1), (2) and (3), for any $H > 1/4$ (see also [8] for a generalized framework). Some other constructions can be found in [8, 16, 19] by means of stochastic analysis methods, and in [21] thanks to complex analysis tools. In all those cases, the barrier $H > 1/4$ remains, and it has long been believed that this was a natural boundary, in terms of regularity, for an accurate rough path construction.

Let us describe now several recent attempts to go beyond the threshold $H = 1/4$. One should first quote the interesting paper [15], where a general construction of a rough path is performed by means of a discretization procedure. However, the rough path constructed in this reference is only defined on dyadic points, and then extended to any real positive number by an abstract analytic result. The complex analysis methods used in [20] also allowed the authors to build a rough path above a process Γ called analytic fBm, which is a complex-valued process whose real and imaginary parts are fBm, for any value of $H \in (0, 1)$. It should be mentioned, however, that $\Re\Gamma$ and $\Im\Gamma$ are not independent, and thus the arguments in [20] cannot be extrapolated to the real-valued fBm. Then a series of brilliant ideas developed in [22, 23] lead to the rough path construction in the real-valued case. We will try now to summarize briefly, in very vague terms, this series of ideas (see Section 3 for a more detailed didactic explanation):

(i) Consider a smooth approximation B^ε of the fBm B and the corresponding approximation $\mathbf{B}^{\mathbf{n},\varepsilon}$ of $\mathbf{B}^{\mathbf{n}}$. Clearly $\mathbf{B}^{\mathbf{n},\varepsilon}$ satisfies relation (2), but may diverge as $\varepsilon \to 0$ whenever $H < 1/4$. Then, one can decompose $\mathbf{B}^{\mathbf{n},\varepsilon}_{st}$ as $\mathbf{B}^{\mathbf{n},\varepsilon}_{st} = \mathbf{A}^{\mathbf{n},\varepsilon}_{st} + \mathbf{C}^{\mathbf{n},\varepsilon}_{st}$, where $\mathbf{C}^{\mathbf{n},\varepsilon}$ is the increment of a function f, namely $\mathbf{C}^{\mathbf{n},\varepsilon}_{st} = f_t - f_s$, and $\mathbf{A}^{\mathbf{n},\varepsilon}$ is obtained as a boundary term in the integrals defining $\mathbf{B}^{\mathbf{n},\varepsilon}$. As explained in Section 3, a typical example of such a decomposition is given (for $n = 2$) by $\mathbf{A}^{\mathbf{2},\varepsilon}_{st}(i_1, i_2) = -B^\varepsilon_s(i_1)\delta B^\varepsilon_{st}(i_2)$ and $\mathbf{C}^{\mathbf{2},\varepsilon}_{st}(i_1, i_2) = \int_s^t B^\varepsilon_u(i_1)\, dB^\varepsilon_u(i_2)$, and in this case $f_t(i_1, i_2) = \int_0^t B^\varepsilon_u(i_1)\, dB^\varepsilon_u(i_2)$. Then it can be easily checked, thanks to the relation $\mathbf{C}^{\mathbf{n},\varepsilon}_{st} = f_t - f_s$, that $\mathbf{C}^{\mathbf{n},\varepsilon}_{st} - \mathbf{C}^{\mathbf{n},\varepsilon}_{su} - \mathbf{C}^{\mathbf{n},\varepsilon}_{ut} = 0$ for any $(s, u, t) \in \mathcal{S}_{3,T}$. This means that replacing $\mathbf{B}^{\mathbf{n},\varepsilon}_{st}$ by $\mathbf{A}^{\mathbf{n},\varepsilon}_{st} = \mathbf{B}^{\mathbf{n},\varepsilon} - \mathbf{C}^{\mathbf{n},\varepsilon}_{st}$ does not affect the multiplicative property (2) of $\mathbf{B}^{\mathbf{n},\varepsilon}$. On the other hand, the boundary term $\mathbf{A}^{\mathbf{n},\varepsilon}_{st}$ is usually easily seen to be convergent as $\varepsilon \to 0$ to some limit $\mathbf{A}^{\mathbf{n}}_{st}$. Then, the limit $\mathbf{A}^{\mathbf{n}}_{st}$ should fulfill the desired multiplicative property, but it does not exhibit the desired Hölder regularity $(kH)^-$. It should also be noticed that $\mathbf{A}^{\mathbf{n},\varepsilon}_{st}$ is not the only function of two variables sharing the multiplicative property with $\mathbf{B}^{\mathbf{n},\varepsilon}$. We refer to Section 3 for further details, but let us mention that another possibility for $n = 2$ is the boundary term $\delta X^\varepsilon_{st}(i_1)X^\varepsilon_t(i_2)$, which is easily seen to satisfy relation (2).

(ii) The essential point in Unterberger's method is then the following: carry out the above program for some given regularizations of the fBm path. Then, it turns out that there is a choice of boundary terms such that their sum satisfies the desired Hölder and multiplicative properties. This idea has been successfully implemented in [22, 23], providing an explicit construction of a rough path associated to B. However, this construction is rather long and intricate, because the changes in the order of integration in the multiple integrals are coded by admissible cuts in some trees associated to multiple integrals. This language, well known by algebraists [3, 6], numerical analysts [2, 12] and theoretical physicists [4], may, however, sound difficult to the noninitiated reader.

The purpose of the current paper is to take up the program initiated in [22], and construct a rough path over B in a rather simple way, using the stochastic integral representation of the fBm as a Volterra Gaussian process. We know that (see [18], Proposition 5.1.3, for a justification) for $H < 1/2$, each component $B(i)$ of B can be written as

$$B_t(i) = \int_{\mathbb{R}} K(t, u)\, dW_u(i), \qquad t \geq 0, \tag{4}$$

where $W = (W(1), \dots, W(d))$ is a d-dimensional Wiener process, and where the Volterra-type kernel K is defined on $\mathbb{R}_+ \times \mathbb{R}_+$ by

$$K(t,u) = c_H\left[\left(\frac{u}{t}\right)^{1/2-H}(t-u)^{H-1/2} + \left(\frac{1}{2}-H\right)u^{1/2-H}\int_u^t v^{H-3/2}(v-u)^{H-1/2}\,dv\right]\mathbf{1}_{\{0<u<t\}}, \tag{5}$$

with a strictly positive constant c_H, whose exact value is irrelevant for our purposes. Then we show that the simple trick described at point (ii) above can be applied in a straightforward way using the Volterra representation, leading to a simple general formula for the multiple integrals $\mathbf{B^n}$. To be more specific, let us describe the main result of this paper.

THEOREM 1.1. *Let B be a d-dimensional fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$, admitting representation* (4). *For $2 \le n \le \lfloor 1/H \rfloor$, any tuple $(i_1, \dots, i_n)$ of elements of $\{1, \dots, d\}$, $1 \le j \le n$ and $(s,t) \in \mathcal{S}_{2,T}$, set*

$$\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n) = (-1)^{j-1}\int_{A^n_j}\prod_{l=1}^{j-1}K(s,u_l)[K(t,u_j)-K(s,u_j)] \times \prod_{l=j+1}^{n}K(t,u_l)\,dW_{u_1}(i_1)\cdots dW_{u_n}(i_n), \tag{6}$$

where the kernel K is given by (5) *and A^n_j is the subset of $[0,t]^n$ defined by*

$$A^n_j = \{(u_1,\dots,u_n) \in [0,t]^n;\ u_j = \min(u_1,\dots,u_n), u_1 > \cdots > u_{j-1} \text{ and } u_{j+1} < \cdots < u_n\}.$$

Notice that the multiple stochastic integral in (6) *is understood in the Stratonovich sense, and is well defined as a $L^2(\Omega)$ random variable as long as $n \le \lfloor 1/H \rfloor$. Set also $\mathbf{B}^{\mathbf{1}}_{st}(i) = B_t(i) - B_s(i)$, and for $2 \le n \le \lfloor 1/H \rfloor$,*

$$\mathbf{B}^{\mathbf{n}}_{st}(i_1,\dots,i_n) = \sum_{j=1}^{n}\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n). \tag{7}$$

Then the family $\{\mathbf{B^n}; 1 \le n \le \lfloor 1/H \rfloor\}$ defines a rough path over B, in the sense that $\mathbf{B^n}$ is almost surely $n\gamma$-Hölder continuous for any $\gamma < H$, and that it satisfies relations (2) *and* (3).

As announced above, formula (6) defines in a compact and simple way the (substitute to) iterated integrals of B with respect to itself. Furthermore, this formula also yields a reasonably short way to estimate the moments of $\mathbf{B}^{\mathbf{n}}_{st}$, and thus its Hölder regularity. It should be mentioned, however, that our construction is not as general as the one proposed in [23], though it can be extended to a broad class of Gaussian Volterra processes. More precisely, the reader can check that the only properties of the kernel K used in this paper are

$$|K(t,u)| \le C[(t-u)^{H-1/2} + u^{H-1/2}] \quad \text{and} \quad |\partial_t K(t,u)| \le C(t-u)^{H-3/2}$$

for any $H \in (0, 1/2)$ and for some constant $C > 0$. It is also worth mentioning at this point that our representation (7) of $\mathbf{B}^{\mathbf{n}}$ is adapted to the past of the path B.

It is hard to compare our main result with the one given in [15], due to the abstract nature of the latter. We can, however, say a few words about the relationship between the processes $\mathbf{B}^{\mathbf{n}}$ we have produced and the pathwise ones constructed in the aforementioned references [5, 8, 16, 21], as well as with the recent objects introduced in [23].

(i) When $1/4 < H \le 1/2$, let us denote by $\mathbf{B}^{\mathbf{2},\mathrm{p}}$ (where p stands for pathwise) the double iterated integral constructed in [5, 8, 16, 21]. Notice that these integrals all coincide as limit of Riemann sums (a fact which is mentioned in [17]). One has then to distinguish two situations:

(1) For $H = 1/2$, a slight extension of our construction also allows to define $\mathbf{B}^{\mathbf{2}}$ for Brownian motion, and it is readily checked in this case that $\mathbf{B}^{\mathbf{2}}$ coincides with the usual Stratonovich double iterated integral.

(2) When $1/4 < H < 1/2$, we know that $\delta\mathbf{B}^{\mathbf{n}} = \delta\mathbf{B}^{\mathbf{n},\mathrm{p}}$, and it can be seen from this relation that $\mathbf{B}^{\mathbf{n}}$ and $\mathbf{B}^{\mathbf{n},\mathrm{p}}$ only differ by the increment of a function f. This nontrivial correction term is identified at Section 5. Notice that the correction term for $\mathbf{B}^{\mathbf{3}}$ could be identified as well, but we did not include these computations for the sake of conciseness.

(ii) For $H \le 1/4$, we shall see that our iterated integrals can be considered under the framework of the rough path constructions by Fourier normal ordering contained in [23]. As mentioned above, our main result gives a more direct an elementary (though less general) representation of the iterated integrals. This representation only uses direct (as opposed to Fourier) coordinates and is adapted with respect to the underlying fBm B. All these considerations will be developed at Section 5.

Here is how our article is divided: some preliminary results, including algebraic integration vocabulary, some estimates on the kernel K and Itô–Stratonovich corrections, are given in Section 2. Then the basic ideas of the construction are implemented in Section 3 on second order iterated integrals. This section is thus intended as a didactic introduction to the construction, and could be enough for a first quick glimpse at the topic. Then we give all the details concerning the general iterated integral definition and prove Theorem 1.1 in Section 4. Finally, Section 5 establishes some links between our integrals and other well established iterated integrals for fBm.

2. Preliminaries.

This section is first devoted to recall some notational conventions for a special subset (called set of increments) of functions of several variables. These conventions are taken from the algebraic integration theory as explained in [10, 11]. We will then recall some basic estimates on iterated Stratonovich integrals with respect to the Wiener process, which turn out to be useful for the remainder of the article.

2.1. *Some algebraic integration vocabulary.*

The current section is not intended as an introduction to algebraic integration, which would be useless for our purposes. However, we shall use in the sequel some notation taken from this method of rough paths analysis, and we shall proceed to recall them now.

The algebraic integration setting is based on the notion of increment, together with an elementary operator δ acting on them. The notion of increment can be introduced in the following way: for an arbitrary real number $T > 0$, a vector space V, and an integer $k \geq 1$, we denote by $\mathcal{S}_{k,T}$ the kth order simplex on $[0, T]$, and by $\mathcal{C}_k(V)$ the set of continuous functions $g : \mathcal{S}_{k,T} \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*(V) = \bigcup_{k \geq 1} \mathcal{C}_k(V)$. The operator δ alluded to above can be seen as an operator acting on k-increments, and is defined as follows on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{8}$$

where $\hat{t}_i$ means that this particular argument is omitted. Then a fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{Z}\mathcal{C}_k(V) = \mathcal{C}_k(V) \cap \operatorname{Ker} \delta$.

Some simple examples of actions of δ, which will be the ones we will really use throughout the paper, are obtained by letting $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $(s, u, t) \in \mathcal{S}_{3,T}$, we have

$$(\delta g)_{st} = g_t - g_s \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}, \tag{9}$$

and in this particular case, it can be trivially checked that for any $g \in \mathcal{C}_1$, one has $\delta\delta g = 0$. Conversely, any $h \in \mathcal{Z}\mathcal{C}_2$ can be written as $h = \delta g$ for an element $g \in \mathcal{C}_1$. In the sequel of the paper, we shall write for two elements $h^1, h^2 \in \mathcal{C}_2$

$$h^1 \overset{\mathcal{Z}\mathcal{C}_2}{=} h^2 \quad \text{iff} \quad h^1 = h^2 + z \qquad \text{with } z \in \mathcal{Z}\mathcal{C}_2. \tag{10}$$

Otherwise stated, $h^1 \overset{\mathcal{Z}\mathcal{C}_2}{=} h^2$ iff $\delta h^1 = \delta h^2$.

Notice that our future discussions will rely on some analytical assumptions made on elements of $\mathcal{C}_k(V)$. Suppose V is equipped with a norm $|\cdot|$. We measure the size of the increments by Hölder norms defined in the following way: for $g \in \mathcal{C}_2(V)$ let

$$\|g\|_\mu \equiv \sup_{(s,t) \in \mathcal{S}_{2,T}} \frac{|g_{st}|}{|t-s|^\mu} \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{g \in \mathcal{C}_2(V); \|g\|_\mu < \infty\}. \tag{11}$$

With this notation, we also set $\mathcal{C}_1^\mu(V) = \{f \in \mathcal{C}_1(V); \|\delta f\|_\mu < \infty\}$ (notice that the sup norm of f is not taken into account in this definition). In the same way, for $h \in \mathcal{C}_3(V)$, set

$$\|h\|_{\gamma,\rho} = \sup_{(s,u,t)\in\mathcal{S}_{3,T}} \frac{|h_{sut}|}{|u-s|^\gamma |t-u|^\rho}, \tag{12}$$

$$\|h\|_\mu \equiv \inf\left\{\sum_i \|h_i\|_{\rho_i,\mu-\rho_i}; h = \sum_i h_i, 0 < \rho_i < \mu\right\}, \tag{13}$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0,\mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V); \|h\|_\mu < \infty\}.$$

In order to avoid ambiguities, we shall denote by $\mathcal{N}[f; \mathcal{C}_j^\mu(V)]$ the μ-Hölder norm (or semi-norm) on the space $\mathcal{C}_j(V)$, for $j = 1, 2, 3$.

The lemma below, borrowed from [10], Lemma 4, will be an essential tool for the analysis of Hölder-type regularity of our increments:

LEMMA 2.1. *Let $\kappa > 0$ and $p \geq 1$. Let $R \in \mathcal{C}_2(\mathbb{R}^l)$, with $\delta R \in \mathcal{C}_3^\kappa(\mathbb{R}^l)$ in the sense given by* (13). *If*

$$\int_{\mathcal{S}_{2,T}} \frac{|R_{uv}|^{2p}}{|u-v|^{2\kappa p+4}}\, du\, dv < \infty,$$

then $R \in \mathcal{C}_2^\kappa(\mathbb{R}^l)$. In particular, there exists a constant $C_{\kappa,p,l} > 0$, such that

$$\mathcal{N}[R; \mathcal{C}_2^\kappa(\mathbb{R}^l)] \leq C_{\kappa,p,l}\left(\int_{\mathcal{S}_{2,T}} \frac{|R_{uv}|^{2p}}{|u-v|^{2\kappa p+4}}\, du\, dv\right)^{1/(2p)} + C_{\kappa,p,l}\mathcal{N}[\delta R; \mathcal{C}_3^\kappa(\mathbb{R}^l)].$$

2.2. *Analytic bounds on the fractional Brownian kernel*. We gather in this section some technical bounds on the kernel K involved in the Volterra representation of B, for which we use the following convention (valid until the end of the article): for two positive quantities a and b, we write $a \lesssim b$ whenever there exists a universal constant C such that $a \leq Cb$.

First, a classical bound on K is the following:

LEMMA 2.2. *Let K be the fBm kernel defined by* (5). *Then for any $0 < u < t$, one has*

$$|K(t,u)| \lesssim (t-u)^{H-1/2} + u^{H-1/2} \quad \text{and} \quad |\partial_t K(t,u)| \lesssim (t-u)^{H-3/2}. \tag{14}$$

The following simple integral estimate on K also turns out to be useful:

LEMMA 2.3. *Let* $0 < v < t \le T$. *Then* $\int_v^t K^2(t, w)\, dw \lesssim (t - v)^{2H}$.

PROOF. Invoking the bound (14) on K, we have

$$\begin{aligned}\int_v^t K^2(t, w)\, dw &\lesssim \int_v^t [(t - w)^{H-1/2} + w^{H-1/2}]^2\, dw \\ &\lesssim \int_v^t (t - w)^{2H-1}\, dw + \int_v^t w^{2H-1}\, dw \\ &\lesssim (t - v)^{2H} + (t^{2H} - v^{2H}).\end{aligned}$$

Furthermore, since $a^\alpha - b^\alpha \le (a - b)^\alpha$ for any $0 \le b < a$ and $\alpha \in (0, 1)$, we end up with $\int_v^t K^2(t, w)\, dw \lesssim (t - v)^{2H}$, which is our claim. □

We shall also use a slightly more elaborated result on K:

LEMMA 2.4. *Let* $0 < s < t \le T$, *assume* $H < 1/2$ *and consider the quantity*

$$I_{st} = \int_0^t [K(t, u_1) - K(s, u_1)]^2 \left(\int_{u_1}^t K^2(t, u_2)\, du_2 \right) du_1,$$

where we recall that we have used the convention $K(t, u) = K(t, u)\mathbf{1}_{[0,t)}(u)$. *Then* $|I_{st}| \lesssim |t - s|^{4H}$.

PROOF. According to the fact that $K(t, u) = 0$ whenever $u \ge t$, we obtain the expression

$$\begin{aligned}I_{st} &= \int_0^s [K(t, u_1) - K(s, u_1)]^2 \left(\int_{u_1}^t K^2(t, u_2)\, du_2 \right) du_1 \\ &\quad + \int_s^t K^2(t, u_1) \left(\int_{u_1}^t K^2(t, u_2)\, du_2 \right) du_1 := I_{st}^1 + I_{st}^2.\end{aligned}$$

Let us bound now the first of those terms: thanks to Lemma 2.3, one can write $\int_{u_1}^t K^2(t, u_2)\, du_2 \lesssim (t - u_1)^{2H}$. Moreover, for $0 \le u < s$ the bound (14) on $\partial_t K(t, u)$ yields

$$|K(t, u) - K(s, u)| = \left| \int_s^t \partial_v K(v, u)\, dv \right| \lesssim (s - u)^{H-1/2} - (t - u)^{H-1/2}, \tag{15}$$

and thus, putting these two estimates together, we obtain

$$I_{st}^1 \lesssim \int_0^s [(s - u)^{H-1/2} - (t - u)^{H-1/2}]^2 (t - u)^{2H}\, du.$$

Performing the changes of variable $v = s - u$ and $y = v/(t - s)$, we end up with

$$I_{st}^1 \lesssim (t - s)^{4H} \int_0^{s/(t-s)} [(1 + y)^{H-1/2} - y^{H-1/2}]^2 (1 + y)^{2H}\, dy.$$

Furthermore, it is easily checked that $\int_0^\infty [(1+y)^{H-1/2} - y^{H-1/2}]^2 (1+y)^{2H}\, dy$ is a convergent integral whenever $H < 1/2$, which gives the desired bound for I_{st}^1. The term I_{st}^2 is in fact easier to handle, and we leave those details to the reader for the sake of conciseness. Then, the estimates on I_{st}^1 and I_{st}^2 yield our claim. □

Finally, the following related integral bound also turns out to be an important estimate for the analysis of nth order iterated integrals:

LEMMA 2.5. *Suppose that* $2kH < 1$. *For* $A > 0$, *set*

$$\beta_A = \int_0^A [y^{H-1/2} - (1+y)^{H-1/2}][y^{H-1/2} + (A-y)^{H-1/2}] y^{2(k-1)H}\, dy.$$

Then $\sup_{A>0} \beta_A < \infty$.

PROOF. We can write $\beta_A \le \alpha_A + \gamma_A$, with

$$\alpha_A = \int_0^\infty [y^{H-1/2} - (1+y)^{H-1/2}] y^{2(k-1)H+H-1/2}\, dy,$$

$$\gamma_A = \int_0^A [y^{H-1/2} - (1+y)^{H-1/2}](A-y)^{H-1/2} y^{2(k-1)H}\, dy.$$

One can check easily, as in the proof of Lemma 2.4, that α_A is finite as long as $2kH < 1$. On the other hand, an obvious change of variables yields

$$\gamma_A = A^{2kH} \int_0^1 h_A(y)(1-y)^{H-1/2} y^{2(k-1)H}\, dy, \tag{16}$$

where the (positive) function h_A is defined on $\mathbb{R}_+$ by $h_A(y) = y^{H-1/2} - (\frac{1}{A} + y)^{H-1/2}$. We now use two elementary estimates

$$h_A(y) \le \left(\frac{1}{2} - H\right) \frac{y^{H-3/2}}{A} \quad \text{and} \quad h_A(y) \le y^{H-1/2},$$

and we obtain

$$\begin{aligned} h_A(y) &= h_A(y)^{2kH} h_A(y)^{1-2kH} \\ &\le \left(\left(\frac{1}{2} - H\right)\frac{1}{A} y^{H-3/2}\right)^{2kH} y^{(H-1/2)(1-2kH)} \\ &= \frac{c_{H,k}\, y^{(1-2k)H-1/2}}{A^{2kH}}, \end{aligned}$$

where $c_{H,k} = (\frac{1}{2} - H)^{2kH}$. Plugging this bound into (16), we get

$$\gamma_A \le c_{H,k} \int_0^1 (1-y)^{H-1/2} y^{-H-1/2}\, dy.$$

This last integral being finite, our claim is now proved. □

2.3. *Contraction of Stratonovich iterated integrals*. An important tool in our analysis of iterated integrals will be a general formula of Itô–Stratonovich corrections for iterated integrals. This kind of result has already been obtained in the literature, and for our purposes, it will be enough to use a particular case of [1], Proposition 1, recalled here for further use. Note that we need an additional notation for this intermediate result: we set dY for the Stratonovich-type differential with respect to a process Y, while the Itô-type differential is denoted by ∂Y.

PROPOSITION 2.6. *Let $Y = (Y(1), \dots, Y(n))$ be a n-dimensional martingale of Gaussian type, defined on an interval $[s,t]$, of the form $Y_u(j) = \int_s^u \psi_v(j)\, dW_v(i_j)$ for a family of $L^2([s,t])$ functions $(\psi(1), \dots, \psi(n))$, a set of indices $(i_1, \dots, i_n)$ belonging to $\{1, \dots, d\}^n$ and where we recall that $(W(1), \dots, W(d))$ is a d-dimensional Wiener process. Then the following decomposition holds true*:

$$\int_{s\le u_1<\cdots<u_n\le t} dY_{u_1}(i_1)\cdots dY_{u_n}(i_n) = \sum_{k=\lfloor n/2\rfloor}^{n} \frac{1}{2^{n-k}} \sum_{\nu\in D_n^k} J_{st}(\nu).$$

In the above formula, the sets D_n^k are subsets of $\{1,2\}^k$ given by

$$D_n^k = \left\{\nu = (n_1, \dots, n_k); \sum_{j=1}^{k} n_j = n\right\},$$

and the Itô-type multiple integrals $J_{st}(\nu)$ are defined as follows:

$$J_{st}(\nu) = \int_{s\le u_1<\cdots<u_k\le t} \partial Z_{u_1}(1)\cdots \partial Z_{u_k}(k),$$

where, setting $\sum_{l=1}^{j} n_l = m(j)$, we have

$$Z(j) = Y\big(i_{m(j)}\big) \qquad \text{if } n_j = 1,$$

and

$$Z_u(j) = \left(\int_s^u \psi_v\big(m(j)-1\big)\psi_v(m(j))\, dv\right)\mathbf{1}_{(i_{m(j)-1}=i_{m(j)})} \qquad \text{if } n_j = 2.$$

The previous Itô–Stratonovich decomposition allows us to bound the second order moment of iterated Stratonovich integrals in the following way:

LEMMA 2.7. *Let $\varphi \in L^2([s,t])$. Consider the Stratonovich iterated integral*

$$I_{st}^n(\varphi) = \int_{s<u_1<\cdots<u_n<t} \prod_{i=1}^{n} \varphi(u_i)\, dW_{u_1}(i_1)\cdots dW_{u_n}(i_n).$$

Then

$$\mathbf{E}[I_{st}^n(\varphi)^2] \le C\left(\int_s^t \varphi(u)^2\, du\right)^n, \tag{17}$$

where the constant C depends on n and the multiindex $(i_1, \dots, i_n)$.

PROOF. By Proposition 2.6, we can decompose the Stratonovich integral $I^n_{st}(\varphi)$ into a sum of Itô integrals

$$I^n_{st}(\varphi) = \sum_{k=\lfloor n/2 \rfloor}^{n} \frac{1}{2^{n-k}} \sum_{\nu \in D^k_n} J_{st}(\nu),$$

and it suffices to consider each Itô integral $J_{st}(\nu)$. Then we proceed by recurrence with respect to k, with the notation of Proposition 2.6. Suppose first that $n_k = 1$. Then,

$$J_{st}(\nu) = \int_s^t J_{su}(\nu')\varphi(u)\, \partial_u W(i_n),$$

where $\nu' = (n_1, \dots, n_{k-1})$. As a consequence,

$$\mathbf{E}[J_{st}(\nu)^2] = \int_s^t \mathbf{E}[J_{su}(\nu')^2]\varphi(u)^2\, du \le \sup_{s \le u \le t} \mathbf{E}[J_{su}(\nu')^2] \int_s^t \varphi(u)^2\, du.$$

On the other hand, if $n_k = 2$, then $J_{st}(\nu) = \int_s^t J_{su}(\nu'')\varphi(u)^2\, du$, with $\nu'' = (n_1, \dots, n_{k-2})$, and again

$$\mathbf{E}[J_{st}(\nu)^2] \le \sup_{s \le u \le t} \mathbf{E}[J_{su}(\nu'')^2] \left(\int_s^t \varphi(u)^2\, du \right)^2.$$

By recurrence we obtain (17), where $C = (\sum_{k=\lfloor n/2 \rfloor}^{n} \frac{|D^k_n|}{2^{n-k}})^2$. □

3. Iterated integrals of order 2. In this section, we will define the element $\mathbf{B}^2$ announced in Theorem 1.1. The study of this particular case will (hopefully) allow us to introduce many of the technical ingredients needed for the general case in a didactic way.

3.1. *Heuristic considerations.* Let us first specify what is meant by an iterated integral of order 2: according to the definitions contained in the Introduction, we are searching for a process $\{\mathbf{B}^2_{st}(i_1, i_2); (s,t) \in \mathcal{S}_{2,T}, 1 \le i_1, i_2 \le d\}$ satisfying:

(i) the regularity condition $\mathbf{B}^2 \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d^2})$;
(ii) the multiplicative property

$$\delta \mathbf{B}^2_{sut}(i_1, i_2) = \mathbf{B}^1_{su}(i_1)\mathbf{B}^1_{ut}(i_2) = [B_u(i_1) - B_s(i_1)][B_t(i_2) - B_u(i_2)], \tag{18}$$

which should be satisfied almost surely for all $(s,u,t) \in \mathcal{S}_{3,T}$ and $1 \le i_1, i_2 \le d$;
(iii) the geometric relation, which can be read here as:

$$\mathbf{B}^2_{st}(i_1, i_2) + \mathbf{B}^2_{st}(i_2, i_1) = \mathbf{B}^1_{st}(i_1)\mathbf{B}^1_{st}(i_2), \qquad (s,t) \in \mathcal{S}_{2,T}, 1 \le i_1, i_2 \le d. \tag{19}$$

In order to construct this kind of element, let us start with some heuristic considerations, similar to the starting point of [22]: assume for the moment that X is a smooth d-dimensional function defined on $[0, T]$. Then the natural notion of iterated integral of order 2 for X is obviously an element $\hat{\mathbf{X}}^{\mathbf{2}}$, defined in the Riemann sense by

$$\begin{aligned}\hat{\mathbf{X}}^{\mathbf{2}}_{st}(i_1, i_2) &= \int_{s\le u_1\le u_2\le t} dX_{u_1}(i_1)\, dX_{u_2}(i_2)\\ &= \int_s^t [X_u(i_1) - X_s(i_1)]\, dX_u(i_2).\end{aligned} \tag{20}$$

We shall now decompose $\hat{\mathbf{X}}^{\mathbf{2}}$ into terms of the form $\mathbf{A}^{\mathbf{2}}$ and $\mathbf{C}^{\mathbf{2}}$ as explained in the Introduction. In our case, this can be done in two ways: first, equation (20) immediately yields

$$\hat{\mathbf{X}}^{\mathbf{2}}_{st}(i_1, i_2) = \hat{\mathbf{A}}^{\mathbf{2},2}_{st} + \hat{\mathbf{C}}^{\mathbf{2},2}_{st}$$

with

$$\hat{\mathbf{A}}^{\mathbf{2},2}_{st} = -X_s(i_1)\delta X_{st}(i_2), \qquad \hat{\mathbf{C}}^{\mathbf{2},2}_{st} = \int_s^t X_u(i_1)\, dX_u(i_2),$$

where we have called those quantities $\hat{\mathbf{A}}^{\mathbf{2},2}$ and $\hat{\mathbf{C}}^{\mathbf{2},2}$ because they involve increments of the second component $X(i_2)$ of X. Notice now that $\hat{\mathbf{C}}^{\mathbf{2},2}$ is the increment of a function f defined as $f_t = \int_0^t X_u(i_1)\, dX_u(i_2)$. Hence, according to convention (10), one can write $\hat{\mathbf{X}}^{\mathbf{2}}(i_1, i_2) \overset{\mathcal{ZC}_2}{=} \hat{\mathbf{A}}^{\mathbf{2},2}$. By inverting the order of integration in u_1, u_2 thanks to Fubini's theorem, we also obtain

$$\hat{\mathbf{X}}^{\mathbf{2}}_{st}(i_1, i_2) = \hat{\mathbf{A}}^{\mathbf{2},1}_{st} + \hat{\mathbf{C}}^{\mathbf{2},1}_{st}$$

with

$$\hat{\mathbf{A}}^{\mathbf{2},1}_{st} = \delta X_{st}(i_1) X_t(i_2), \qquad \hat{\mathbf{C}}^{\mathbf{2},1}_{st} = -\int_s^t X_u(i_2)\, dX_u(i_1),$$

and thus $\hat{\mathbf{X}}^{\mathbf{2}}(i_1, i_2) \overset{\mathcal{ZC}_2}{=} \hat{\mathbf{A}}^{\mathbf{2},1}$.

Let us go back now to the case of the d-dimensional fBm B. If we wish the iterated integral $\mathbf{B}^{\mathbf{2}}$ we are constructing to behave in a similar manner as a Riemann-type integral, then, by the Chen property, one should have $\delta\mathbf{B}^{\mathbf{2}} = \delta\mathbf{A}^{\mathbf{2},i}$, for $i = 1, 2$, that is,

$$\mathbf{B}^{\mathbf{2}}(i_1, i_2) \overset{\mathcal{ZC}_2}{=} \mathbf{A}^{\mathbf{2},2} \quad \text{and} \quad \mathbf{B}^{\mathbf{2}}(i_1, i_2) \overset{\mathcal{ZC}_2}{=} \mathbf{A}^{\mathbf{2},1},$$

with $\mathbf{A}^{\mathbf{2},2}_{st} = -B_s(i_1)\delta B_{st}(i_2)$ and $\mathbf{A}^{\mathbf{2},1}_{st} = \delta B_{st}(i_1) B_t(i_2)$. This means in particular, according to the fact that $\delta|_{\mathcal{ZC}_2} = 0$, that both $\mathbf{A}^{\mathbf{2},1}$ and $\mathbf{A}^{\mathbf{2},2}$ satisfy the multiplicative relation (18), as it can be easily checked by direct computations. However, this naive decomposition has an important drawback: the increments $\mathbf{A}^{\mathbf{2},1}$ and $\mathbf{A}^{\mathbf{2},2}$

only belong to $\mathcal{C}_2^{\gamma}$, instead of $\mathcal{C}_2^{2\gamma}$, for any $\gamma < H$ (this point was also stressed in [22]).

Our construction diverges from [22] in the way we cope with the regularity problem mentioned above. Indeed, we start from the following observation: invoking the representation (4) of B, one can write

$$\begin{aligned}\mathbf{A}_{st}^{\mathbf{2},2} &= -B_s(i_1)\delta B_{st}(i_2)\\ &= -\int_{\mathbb{R}} K(s,u_1)\,dW_{u_1}(i_1)\int_{\mathbb{R}}[K(t,u_2)-K(s,u_2)]\,dW_{u_2}(i_2)\\ &= -\int_{\mathbb{R}^2} K(s,u_1)[K(t,u_2)-K(s,u_2)]\,dW_{u_1}(i_1)\,dW_{u_2}(i_2),\end{aligned}$$

where we recall that the stochastic differentials dW are defined in the Stratonovich sense. In the same way, we get

$$\mathbf{A}_{st}^{\mathbf{2},1} = \int_{\mathbb{R}^2}[K(t,u_1)-K(s,u_1)]K(t,u_2)\,dW_{u_1}(i_1)\,dW_{u_2}(i_2).$$

The idea in order to transform $\mathbf{A}^{\mathbf{2},1}, \mathbf{A}^{\mathbf{2},2}$ into $\mathcal{C}_2^{2\gamma}$ increments is then to replace the integrals over $\mathbb{R}^2$ above by integrals on the simplex, as mentioned in the Introduction. Namely, we set now

$$\hat{\mathbf{B}}_{st}^{\mathbf{2},1}(i_1,i_2) = \int_{u_1<u_2}[K(t,u_1)-K(s,u_1)]K(t,u_2)\,dW_{u_1}(i_1)\,dW_{u_2}(i_2), \tag{21}$$

$$\hat{\mathbf{B}}_{st}^{\mathbf{2},2}(i_1,i_2) = -\int_{u_2<u_1} K(s,u_1)[K(t,u_2)-K(s,u_2)]\,dW_{u_1}(i_1)\,dW_{u_2}(i_2), \tag{22}$$

and notice that these formulas are a particular case of (6) for $n=2$. We shall see that $\hat{\mathbf{B}}^{\mathbf{2},1}(i_1,i_2)$ and $\hat{\mathbf{B}}^{\mathbf{2},2}(i_1,i_2)$ are elements of $\mathcal{C}_2^{2\gamma}$, but they do not satisfy the multiplicative and geometric property anymore. However, it is now easily conceived, by some symmetry arguments, that the sum of these last two terms do satisfy the desired algebraic properties again. Indeed, we set now

$$\mathbf{B}_{st}^{\mathbf{2}}(i_1,i_2) = \hat{\mathbf{B}}_{st}^{\mathbf{2},1}(i_1,i_2) + \hat{\mathbf{B}}_{st}^{\mathbf{2},2}(i_1,i_2), \tag{23}$$

and we claim that $\mathbf{B}^{\mathbf{2}}$ is a $\mathcal{C}_2^{2\gamma}(\mathbb{R}^{d^2})$ increment which fulfills relations (18) and (19). The remainder of this section is devoted to prove these claims.

3.2. *Properties of the second order increment.* It is obviously essential for the following developments to check that $\mathbf{B}^{\mathbf{2}}$ is a well defined object in $L^2(\Omega)$. The next proposition asserts the existence of $\mathbf{B}_{st}^{2}$ as a L^2 random variable for all s,t in the interval $[0,T]$.

PROPOSITION 3.1. *Let $H<1/2$, $(s,t)\in\mathcal{S}_{2,T}$ and $\mathbf{B}_{st}^{\mathbf{2}}$ be the matrix valued random variable defined by* (23). *Then $\mathbf{B}_{st}^{\mathbf{2}}(i_1,i_2)\in L^2(\Omega;\mathbb{R}^{d^2})$ and $\mathbf{E}[|\mathbf{B}_{st}^{\mathbf{2}}|^2]\lesssim (t-s)^{4H}$.*

PROOF. Assume first $i_1 \neq i_2$. We shall focus on the relation $\mathbf{E}[(\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_1, i_2))^2] \lesssim (t-s)^{4H}$, the bound on $\hat{\mathbf{B}}^{\mathbf{2},2}_{st}$ being obtained in a similar way. Now Stratonovich and Itô-type integrals coincide when $i_1 \neq i_2$, and according to expression (21) we have

$$\begin{aligned}\mathbf{E}[(\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_1, i_2))^2] &= \int_{u_1<u_2} \big[K(t,u_1)\mathbf{1}_{[0,t]}(u_1) - K(s,u_1)\mathbf{1}_{[0,s]}(u_1)\big]^2 \\ &\qquad \times K^2(t,u_2)\mathbf{1}_{[0,t]}(u_2)\, du_1\, du_2,\end{aligned}$$

which is exactly the quantity I_{st} studied at Lemma 2.4. The desired bound follows from Lemma 2.4.

Let us now treat the case $i_1 = i_2 = i$, still concentrating our efforts on the inequality $\mathbf{E}[(\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i,i))^2] \lesssim (t-s)^{4H}$. In this context, Proposition 2.6 yields the decomposition $\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i,i) = M_{st} + V_{st}$, with

$$M_{st} = \int_{u_1<u_2} [K(t,u_1) - K(s,u_1)]K(t,u_2)\, \partial W_{u_1}(i)\, \partial W_{u_2}(i),$$
$$V_{st} = \frac{1}{2}\int_0^t [K(t,u) - K(s,u)]K(t,u)\, du,$$

where we stress the fact that V_{st} is a deterministic correction term. It is thus obviously enough to obtain the bounds $\mathbf{E}[M_{st}^2] \lesssim (t-s)^{4H}$ and $V_{st}^2 \lesssim (t-s)^{4H}$ separately, the first of these bounds being obtained by evaluating I_{st} in Lemma 2.4 again. As far as V_{st} is concerned, we make the decomposition

$$V_{st} = \frac{1}{2}\int_0^s [K(t,u) - K(s,u)]K(t,u)\, du + \int_s^t K(t,u)^2\, du.$$

The second term is bounded by a constant times $(t-s)^{2H}$ by Lemma 2.3. For the first term we use the estimate

$$|K(t,u) - K(s,u)| \lesssim (t-s)^{2H}(s-u)^{-H-1/2},$$

which trivially finishes the proof. □

By standard arguments (see [20]) it can be proved that the estimates in Proposition 3.1 imply that $\mathbf{B^2} \in \mathcal{C}_2^{2H-}(\mathbb{R}^{d^2})$.

We are now equipped with the continuous version of $\mathbf{B^2}$ exhibited in the last proposition, with which we will work without further mention, and we are now ready to prove the algebraic relations satisfied by our second order increment.

PROPOSITION 3.2. *The increment* $\mathbf{B^2}$ *defined by* (23) *satisfies relations* (18) *and* (19).

PROOF. Recall that we are now dealing with a continuous version of $\mathbf{B^2}$. In fact, one can easily modify the arguments of [20] in order to get a continuous version of the pair $(\mathbf{B^1}, \mathbf{B^2})$. This means that it is enough to check relations (18) and (19) for some fixed $0 \le s < u < t \le T$.

Let us then verify (18) for $s, u, t \in [0, T]$ such that $s < u < t$. It is readily seen, by writing the definitions of $\mathbf{B}^{\mathbf{2},1}_{st}(i_1, i_2)$, $\mathbf{B}^{\mathbf{2},1}_{su}(i_1, i_2)$ and $\mathbf{B}^{\mathbf{2},1}_{ut}(i_1, i_2)$, that

$$\delta \mathbf{B}^{\mathbf{2},1}_{sut}(i_1, i_2) = \int_{u_1 < u_2} [K(u, u_1) - K(s, u_1)] \times [K(t, u_2) - K(u, u_2)]\, dW_{u_1}(i_1)\, dW_{u_2}(i_2),$$

the right-hand side of this equality being well defined as a L^2 random variable (a fact which can be shown similarly to Proposition 3.1). Along the same lines, we also get

$$\delta \mathbf{B}^{\mathbf{2},2}_{sut}(i_1, i_2) = \int_{u_1 > u_2} [K(u, u_1) - K(s, u_1)] \times [K(t, u_2) - K(u, u_2)]\, dW_{u_1}(i_1)\, dW_{u_2}(i_2),$$

and thus

$$\begin{aligned}\delta \mathbf{B}^{\mathbf{2}}_{sut}(i_1, i_2) &= \delta \mathbf{B}^{\mathbf{2},1}_{sut}(i_1, i_2) + \delta \mathbf{B}^{\mathbf{2},2}_{sut}(i_1, i_2) \\ &= \int_{\mathbb{R}^2} [K(u, u_1) - K(s, u_1)] \\ &\qquad \times [K(t, u_2) - K(u, u_2)]\, dW_{u_1}(i_1)\, dW_{u_2}(i_2) \\ &= \mathbf{B}^{\mathbf{1}}_{su}(i_1) \mathbf{B}^{\mathbf{1}}_{ut}(i_2),\end{aligned}$$

which is relation (18).

As far as relation (19) is concerned, reorder the integration indices in (21) in order to get

$$\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_2, i_1) = \int_{u_2 < u_1} K(t, u_1)[K(t, u_2) - K(s, u_2)]\, dW_{u_1}(i_1)\, dW_{u_2}(i_2).$$

Add this expression to (22), which yields

$$\begin{aligned}&\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_2, i_1) + \hat{\mathbf{B}}^{\mathbf{2},2}_{st}(i_1, i_2) \\ &\quad = \int_{u_2 < u_1} [K(t, u_1) - K(s, u_2)] \\ &\qquad \times [K(t, u_2) - K(s, u_2)]\, dW_{u_1}(i_1)\, dW_{u_2}(i_2).\end{aligned} \tag{24}$$

Exactly in the same way, we get

$$\begin{aligned}&\hat{\mathbf{B}}^{\mathbf{2},1}_{st}(i_1, i_2) + \hat{\mathbf{B}}^{\mathbf{2},2}_{st}(i_2, i_1) \\ &\quad = \int_{u_1 < u_2} [K(t, u_1) - K(s, u_2)] \\ &\qquad \times [K(t, u_2) - K(s, u_2)]\, dW_{u_1}(i_1)\, dW_{u_2}(i_2).\end{aligned} \tag{25}$$

Putting together equations (24) and (25), our claim (19) is now readily checked. $\square$

Finally, let us close this section by giving the proof of the announced regularity result on $\mathbf{B^2}$.

PROPOSITION 3.3. *The increment* $\mathbf{B^2}$ *is almost surely an element of* $\mathcal{C}_2^{2\gamma}(\mathbb{R}^{d^2})$, *for any* $\gamma < H$.

PROOF. Consider a fixed Hölder exponent $\gamma < H$. The proof of this result is based on Lemma 2.1, which can be read here as $\mathcal{N}[\mathbf{B^2}; \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d^2})] \lesssim A + D$, with

$$A = \left(\int_{\mathcal{S}_{2,T}} \frac{|\mathbf{B}^{\mathbf{2}}_{uv}|^{2p}}{|u-v|^{4\gamma p+4}}\, du\, dv\right)^{1/(2p)} \quad \text{and} \quad D = \mathcal{N}[\delta\mathbf{B^2}; \mathcal{C}_3^{2\gamma}(\mathbb{R}^{d^2})].$$

Let us first deal with the term D above: we have seen that $\mathbf{B^2}$ satisfies the multiplicative property (18), which can be summarized as $\delta\mathbf{B^2} = \delta B \otimes \delta B$. Furthermore, $B \in \mathcal{C}_1^{\gamma}(\mathbb{R}^d)$ for any $\gamma < H$, and thus, for any $1 \le i_1, i_2 \le d$ and $0 \le s < u < t \le T$

$$|\delta\mathbf{B}^{\mathbf{2}}_{sut}(i_1, i_2)| = |\delta B_{su}(i_1)||\delta B_{ut}(i_2)| \le \mathcal{N}^2[B; \mathcal{C}_1^{\gamma}(\mathbb{R}^d)]|u-s|^{\gamma}|t-u|^{\gamma}.$$

In other words, the quantity $\|\delta\mathbf{B^2}\|_{\gamma,\gamma}$ defined by (12) is almost surely finite, and according to definition (13), we obtain that D is also almost surely finite.

We will now show that A is finite almost surely when p is large enough, by proving that $\mathbf{E}[A] < \infty$. Indeed, invoking Jensen's inequality we obtain

$$\begin{aligned}\mathbf{E}[A] &\le \left(\int_{\mathcal{S}_{2,T}} \frac{\mathbf{E}[|\mathbf{B}^{\mathbf{2}}_{uv}|^{2p}]}{|u-v|^{4\gamma p+4}}\, du\, dv\right)^{1/(2p)} \\ &\lesssim \left(\int_{\mathcal{S}_{2,T}} \frac{\mathbf{E}^p[|\mathbf{B}^{\mathbf{2}}_{uv}|^{2}]}{|u-v|^{4\gamma p+4}}\, du\, dv\right)^{1/(2p)},\end{aligned} \tag{26}$$

where we have used the fact that $\mathbf{B^2}$ belongs to the second chaos of W, on which all the L^p norms are equivalent. On the other hand, Proposition 3.1 gives $\mathbf{E}^p[|\mathbf{B}^{\mathbf{2}}_{uv}|^2] \lesssim |u-v|^{4pH}$, and plugging this inequality into (26), we obtain that $\mathbf{E}[A]$ is finite as long as $p > 1/(H-\gamma)$. $\square$

In conclusion, putting together the last two propositions, we have constructed an element $\mathbf{B^2}$ which satisfies the properties (i)–(iii) given at the beginning of the section, for any $H < 1/2$.

4. General case. The aim of this section is to prove Theorem 1.1 in its full generality. Recall that we define our substitute $\mathbf{B^n}$ to nth order integrals in the following way: for $2 \le n \le \lfloor 1/H \rfloor$, any tuple $(i_1, \dots, i_n)$ of elements of $\{1, \dots, d\}$, $1 \le j \le n$ and $(s,t) \in \mathcal{S}_{2,T}$, set

$$\begin{aligned} &\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n) \\ &\quad = (-1)^{j-1} \int_{A^n_j} \prod_{l=1}^{j-1} K(s,u_l)[K(t,u_j) - K(s,u_j)] \\ &\qquad \times \prod_{l=j+1}^{n} K(t,u_l)\, dW_{u_1}(i_1)\cdots dW_{u_n}(i_n), \end{aligned} \tag{27}$$

where the kernel K is given by (5) and A^n_j is the subset of $[0,t]^n$ defined by

$$\begin{aligned} A^n_j = \{&(u_1,\dots,u_n) \in [0,t]^n; \\ &u_j = \min(u_1,\dots,u_n), u_1 > \cdots > u_{j-1} \text{ and } u_{j+1} < \cdots < u_n\}. \end{aligned}$$

The 1-increment $\mathbf{B}^{\mathbf{n}}$ is then given by

$$\mathbf{B}^{\mathbf{n}}_{st}(i_1,\dots,i_n) = \sum_{j=1}^{n-1} \hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n). \tag{28}$$

It is obviously harder to reproduce the heuristic considerations leading to this expression than in Section 3.1. Let us just mention that the same kind of changes in the order of integration allows us to produce some 1-increments similar to $\mathbf{A}^{\mathbf{2},1}, \mathbf{A}^{\mathbf{2},2}$. Then the reordering trick yields some terms of the form $\hat{\mathbf{B}}^{\mathbf{n},j}_{st}$. After observing the form of several of these terms, the general expression (27) is then intuited in a natural way.

Notation: in order to write shorter formulas in the computations below, we use the following conventions in the sequel, whenever possible:

(i) A product of kernels of the form $\prod_{j=1}^n K(\tau_j, u_j)$ will simply be denoted by $\prod_{j=1}^n K_{\tau_j}$, meaning that the variable u_j has to be understood according to the position of the kernel K in the product.

(ii) In the same context, we will also set δK_{st} for a quantity of the form $K(t,u_j) - K(s,u_j)$.

(iii) Furthermore, when all the τ_j are equal to the same instant t, we write $\prod_{j=1}^n K(t,u_j) = K_t^{\otimes n}$.

(iv) Finally, we will also shorten the notation for the increments of the Wiener process W, and simply write dW for $\prod_{j=1}^n dW_{u_j}(i_j)$.

All these conventions allow us, for instance, to summarize formula (27) into

$$\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1,\dots,i_n) = (-1)^{j-1} \int_{A^n_j} K_s^{\otimes(j-1)} \delta K_{st} K_t^{\otimes(n-j)}\, dW. \tag{29}$$

4.1. *Moments of the* **n***th order integrals*. As in Section 3.2, an important step of our analysis is a control of the second moment of $\mathbf{B^n}$. This is given in the following proposition.

PROPOSITION 4.1. *For $n \le \lfloor \frac{1}{H} \rfloor$, let $\mathbf{B}^n_{st}$ be defined by* (28). *Then for $(s,t) \in \mathcal{S}_{2,T}$, we have*

$$\mathbf{E}[|\mathbf{B}^n_{st}|^2] \le C(t-s)^{2nH},$$

for a strictly positive constant C.

PROOF. Thanks to decomposition (28), it suffices to show that for any fixed family of indexes $i_1, \dots, i_n \in \{1, \dots, d\}$ and for any $1 \le j \le n-1$, we have

$$\mathbf{E}[|\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1, \dots, i_n)|^2] \le C(t-s)^{2nH}.$$

Invoking now expression (27) for $\hat{\mathbf{B}}^{\mathbf{n},j}$ and decomposing the integral over the region A_j appearing in the definition of $\hat{\mathbf{B}}^{\mathbf{n},j}_{st}(i_1, \dots, i_n)$ into sums of integrals over the simplex by means of Fubini's theorem, it suffices to show an inequality of the type

$$\mathbf{E}[(Q_{st})^2] \le C(t-s)^{2nH} \qquad \text{with } Q_{st} = \int_{0<u_1<\cdots<u_n<t} \delta K_{st} \prod_{i=2}^{n} K_{\tau_i}\, dW. \tag{30}$$

Notice that in the expression above, we made use of the notation introduced at the beginning of the current section, and for $i = 1, \dots, n$, we assume $\tau_i = s$ or t. We concentrate our efforts now in proving (30).

Let us further decompose Q into $Q = Q^1 + Q^2$, where

$$\begin{aligned} Q^1_{st} &= \int_{s<u_1<\cdots<u_n<t} K_t^{\otimes n}\, dW \quad \text{and} \\ Q^2_{st} &= \int_{0<u_1<\cdots<u_n<t, u_1<s} \delta K_{st} \prod_{i=2}^{n} K_{\tau_i}\, dW, \end{aligned} \tag{31}$$

as in the proof of Lemma 2.4. Notice that in Q^1_{st} we have assumed that $\tau_i = t$ for all i, since otherwise this term vanishes. Moreover, the term Q^1_{st} can be handled using the properties of the multiple Stratonovich integrals established in Lemma 2.7, and applying the estimate obtained in Lemma 2.3. This yields easily the relation $\mathbf{E}[(Q^1_{st})^2] \lesssim (t-s)^{2nH}$.

Concerning Q^2_{st}, one can write $Q^2_{st} = \sum_{j=1}^{n} B^j_{st}$ where

$$B^j_{st} = \int_{0<u_1<\cdots<u_j<s<u_{j+1}<\cdots<u_n<t} \delta K_{st} \prod_{i=2}^{n} K_{\tau_i}\, dW.$$

Notice that in the above equation $\tau_i = t$ if $i = j+1, \dots, n$, since we have again $B^j_{st} = 0$ otherwise. Each term B^j_{st} can thus be written as the product of two factors: $B^j_{st} = C^j_{st} D^j_{st}$, where for $j \ge 2$

$$C^j_{st} = \int_{0<u_1<\cdots<u_j<s} \delta K_{st} \prod_{i=2}^{j} K_{\tau_i}\, dW$$

and

$$D^j_{st} = \int_{s<u_{j+1}<\cdots<u_n<t} K_t^{\otimes(n-j)}\, dW,$$

and for $j=1$, $C^1_{st} = \int_0^s \delta K_{st}\, dW$ and D^1_{st} is given by the above formula.

The random variables C^j_{st} and D^j_{st} are independent, and $\mathbf{E}[(D^j_{st})^2]$ can be bounded easily like $\mathbf{E}[(Q^1_{st})^2]$. Hence we obtain

$$\mathbf{E}[(B^j_{st})^2] = \mathbf{E}[(C^j_{st})^2]\mathbf{E}[(D^j_{st})^2] \le C\mathbf{E}[(C^j_{st})^2](t-s)^{2(n-j)H}. \tag{32}$$

In order to bound the second moment of C^j_{st}, we express this factor as a sum of Itô integrals by means of Proposition 2.6. To do this, we give up for a moment our convention on products of increments, and we define, for $u \in [0,s]$ and $l = 2, \dots, j$, the processes

$$Y_u(1) = \int_0^u [K(t,v) - K(s,v)]\, dW_v(i_1) \quad \text{and} \quad Y_u(l) = \int_0^u K(\tau_l, v)\, dW_v(i_l).$$

Then, the processes $\{Y_u(l); 0 \le u \le s\}$ are Gaussian martingales and

$$C^j_{st} = \int_{0<u_1<\cdots<u_j<s} dY_{u_1}(1)\, dY_{u_1}(2) \cdots dY_{u_l}(l).$$

Thus, a direct application of Proposition 2.6 yields

$$C^j_{st} = \sum_{k=\lfloor j/2 \rfloor}^{j} \frac{1}{2^{j-k}} \sum_{\nu \in D^k_j} J_{0s}(\nu),$$

where

$$J_{0s}(\nu) = \int_{0<u_1<\cdots<u_k<s} \partial Z_{u_1}(1) \cdots \partial Z_{u_k}(k),$$

for $\nu = (j_1, \dots, j_k)$. Thus, setting $\sum_{l=1}^h j_l = m(h)$, we have $Z(h) = Y(i_{m(h)})$ if $j_h = 1$, and $Z_u(h) = \langle Y(m(h)-1), Y(m(h)) \rangle_u$ if $j_h = 2$ and $i_{m(h)-1} = i_{m(h)}$, where $\langle \cdot, \cdot \rangle$ designates the bracket of two continuous martingales. We are going to estimate $\mathbf{E}[J_{0s}(\nu)^2]$ using a recursive argument. This will be done in several steps.

Step 1: *suppose* $j_k, j_{k-1}, \dots, j_1 = 2$. Then $j = 2k$, and we can assume that $i_m = i_{m-1}$ for $m = 2, 4, \dots, 2k$, otherwise $J_{0s}(\nu) = 0$. The term $J_{0s}(\nu)$ is deterministic and it can be expressed as follows:

$$J_{0s}(\nu) = \int_{0<u_1<\cdots<u_k<s} [K(t,u_1) - K(s,u_1)]K(\tau_2,u_1) \times \prod_{h=2}^{k} K(\tau_{2h-1},u_h)K(\tau_{2h},u_h)\,du_1\cdots du_k.$$

As a consequence, owing to (14) and (15), we have

$$|J_{0s}(\nu)| \le C \int_{0<u_1<\cdots<u_k<s} \varphi_{u_1}^{(1)} \prod_{h=2}^{k} \varphi_{u_h}^{(2)}\,du_1\cdots du_k, \tag{33}$$

where

$$\varphi_{u_1}^{(1)} = [(s-u_1)^{H-1/2} - (t-u_1)^{H-1/2}][(s-u_1)^{H-1/2} + u_1^{H-1/2}]$$

and

$$\varphi_{u_h}^{(2)} = [(s-u_h)^{H-1/2} + u_h^{H-1/2}]^2.$$

Moreover, the integral of $\prod_{h=2}^{k} \varphi_{u_h}^{(2)}$ is easily bounded: indeed, we have

$$\begin{aligned}
&\int_{u_1<u_2<\cdots<u_k<s} \prod_{h=2}^{k} \varphi_{u_h}^{(2)}\,du_2\cdots du_k \\
&\quad \le \int_{u_1<u_2<\cdots<u_k<s} \prod_{h=2}^{k} [(s-u_h)^{H-1/2} + (u_h-u_1)^{H-1/2}]^2\,du_2\cdots du_k \\
&\quad \le C\int_{[u_1,s]^{k-1}} \prod_{h=2}^{k} [(s-u_h)^{2H-1} + (u_h-u_1)^{2H-1}]\,du_2\cdots du_k \\
&\quad \le C(s-u_1)^{2(k-1)H},
\end{aligned}$$

with the convention $u_{k+1} = s$. Therefore, plugging this inequality into (33) and making the change of variables $s - u_1 = v$ and $y = \frac{v}{t-s}$, we get

$$\begin{aligned}
|J_{0s}(\nu)| &\le C\int_0^s [(s-u_1)^{H-1/2} - (t-u_1)^{H-1/2}][(s-u_1)^{H-1/2} + u_1^{H-1/2}] \\
&\qquad \times (s-u_1)^{2(k-1)H}\,du_1 \\
&= C\int_0^s [v^{H-1/2} - (t-s+v)^{H-1/2}][v^{H-1/2} + (s-v)^{H-1/2}] \\
&\qquad \times v^{2(k-1)H}\,dv
\end{aligned}$$

$$= C(t-s)^{2kH} \int_0^{s/(t-s)} [y^{H-1/2} - (1+y)^{H-1/2}] \times \left[y^{H-1/2} + \left(\frac{s}{t-s} - y\right)^{H-1/2}\right] y^{2(k-1)H}\, dy.$$

We are now in a position to use Lemma 2.5 with $A = s/(t-s)$, and we obtain

$$|J_{0s}(\nu)| \le C(t-s)^{2kH}, \tag{34}$$

which implies that $J_{0s}(\nu)^2 \le C(t-s)^{2jH}$, owing to the fact that $2k = j$.

Step 2: *suppose that* $j_k = 1$. Then Proposition 2.6 gives

$$J_{0s}(\nu) = \int_{0<u_1<\cdots<u_k<s} \partial Z_{u_1}(1) \cdots \partial Z_{u_{k-1}}(k-1) K(\tau_j, u_k)\, \partial W_u(i_j)$$

and

$$\mathbf{E}[J_{0s}(\nu)^2] = \int_0^s \mathbf{E}(J_{0u}(\nu')^2) K(\tau_j, u)^2\, du \le \int_0^s \mathbf{E}(J_{0u}(\nu')^2)\big((s-u)^{2H-1} + u^{2H-1}\big)\, du,$$

with $\nu' = (j_1, \dots, j_{k-1})$. This relation allows us to set an induction procedure, as we shall see later.

Step 3: *suppose that* $j_k, j_{k-1}, \dots, j_{b+1} = 2$ *and* $j_b = 1$, *where* $b \ge 2$. We assume that $i_{m(h)} = i_{m(h)-1}$ for $h = b+1, \dots, k$. Here again, Proposition 2.6 implies

$$J_{0s}(\nu) = \int_{0<u_1<\cdots<u_k<s} \partial Z_{u_1}(1) \cdots \partial Z_{u_b}(b) \times \prod_{h=b+1}^{k} K\big(\tau_{m(h)-1}, u_h\big) K\big(\tau_{m(h)}, u_h\big)\, du_1 \cdots du_k,$$

and Fubini's theorem yields

$$J_{0s}(\nu) = \int_0^s J_{0u_b}(\nu') K\big(\tau_{m(h)}, u_b\big) G(u_b)\, dW_{u_b}\big(i_{m(h)}\big),$$

with $\nu' = (j_1, \dots, j_{b-1})$, and where

$$G(u_b) = \int_{0<u_b<u_{b+1}<\cdots<u_k<s} \prod_{h=b+1}^{k} K\big(\tau_{m(h)-1}, u_h\big) \times K\big(\tau^{m(h)}, u_h\big)\, du_{b+1} \cdots du_k.$$

As for the previous bound (34) we obtain

$$|G(u_b)| \le C(s-u_b)^{2(k-b)H}.$$

Therefore

$$\mathbf{E}[J_{0s}(\nu)^2] = \int_0^s \mathbf{E}[J_{0u_b}(\nu')^2] K\big(\tau_{m(h)}, u_b\big)^2 G(u_b)^2\, du_b$$
$$\le C \int_0^s \mathbf{E}[J_{0u_b}(\nu'')^2][(s-u_b)^{2H-1} + u_b^{2H-1}](s-u_b)^{4(k-b)H}\, du_b.$$

Notice that the above inequality includes the inequality obtained in Step 2, which corresponds to the case $b = k$.

Step 4: *suppose that* $j_k, j_{k-1}, \dots, j_{b+1} = 2$, $j_b = 1$, $j_{b-1}, j_{b-2}, \dots, j_{c+1} = 2$ *and* $j_c = 1$, *where* $2 \le c \le b$. We assume also that $i_{m(h)} = i_{m(h)-1}$ for $h = c+1, \dots, b-1, b+1, \dots, k$. By the same arguments as in Step 2 we obtain

$$\mathbf{E}[J_{0s}(\nu)^2] \le C \int_{0<u_c<u_b<s} \mathbf{E}[J_{0u_c}(\nu')^2][(u_b-u_c)^{2H-1} + u_c^{2H-1}]$$
$$\times (u_b-u_c)^{4(b-c)H}[(s-u_b)^{2H-1} + u_b^{2H-1}]$$
$$\times (s-u_b)^{4(k-b)H}\, du_c\, du_b,$$

with $\nu' = (j_1, \dots, j_{c-1})$. Replacing u_b^{2H-1} by $(u_b-u_c)^{2H-1}$ and integrating with respect to u_b yields

$$\mathbf{E}[J_{0s}(\nu)^2] \le C \int_0^s \mathbf{E}[J_{0u_c}(\nu')^2][(s-u_c)^{2H-1} + u_c^{2H-1}]$$
$$\times (s-u_c)^{4(k-c)H+2H}\, du_c.$$

Step 5: *iteration scheme*. Iterating the argument in Step 4, we reduce the size of ν' until we obtain a multiindex of length r such that $\nu' = (1, 2, \dots, 2)$ or $\nu' = (2, 2, \dots, 2)$, with $j_{r+1} = 1$, and we obtain an estimate of the form

$$\mathbf{E}[J_{0s}(\nu)^2] \le C \int_0^s \mathbf{E}[J_{0u}(\nu')^2][(s-u)^{2H-1} + u^{2H-1}] \times (s-u)^{2H\sum_{l=r+2}^k j_l}\, du. \tag{35}$$

Suppose first that $\nu' = (1, 2, \dots, 2)$. Then,

$$J_{0s}(\nu') = \int_{0<u_1<\cdots<u_r<u} [K(t,u_1) - K(s,u_1)]$$
$$\times \prod_{h=2}^r K\big(\tau_{m(h)-1}, u_h\big) K\big(\tau_{m(h)}, u_h\big)\, dW_{u_1}(i_1)\, du_2 \cdots du_r,$$

and by Fubini's theorem

$$J_{0s}(\nu') = \int_0^u [K(t,u_1) - K(s,u_1)] F(u_1)\, dW_{u_1}(i_1),$$

where

$$F(u_1) = \int_{u_1<u_2<\cdots<u_r<u} \prod_{h=2}^{r} K(\tau_{m(h)-1}, u_h) K(\tau_{m(h)}, u_h)\, du_2 \cdots du_r.$$

As in the proof of (34) we get

$$|F(u_1)| \le C(u-u_1)^{2(r-1)H}.$$

Therefore,

$$\begin{aligned} \mathbf{E}[J_{0s}(\nu')^2] \le C \int_0^u & [(t-u_1)^{H-1/2} - (s-u_1)^{H-1/2}]^2 \\ & \times (u-u_1)^{4(r-1)H}\, du_1. \end{aligned} \tag{36}$$

Substituting (36) into (35) yields, after integrating in the variable u,

$$\mathbf{E}[J_{0s}(\nu)^2] \le C \int_0^s [(t-u)^{H-1/2} - (s-u)^{H-1/2}]^2 (s-u)^{2(j-1)H}\, du.$$

Performing the changes of variables $v = s - u$ and $y = v/(t-s)$, we end up with

$$\begin{aligned} \mathbf{E}[J_{0s}(\nu)^2] &\le C(t-s)^{2jH} \int_0^{s/(t-s)} [(1+t)^{H-1/2} - y^{H-1/2}]^2 y^{2(j-1)H}\, dy \\ &\le C(t-s)^{2jH}, \end{aligned} \tag{37}$$

where the last step is obtained thanks to a slight variation of Lemma 2.5.

If $\nu' = (2, 2, \dots, 2)$, then we proceed as in Step 1 and we obtain

$$\begin{aligned} |J_{0u}(\nu')| \le C \int_0^u & [(s-u_1)^{H-1/2} - (t-u_1)^{H-1/2}] \\ & \times [(u-u_1)^{H-1/2} + u_1^{H-1/2}] \\ & \times (u-u_1)^{2(r-1)H}\, du_1. \end{aligned} \tag{38}$$

Substituting (38) into (35), integrating first in the variable u and using the same arguments as in Step 1 we obtain also the estimate

$$\mathbf{E}[J_{0s}(\nu)^2] \le C(t-s)^{2jH}. \tag{39}$$

Step 6: *conclusion*. Our bounds (37) and (39) on $J_{0s}(\nu)$ yield the same kind of estimate for the term C_{st}^j. Thus relation (32) gives $B_{st}^j \lesssim (t-s)^{2nH}$. This estimate can now be plugged into the definition (31) of Q^2, then in the definition of Q, which leads to our claim (30). The proof is now complete. □

4.2. *Proof of Theorem* 1.1. Before we prove our main theorem, we need a last elementary technical ingredient, which relies on the notational convention given at the beginning of the current section.

LEMMA 4.2. *For $n \geq 3$, $j = 2, \dots, n-1$ and $0 \leq s < t \leq T$, set*

$$M_{st}^{n,j} = K_s^{\otimes(j-1)} \delta K_{st} K_t^{\otimes(n-j)}.$$

Recall that for an element $M \in \mathcal{C}_2$, δM is defined by (9). *Then*

$$\delta M_{sut}^{n,j} = -\sum_{m=1}^{j-1} K_s^{\otimes(m-1)} \delta K_{su} K_u^{\otimes(j-1-m)} \delta K_{ut} K_t^{\otimes(n-j)} + K_s^{\otimes(j-1)} \delta K_{su} \sum_{m=1}^{n-j} K_u^{\otimes(m-1)} \delta K_{ut} K_t^{\otimes(n-j-m)}.$$

The relation still holds true for $j \in \{1, n\}$ and $n = 2$, with the convention $K^{\otimes 0} = \mathbf{1}$ and $\delta K^{\otimes 0} = 0$.

PROOF. This proof is completely elementary, and included here for the sake of completeness, since it uses heavily the notation of Section 2.1.

First, if a, b, c are 3 increments in $\mathcal{C}_1$, and if we define $N \in \mathcal{C}_2$ by $N_{st} = a_s \delta b_{st} c_t$, then a simple application of Definition (9) gives

$$\delta N_{sut} = -\delta a_{su} \delta b_{ut} c_t + a_s \delta b_{su} \delta c_{ut}.$$

Our claim is thus proved by applying this relation to $a = K^{\otimes(j-1)}$, $b = K$, $c = K^{\otimes(n-j)}$, and observing that $[\delta K^{\otimes l}]_{st} = \sum_{p=1}^{l} K_s^{\otimes(p-1)} \delta K_{st} K_t^{\otimes(l-p)}$. □

PROOF OF THEOREM 1.1. The structure of the proof is the same as in the second order case of Section 3.2: we first reduce the algebraic relations (2) and (3) to the case of some fixed s, u, t by standard considerations. Then we first focus on (2).

Step 1: *proof of the multiplicative property* (2). Fix $(s, u, t) \in \mathcal{S}_{3,T}$. Recall that $\hat{\mathbf{B}}_{st}^{\mathbf{n},j}$ is defined by (29). Therefore, invoking Lemma 4.2, $\delta \hat{\mathbf{B}}^{\mathbf{n},j}$ is given by

$$\begin{aligned} \delta \hat{\mathbf{B}}_{sut}^{\mathbf{n},j}(i_1, \dots, i_n) &= (-1)^j \int_{A_j^n} \sum_{m=1}^{j-1} K_s^{\otimes(m-1)} \delta K_{su} K_u^{\otimes(j-1-m)} \delta K_{ut} K_t^{\otimes(n-j)} \, dW \\ &\quad + (-1)^{j-1} \int_{A_j^n} K_s^{\otimes(j-1)} \delta K_{su} \sum_{m=1}^{n-j} K_u^{\otimes(m-1)} \delta K_{ut} K_t^{\otimes(n-j-m)} \, dW. \end{aligned} \tag{40}$$

On the other hand, set $Z_{sut} = \sum_{n_1=1}^{n-1} \mathbf{B}_{su}^{\mathbf{n_1}} \mathbf{B}_{ut}^{\mathbf{n-n_1}}$. One can easily check that

$$
\begin{aligned}
Z_{sut} &= \sum_{n_1=1}^{n-1} \sum_{k=1}^{n_1} \sum_{h=1}^{n-n_1} \hat{\mathbf{B}}_{su}^{\mathbf{n_1},k} \hat{\mathbf{B}}_{ut}^{\mathbf{n-n_1},h} \\
&= \sum_{n_1=1}^{n-1} \sum_{k=1}^{n_1} \sum_{h=1}^{n-n_1} (-1)^{k+h} \int_{A_{k,h}(n_1)} K_s^{\otimes(k-1)} \delta K_{su} K_u^{\otimes(n_1-k+h-1)} \delta K_{ut} \\
&\qquad \times K_t^{\otimes(n-n_1-h)} \, dW,
\end{aligned}
\tag{41}
$$

where $A_{k,h}(n_1)$ is the set defined by

$$
\begin{aligned}
A_{k,h}(n_1) &= A_k^{n_1} \times A_h^{n-n_1} \\
&= \{(u_1, \ldots, u_n); u_k < u_{k+1} < \cdots < u_{n_1}, u_k < u_{k-1} < \cdots < u_1, \\
&\qquad u_{n_1+h} < u_{n_1+h+1} < \cdots < u_n, u_{n_1+h} < u_{n_1+h-1} < \cdots < u_{n_1+1}\}.
\end{aligned}
$$

We want to show that (41) and (40) coincide.

In order to follow the computations below, it might be useful to keep in mind an illustration of the coordinate ordering on a set of the form $A_{k,h}(m)$, for which an example is provided at Figure 1 (note that the ordering between u_m and u_{m+1} is not specified).

Notice that on the set $A_{k,h}(n_1) \cap \{u_k < u_{n_1+h}\}$ the minimum of the coordinates is u_k, and on the set $A_{k,h}(n_1) \cap \{u_{n_1+h} < u_k\}$ the minimum is u_{n_1+h}. Define

$$
A_{h,k}^1(n_1) = A_{k,h}(n_1) \cap \{u_k < u_{n_1+h}\} \quad \text{and} \quad A_{h,k}^2(n_1) = A_{k,h}(n_1) \cap \{u_{n_1+h} < u_k\}.
$$

Consider now the decomposition $Z = Z^1 + Z^2$, where

$$
\begin{aligned}
Z_{sut}^i = \sum_{n_1=1}^{n-1} \sum_{k=1}^{n_1} \sum_{h=1}^{n-n_1} (-1)^{k+h} \int_{A_{k,h}^i(n_1)} & K_s^{\otimes(k-1)} \delta K_{su} K_u^{\otimes(n_1-k+h-1)} \delta K_{ut} \\
& \times K_t^{\otimes(n-n_1-h)} \, dW.
\end{aligned}
$$

We fix j and we try to compute the contribution of Z_{sut}^i on the set A_j^n for $i = 1, 2$. This contribution will be the sum of the integrals on the set $A_j^n \cap A_{k,h}^i(n_1)$, for each $k = 1, \ldots, n_1$, $h = 1, \ldots, n - n_1$ and for each $n_1 = 1, \ldots, n - 1$.

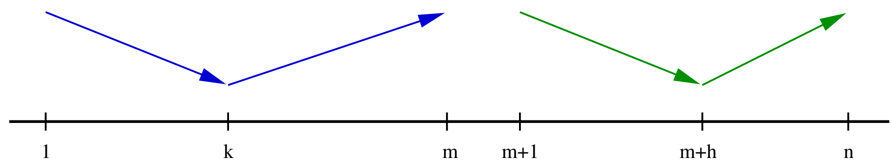

FIG. 1. *Coordinates ordering on $A_{k,h}(m)$.*

Notice first that the intersection $A_j^n \cap A_{k,h}^1(n_1)$ is nonempty only if $k = j$, $h = 1$ and $u_{n_1} < u_{n_1+1}$ which also implies $j \le n_1$. Moreover, in this case we have $A_{j,1}^1(n_1) \cap \{u_{n_1} < u_{n_1+1}\} = A_j^n$. In this way we obtain that the contribution of Z_{sut}^1 on A_j^n is

$$
\begin{aligned}
&(-1)^{j-1} \sum_{n_1=j}^{n-1} \int_{A_j^n} K_s^{\otimes(j-1)} \delta K_{su} K_u^{\otimes(n_1-j)} \delta K_{ut} K_t^{\otimes(n-n_1-1)}\, dW \\
&\quad = (-1)^{j-1} \sum_{m=1}^{n-j} \int_{A_j^n} K_s^{\otimes(j-1)} \delta K_{su} K_u^{\otimes(m-1)} \delta K_{ut} K_t^{\otimes(n-m-j)}\, dW,
\end{aligned}
\tag{42}
$$

where we have used the simple change of variables $n_1 - j = m - 1$. In the same manner, on the set $A_j^n \cap A_{k,h}^2(n_1)$ we have $k = n_1$, $n_1 + h = j$, which also implies $n_1 \le j - 1$. Therefore, the contribution of Z_{sut}^2 on A_j^n is

$$
(-1)^j \sum_{n_1=1}^{j-1} \int_{A_j^n} K_s^{\otimes(n_1-1)} \delta K_{su} K_u^{\otimes(j-1-n_1)} \delta K_{ut} K_t^{\otimes(n-j)}\, dW. \tag{43}
$$

One can now easily verify that the sum of (42) and (43) is equal to the term (40).

It remains to prove that the contribution of Z_{sut} to the set $(\bigcup_j A_j^n)^c$ is zero. For this, observe that $(\bigcup_j A_j^n)^c$ can be split into slices $D_{k,p,h}$ of the following form: for $1 \le k \le p \le n-1$, we assume that $u_k < u_{k-1} < \cdots < u_1$ and $u_k < u_{k+1} < \cdots < u_p$ but $u_p > u_{p+1}$. Suppose also that $1 \le h \le n - p$ and that u_{p+h} is the minimum of the coordinates $u_{p+1}, \dots, u_n$. Then, for $D_{k,p,h}$ to be a subset of $\bigcup_{n_1=1}^n \bigcup_{k,h} A_{k,h}(n_1)$, we need the further condition $u_{p+h} < u_{p+h+1} < \cdots < u_n$ and $u_{p+h} < u_{p+h-1} < \cdots < u_{p+1}$. With all these constraints in mind, it is easily seen that $D_{k,p,h}$ corresponds to two possible choices of set $A_{k,h}(n_1)$. Indeed, we have

$$
D_{k,p,h} = A_{k,h}(p) = A_{k,h+1}(p-1).
$$

Going back now to the expression (41) of Z_{sut}, it is readily checked that the two contributions, respectively, on $A_{k,h}(p)$ and $A_{k,h+1}(p-1)$, yield two terms with opposite sign, which cancel out in the sum.

Step 2: *proof of the geometric property* (3). Fix n, m such that $n + m \le \lfloor 1/\gamma \rfloor$ and let $(s,t) \in \mathcal{S}_{2,T}$. Consider the product

$$
\begin{aligned}
&\mathbf{B}_{st}^{\mathbf{n}}(i_1, \dots, i_n) \mathbf{B}_{st}^{\mathbf{m}}(j_1, \dots, j_m) \\
&\quad = \sum_{j=1}^n \sum_{h=1}^m (-1)^{j+h} \left(\int_{A_j^n} K_s^{\otimes(j-1)} \delta K_{st} K_t^{\otimes(n-j)}\, dW \right) \\
&\qquad \times \left(\int_{A_h^m} K_s^{\otimes(h-1)} \delta K_{st} K_t^{\otimes(m-h)}\, dW \right),
\end{aligned}
$$

where we have used notation (29) and where we recall that the sets A_j^n and A_h^m are defined by

$$A_j^n = \{u \in [0,t]^n : u_j < u_{j-1} < \cdots < u_1, u_j < u_{j+1} < \cdots < u_n\},$$
$$A_h^m = \{v \in [0,t]^m : v_h < v_{h-1} < \cdots < v_1, v_h < v_{h+1} < \cdots < v_m\}.$$

The product of the two Stratonovich integrals can be expressed as a Stratonovich integral on the region $A_j^n \times A_h^m$ with respect to the differential

$$dW_{u_1}(i_1) \cdots dW_{u_n}(i_n)\, dW_{v_1}(j_1) \cdots dW_{v_m}(j_m).$$

We will make use of the notation $z = (u, v)$, where $z_\alpha = u_\alpha$, for $\alpha = 1, \dots, n$ and $z_\alpha = v_{\alpha-n}$ for $\alpha = n+1, \dots, n+m$. As in Step 1, the region $A_j^n \times A_h^m$ can be first decomposed into the union of the disjoint regions $D_{j,h}$ and $E_{j,h}$, corresponding, respectively, to the additional constraints $\{u_j < v_h\}$ and $\{u_j > v_h\}$ (notice that this decomposition is valid up to the set $\{u_j = v_h\}$, whose contribution to the stochastic integral is null).

Consider first the case $\{u_j < v_h\}$. On $D_{j,h}$ the minimum of all the coordinates z_α is z_j. Then $D_{j,h}$ can be further decomposed into the disjoint union of the sets

$$\begin{aligned} D_{j,h,1}^{\pi} = \{z \in [0,t]^{n+m} : {} & z_j < z_{\alpha_{j+h-2}} < \cdots < z_{\alpha_1}, \\ & z_j < z_{\beta_1} < \cdots < z_{\beta_{n-j+1+m-h}}\} \\ \cap \{z_{n+h} < z_{n+h-1}\}, \end{aligned}$$

where

$$\pi(1, \dots, n+m) = (\alpha_1, \dots, \alpha_{j+h-2}, j, \beta_1, \dots, \beta_{n-j+1+m-h})$$

runs over all permutations of the coordinates $1, \dots, n+m$ such that $\pi(j+h-1) = j$ and:

(i) $\alpha_1, \dots, \alpha_{j+h-2}$ is a permutation of the coordinates $1, \dots, j-1$ and $n+1, \dots, n+h-1$ that preserves the orderings of the indices $1, \dots, j-1$ and $n+1, \dots, n+h-1$.

(ii) $\beta_1, \dots, \beta_{n-j+1+m-h}$ is a permutation of the coordinates $j+1, \dots, n$ and $n+h, \dots, n+m$ that preserves the orderings of the indices $j+1, \dots, n$ and $n+h, \dots, n+m$.

Notice that α is the inverse of a shuffle since it splits an ordered list into two ordered sublists. The same remark applies to β.

Moreover, $D_{j,h}$ can be also be decomposed into the disjoint union of the sets

$$\begin{aligned} D_{j,h,2}^{\tilde\pi} = \{z \in [0,t]^{n+m} : {} & z_j < z_{\alpha_{j+h-1}} < \cdots < z_{\alpha_1}, \\ & z_j < z_{\beta_1} < \cdots < z_{\beta_{n-j+m-h}}\} \\ \cap \{z_{n+h} < z_{n+h+1}\}, \end{aligned}$$

where

$$\tilde{\pi}(1,\dots,n+m)=(\alpha_1,\dots,\alpha_{j+h-1},j,\beta_1,\dots,\beta_{n-j+m-h})$$

runs over all permutations of the coordinates $1,\dots,n+m$ such that $\tilde{\pi}(j+h)=j$ and:

(i) $\alpha_1,\dots,\alpha_{j+h-1}$ is a permutation of the coordinates $1,\dots,j-1$ and $n+1,\dots,n+h$ that preserves the orderings of the indices $1,\dots,j-1$ and $n+1,\dots,n+h$.

(ii) $\beta_1,\dots,\beta_{n-j+m-h}$ is a permutation of the coordinates $j+1,\dots,n$ and $n+h+1,\dots,n+m$ that preserves the orderings of the indices $j+11,\dots,n$ and $n+h+1,\dots,n+m$.

Then, on the set $D_{j,h}$ we write

$$\begin{aligned}&K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}K_s^{\otimes(h-1)}\delta K_{st}K_t^{\otimes(m-h)}\\&\quad=K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}K_s^{\otimes(h-1)}K_t^{\otimes(m-h+1)}\\&\qquad-K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}K_s^{\otimes h}K_t^{\otimes(m-h)},\end{aligned}$$

and the integral

$$I_{j,h}:=\int_{D_{j,h}}K_s^{\otimes(j-1)}\delta K_{st}K_t^{\otimes(n-j)}K_s^{\otimes(h-1)}\delta K_{st}K_t^{\otimes(m-h)}\,dW$$

can be expressed as the sum $I_{j,h}=I_{j,h}^{+}+I_{j,h}^{-}$, with

$$\begin{aligned}I_{j,h}^{+}=\sum_{\pi}\int_{D_{j,h,1}^{\pi}}&(-1)^{j+h-2}\\&\times\prod_{l=1}^{j+h-2}K(s,z_{\alpha_l})\delta K_{st}(z_j)\\&\times\prod_{l=1}^{n-j+1+m-h}K(t,z_{\beta_l})\,dW_{z_1}(i_1)\cdots dW_{z_{n+m}}(i_{n+m})\end{aligned}$$

and

$$\begin{aligned}I_{j,h}^{-}=\sum_{\tilde{\pi}}\int_{D_{j,h,2}^{\tilde{\pi}}}&(-1)^{j+h-1}\\&\times\prod_{l=1}^{j+h-1}K(s,z_{\alpha_l})\delta K_{st}(z_j)\\&\times\prod_{l=1}^{n-j+m-h}K(t,z_{\beta_l})\,dW_{z_1}(i_1)\cdots dW_{z_{n+m}}(i_{n+m}).\end{aligned}$$

Let us handle first the term $I^+_{j,h}$: consider the permutation $\sigma=\pi^{-1}$ of $1,\dots,n+m$ which maps $\alpha_1,\dots,\alpha_{j+h-2}$ into $1,\dots,j+h-2$ and $\beta_1,\dots,\beta_{n-j+1+m-h}$ into $j+h,\dots,n+m$, with the additional condition $\sigma(j)=j+h-1$. If we make this permutation in the coordinates of $I^+_{j,h}$ we obtain

$$I^+_{j,h}=\int_{A^{n+m}_{j+h-1}\cap\{z_\nu<z_\eta\}}(-1)^{j+h-2}\times\prod_{l=1}^{j+h-2}K(s,z_l)\delta K_{st}(z_{j+h-1})\times\prod_{l=j+h}^{n+m}K(t,z_l)\,dW_{z_1}(k_1)\cdots dW_{z_{n+m}}(k_{n+m}),$$

where $k_1,\dots,k_{n+m}$ is a permutation of the indexes $i_1,\dots,i_n,j_1,\dots,j_m$ defined by $k_\ell=i_{\sigma(\ell)}$ if $1\le\sigma(\ell)\le n$ and $k_\ell=j_{\sigma(\ell)}$ if $n+1\le\sigma(\ell)\le n+m$, and where ν,η are defined by

$$\nu=\min\{i\ge j+h:k_i\in\{j_1,\dots,j_m\}\},$$
$$\eta=\max\{i\le j+h-2:k_i\in\{j_1,\dots,j_m\}\}.$$

In the same way, we can consider a permutation $\sigma=\tilde\pi^{-1}$ in the coordinates z_i which maps $\alpha_1,\dots,\alpha_{j+h-1}$ into $1,\dots,j+h-1$ and $\beta_1,\dots,\beta_{n-j+m-h}$ into $j+h+1,\dots,n+m$, and $\sigma(j)=j+h$. If we make this permutation in the coordinates of $I^-_{j,h}$ we obtain

$$I^-_{j,h}=\int_{A^{n+m}_{j+h}\cap\{z_\nu>z_\eta\}}(-1)^{j+h-1}\times\prod_{l=1}^{j+h-1}K(s,z_l)\delta K_{st}(z_{j+h-1})\times\prod_{l=j+h+1}^{n+m}K(t,z_l)\,dW_{z_1}(k_1)\cdots dW_{z_{n+m}}(k_{n+m}),$$

where again $k_1,\dots,k_{n+m}$ is a permutation of the indexes $i_1,\dots,i_n,j_1,\dots,j_m$ defined by $k_\ell=i_{\sigma(\ell)}$ if $1\le\sigma(\ell)\le n$ and $k_\ell=j_{\sigma(\ell)}$ if $n+1\le\sigma(\ell)\le n+m$, and where ν,η are now defined by

$$\nu=\min\{i\ge j+h+1:k_i\in\{j_1,\dots,j_m\}\},$$
$$\eta=\max\{i\le j+h-1:k_i\in\{j_1,\dots,j_m\}\}.$$

When we sum these integrals over all permutations σ of the above type, that is $\sigma=\pi^{-1}$ or $\sigma=\tilde\pi^{-1}$, and also over j and h, we obtain $\sum_{\bar k\in\mathrm{Sh}(\bar\imath,\bar\jmath)}\mathbf{B}^{\mathbf{n+m},1}_{st}(k_1,\dots,$

$k_{n+m})$, where

$$
\begin{aligned}
\mathbf{B}_{st}^{\mathbf{n+m},1}&(k_1,\dots,k_{n+m})\\
&=\sum_{p=1,k_p\in\{i_1,\dots,i_n\}}^{n+m}\int_{A_p^{n+m}}(-1)^{p-1}\\
&\qquad\qquad\times\prod_{l=1}^{p-1}K(s,z_l)\delta K_{st}(z_p)\\
&\qquad\qquad\times\prod_{l=p+1}^{n+m}K(t,z_l)\,dW_{z_1}(k_1)\cdots dW_{z_{n+m}}(k_{n+m}).
\end{aligned}
$$

In a similar manner we could show that the sum of the integrals over $E_{j,h}$ give rise to $\sum_{\bar{k}\in \mathrm{Sh}(\bar{\imath},\bar{\jmath})}\mathbf{B}_{st}^{\mathbf{n+m},2}(k_1,\dots,k_{n+m})$, for $h=1,\dots,m$, where

$$
\begin{aligned}
\mathbf{B}_{st}^{\mathbf{n+m},2}&(k_1,\dots,k_{n+m})\\
&=\sum_{p=1,k_p\in\{j_1,\dots,j_m\}}^{n+m}\int_{A_p^{n+m}}(-1)^{p-1}\\
&\qquad\qquad\times\prod_{l=1}^{p-1}K(s,z_l)\delta K_{st}(z_p)\\
&\qquad\qquad\times\prod_{l=p+1}^{n+m}K(t,z_l)\,dW_{z_1}(k_1)\cdots dW_{z_{n+m}}(k_{n+m}).
\end{aligned}
$$

Taking into account the two contributions $\mathbf{B}_{st}^{\mathbf{n+m},1}$ and $\mathbf{B}_{st}^{\mathbf{n+m},2}$, the proof of the geometric property is now easily finished.

Step 3: *proof of the regularity property*. As in Proposition 3.3, the fact that $\mathbf{B}^{\mathbf{n}}$ belongs to $\mathcal{C}_2^{n\gamma}$ for any $\gamma<H$ is an easy consequence of the moment estimate of Proposition 4.1, plus a simple induction procedure.

Indeed, assume that $\mathbf{B}^{\mathbf{k}}\in\mathcal{C}_2^{k\gamma}((\mathbb{R}^d)^{\otimes k})$ for any $k\le n-1$. Then Lemma 2.1 gives here that $\mathcal{N}[\mathbf{B}^{\mathbf{n}};\mathcal{C}_2^{n\gamma}(\mathbb{R}^{d^2})]\lesssim A+D$, with

$$
A=\left(\int_{\mathcal{S}_{2,T}}\frac{|\mathbf{B}_{uv}^{\mathbf{n}}|^{2p}}{|u-v|^{2n\gamma p+4}}\,du\,dv\right)^{1/(2p)}\quad\text{and}\quad D=\mathcal{N}[\delta\mathbf{B}^{\mathbf{n}};\mathcal{C}_3^{n\gamma}(\mathbb{R}^{d^2})].
$$

Furthermore, since we have seen that $\mathbf{B}^{\mathbf{n}}$ satisfies the multiplicative property (2), then D is easily shown to be almost surely finite thanks to our induction hypothesis. Finally, the quantity $\mathbf{E}[A]$ can be bounded along the same lines as in Proposition 3.3, except that Proposition 4.1 is used instead of Proposition 3.1. □

5. Relationship with other iterated integrals. This section is devoted to a comparison of the rough path above fBm we have just constructed with other existing iterated integrals. We first treat the case of canonical (or pathwise) integrals defined in [5, 9], focusing on the double iterated integral case. Then we shall try to replace our construction into the general context of Fourier normal ordering as introduced in [23].

5.1. *Comparison with the canonical double iterated integral.* Consider $1/4 < H < 1$. We wish to compare $\mathbf{B^2}$ defined by (23) with the increment $\mathbf{B}^{\mathbf{2},\mathrm{p}}$, where

$$\mathbf{B}^{\mathbf{2},\mathrm{p}}_{st} := \int_{s<u_1<u_2<t} dB_{u_1}(i_1)\, dB_{u_2}(i_2) \tag{44}$$

is interpreted in the following way:

(i) If $1/2 < H < 1$, $\mathbf{B}^{\mathbf{2},\mathrm{p}}_{st}$ is defined in the Young sense (or equivalently in the Stratonovich sense of Malliavin calculus–see [18]).

(ii) If $H = 1/2$, $\mathbf{B}^{\mathbf{2},\mathrm{p}}_{st}$ corresponds to a Stratonovich integral with respect to Brownian motion.

(iii) When $1/4 < H < 1/2$, $\mathbf{B}^{\mathbf{2},\mathrm{p}}_{st}$ is defined by a limiting procedure in [5, 9], but is also shown in [5] to correspond to a Stratonovich integral in the Malliavin calculus sense.

In all those cases, $\mathbf{B}^{\mathbf{2},\mathrm{p}}$ can thus be defined thanks to Malliavin calculus tools, and is also thought of as the canonical double iterated integral for B. We shall keep this definition in mind in the sequel, and refer to [18] for further definitions of Malliavin calculus. Notice that "p" in in our notation $\mathbf{B}^{\mathbf{2},\mathrm{p}}$ stands for pathwise.

Our comparison result for double iterated integrals can be read as follows:

PROPOSITION 5.1. *Consider a d-dimensional fBm B with Hurst index $1/4 < H < 1$. Let $\mathbf{B^2}$ be the increment defined by* (23)*, and $\mathbf{B}^{\mathbf{2},\mathrm{p}}$ defined by* (44)*. For $0 < b < a < t$, set $\psi_t(a,b) = \int_a^t K(v,a)\partial_v K(v,b)\, dv$. Then for $H \in (1/4, 1) \setminus \{1/2\}$, we have $\mathbf{B^2} - \mathbf{B}^{\mathbf{2},\mathrm{p}} = \delta f$, where $f : \mathbb{R}_+ \to \mathbb{R}^{d^2}$ is the process defined by*

$$\begin{aligned} f_t(i_1, i_2) &= \int_{0<u_1<u_2<t} \psi_t(u_2, u_1)\, dW_{u_1}(i_1)\, dW_{u_2}(i_2) \\ &\quad - \int_{0<u_2<u_1<t} \psi_t(u_1, u_2)\, dW_{u_1}(i_1)\, dW_{u_2}(i_2). \end{aligned} \tag{45}$$

In particular, $f(i_1, i_2) \equiv 0$ if $i_1 = i_2$. For $H = 1/2$, one gets the relation $\mathbf{B^2} - \mathbf{B}^{\mathbf{2},\mathrm{p}} = 0$.

REMARK 5.2. Consider the antisymmetric parts $\mathbf{B}^{\mathbf{2},a}$ and $\mathbf{B}^{\mathbf{2},\mathrm{p},a}$ of $\mathbf{B^2}$ and $\mathbf{B}^{\mathbf{2},\mathrm{p}}$, respectively, considered as matrix-valued increments. These objects are usually referred to as Lévy areas of B. Then it is readily checked that $\mathbf{B}^{\mathbf{2},a} - \mathbf{B}^{\mathbf{2},\mathrm{p},a} = \delta f$ as well.

PROOF OF PROPOSITION 5.1. It is easily shown, thanks to Proposition 3.2, that $\delta\mathbf{B}^{\mathbf{2}} = \delta\mathbf{B}^{\mathbf{2},\mathrm{p}}$. We thus know that $\mathbf{B}^{\mathbf{2}} - \mathbf{B}^{\mathbf{2},\mathrm{p}} = \delta f$ for a certain function $f \in \mathcal{C}_1$. Furthermore, a possible choice for f (unique up to constants) is simply

$$f_t = \mathbf{B}^{\mathbf{2}}_{0t} - \mathbf{B}^{\mathbf{2},\mathrm{p}}_{0t}.$$

We shall try to simplify the latter expression, and distinguish 3 cases:

Case 1: $H = 1/2$. In this situation the computations differ slightly from the case $1/4 < H < 1/2$, since in $K_t(u) = \mathbf{1}_{[0,t]}(u)$ instead of the expression given by (5). However, the relation $\mathbf{B}^{\mathbf{2}} - \mathbf{B}^{\mathbf{2},\mathrm{p}} = 0$ is easily verified directly.

Case 2: $1/2 < H < 1$. We treat this situation first, since it is technically simpler than the rougher case $H < 1/2$. The kernel K is given here by [18], equation (5.8), instead of (5), but still satisfies a relation of the form (14), which allows to translate many of the bounds in Section 3. In particular, both increments $\mathbf{B}^{\mathbf{2},\mathrm{p}}$ and $\mathbf{B}^{\mathbf{2}}$ are well defined. However, when $H > 1/2$ we cannot assume $\delta f := \mathbf{B}^{\mathbf{2}} - \mathbf{B}^{\mathbf{2},\mathrm{p}}$ lies in $\mathcal{C}_1^{2\gamma}$, since Lemma 2.4 cannot be applied anymore (additionally, $f \in \mathcal{C}_1^{2\gamma}$ would mean $f \equiv$ Constant). We shall thus only work with $f \in \mathcal{C}_1^{\gamma}$.

In order to find an amenable expression for f, decompose again $\mathbf{B}^{\mathbf{2}}_{0t}$ into $\hat{\mathbf{B}}^{\mathbf{2},1}_{0t} + \hat{\mathbf{B}}^{\mathbf{2},2}_{0t}$. Thanks to the fact that $K(0,\cdot) \equiv 0$, it is then easily seen from equation (22) that $\hat{\mathbf{B}}^{\mathbf{2},2}_{0t} = 0$. Thus, reading (22) in our particular situation yields

$$\mathbf{B}^{\mathbf{2}}_{0t}(i_1, i_2) = \int_{u_1 < u_2} K_t(u_1) K_t(u_2)\, dW_{u_1}(i_1)\, dW_{u_2}(i_2). \tag{46}$$

For $H > 1/2$, a suitable expression for $\mathbf{B}^{\mathbf{2},\mathrm{p}}_{0t}$, obtained by means of a Fubini-type arguments, is

$$\begin{aligned}
\mathbf{B}^{\mathbf{2},\mathrm{p}}_{0t}(i_1, i_2) &= \int_0^t \left(\int_{u_2}^t \partial_v K_v(u_2) B_v(i_1)\, dv \right) dW_{u_2}(i_2) \\
&= \int_{[0,t]^2} \left(\int_{u_1 \vee u_2}^t \partial_v K_v(u_2) K_v(u_1)\, dv \right) dW_{u_1}(i_1)\, dW_{u_2}(i_2) \\
&:= J^1_{st} + J^2_{st},
\end{aligned}$$

where

$$\begin{aligned}
J^1_{st} &= \int_{0 < u_1 < u_2 < t} \left(\int_{u_2}^t \partial_v K_v(u_2) K_v(u_1)\, dv \right) dW_{u_1}(i_1)\, dW_{u_2}(i_2), \\
J^2_{st} &= \int_{0 < u_2 < u_1 < t} \left(\int_{u_1}^t \partial_v K_v(u_2) K_v(u_1)\, dv \right) dW_{u_1}(i_1)\, dW_{u_2}(i_2).
\end{aligned}$$

Owing to a simple integration by parts argument, we have

$$\int_{u_2}^t \partial_v K_v(u_2) K_v(u_1)\, dv = K_t(u_1) K_t(u_2) - \int_{u_2}^t K_v(u_2) \partial_v K_v(u_1)\, dv,$$

and hence

$$J_{st}^1 = \int_{0<u_1<u_2<t} \Big[K(t,u_1)K(t,u_2) - \int_{u_2}^t K_v(u_2)\partial_v K_v(u_1)\,dv \Big]\, dW_{u_1}(i_1)\, dW_{u_2}(i_2)$$

$$= \mathbf{B}_{0t}^{\mathbf{2}}(i_1,i_2) - \int_{0<u_1<u_2<t} \Big(\int_{u_2}^t K_v(u_2)\partial_v K_v(u_1)\,dv \Big)\, dW_{u_1}(i_1)\, dW_{u_2}(i_2).$$

Gathering all the expressions we have obtained so far and recalling our notation $\psi_t(a,b) = \int_a^t K(v,a)\ \partial_v K(v,b)\,dv$ for $0 < b < a < t$, the proof of (45) is now readily completed.

Case 3: $1/4 < H < 1/2$. Many of the computations of Case 2 can be reproduced here, and we will just outline the main differences.

Since $H < 1/2$, Lemma 2.4 and the results in [5, 9] assert that f is an element of $\mathcal{C}_1^{2\gamma}$ in the current situation. Moreover, (46) is still valid for $H < 1/2$, so that we only have to find an alternative expression for $\mathbf{B}_{0t}^{\mathbf{2},\mathrm{p}}$.

Thanks to expression (5.29) in [18], one can write

$$\mathbf{B}_{0t}^{\mathbf{2},\mathrm{p}}(i_1,i_2) = \int_0^t [K_t^* B(i_1)]_{u_2}\, dW_{u_2}(i_2) := L_{st}^1 + L_{st}^2,$$

where

$$L_{st}^1 = \int_0^t \Big(\int_{u_2}^t \partial_v K_v(u_2) \delta B_{u_2 v}(i_1)\,dv \Big)\, dW_{u_2}(i_2),$$

$$L_{st}^2 = \int_0^t K_t(u_2) B_{u_2}(i_1)\, dW_{u_2}(i_2).$$

Then the same kind of arguments as for Case 2 (Fubini-type relations and integration by parts for K) yield $L_{st}^1 = L_{st}^{11} + L_{st}^{12}$, with

$$L_{st}^{11} = \int_{0<u_1<u_2<t} [K_t(u_2)\delta K_{tv}(u_1) - \psi_t(u_2,u_1)]\, dW_{u_1}(i_1)\, dW_{u_2}(i_2),$$

$$L_{st}^{12} = \int_{0<u_2<u_1<t} \psi_t(u_1,u_2)\, dW_{u_1}(i_1)\, dW_{u_2}(i_2).$$

It is also easily checked that

$$L_{st}^2 = \int_{0<u_1<u_2<t} K_t(u_2) K_{u_2}(u_1)\, dW_{u_1}(i_1)\, dW_{u_2}(i_2).$$

Recalling then $\mathbf{B}_{0t}^{\mathbf{2},\mathrm{p}}(i_1,i_2) = L_{st}^{11} + L_{st}^{12} + L_{st}^2$ we end up, after some elementary algebraic manipulations, with expression (45). □

5.2. *Comparison with the construction by Fourier normal ordering*. It is impossible to reproduce here the elegant formalism on which [23] is based. We will thus just content ourselves with giving some hints on the possibility to link our construction with the general Fourier normal ordering program described in the latter reference.

One of the starting points in [23] is that any iterated integral with respect to a function X can be encoded by a tree whose vertices are decorated by $\{1, \dots, d\}$ if X is $\mathbb{R}^d$-valued. A Hopf algebra structure is usually added to this set of trees after the pioneering work of Connes and Kreimer [4], the resulting structure being denoted by $\mathbf{H}$.

In case of a smooth function X, consider $\mathbf{X^n}(i_1, \dots, i_n)$ defined by (1) in the Riemann sense. Let also $\sigma \in \Sigma_n$ be a permutation of $\{1, \dots, n\}$. When one wishes to express $\mathbf{X^n}(i_{\sigma(1)}, \dots, i_{\sigma(n)})$ in terms of integrals involving the indices $i_1, \dots, i_n$ in this exact order, one is naturally led to use operations on trees and forests, encoded in the Hopf algebra structure alluded to above. After a huge amount of formalization explained in [23], this allows us to write, for $0 \le s < t \le T$,

$$\mathbf{X}^{\mathbf{n}}_{st}(i_1, \dots, i_n) = [(\chi_X^s \circ S) * \chi_X^t](\mathbb{T}_n), \tag{47}$$

where $\mathbb{T}_n$ designates the trunk tree of order n decorated by $i_1, \dots, i_n$, χ_X^s is a character defined on $\mathbf{H}$, S stands for the antipode operation characteristic of Hopf algebras and $*$ is a certain convolution product defined on $\mathbf{H}$. Notice that the equivalent of decomposition (47) in [23] involves some so-called *skeleton integrals*, which refer to Fourier transform techniques. Our character χ_X^s is defined in direct coordinates, in concordance with the Volterra-type representation we have chosen.

Still in case of a smooth function X, a further analysis of the terms χ_X^s allows the decomposition (valid for a multiindex $(j_1, \dots, j_n)$ assimilated with its associated trunk tree)

$$\chi_X^s(j_1, \dots, j_n) = \sum_{\sigma \in \Sigma_n} I_s(\mathbb{T}^\sigma), \tag{48}$$

where $\mathbb{T}^\sigma$ is a forest called permutation graph (see [23], Lemma 1.5). This kind of decomposition is the one which has to be generalized to nonsmooth situations. In our context, $I_s(\mathbb{T}^\sigma)$ is obviously a Wiener multiple integral weighted by the kernel K, whose generic form is given by

$$I_s(\mathbb{T}^\sigma) = \int_{u_{\sigma(1)} < \cdots < u_{\sigma(n)}} \prod_{j=1}^n K_{a_j}(u_j)\, dW_{u_j}(i_j),$$

where each $a_j = s$ or t according to the permutation graph under consideration.

The algorithm set up in [23] in order to cope with nonsmooth situations basically replaces the integrals $I_s(\mathbb{T}^\sigma)$ for any $\mathbb{T}^\sigma$ having more than two vertices by something smoothed in Fourier coordinates. Our approach is simpler (and rougher), in the sense that we replace all those integrals by 0. We are thus just left with the permutation graph $\mathbb{T}^{\sigma_0}$ corresponding to $\sigma_0 : (1, \dots, n) \mapsto (n, \dots, 1)$, which is the only one containing trees reduced to a root (see [7] for further explanations). It can then be shown that, reading [23], Lemma 3.6, in this context leads to our definition (28) of the multiple iterated integral with respect to B. In a sense, our construction is thus included in the broader context of [23]. Nevertheless, let us insist on the fact that we provide a simple and direct alternative approach to the problem.

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