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ROUGH VOLTERRA EQUATIONS 1: THE ALGEBRAIC INTEGRATION SETTING

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We define and solve Volterra equations driven by an irregular signal, by means of a variant of the rough path theory called algebraic integration. In the Young case, that is for a driving signal with Hölder exponent $\gamma > 1/2$, we obtain a global solution, and are able to handle the case of a singular Volterra coefficient. In case of a driving signal with Hölder exponent $1/3 < \gamma \le 1/2$, we get a local existence and uniqueness theorem. The results are easily applied to the fractional Brownian motion with Hurst coefficient $H > 1/3$.

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1. Introduction

This paper is the first of a series of two papers dealing with Volterra equations driven by rough paths. For an arbitrary positive constant T, this kind of equation can be written, in its general form, as:

$$y_t = a + \int_0^t \sigma(t, u, y_u)\, dx_u, \quad \text{for } s \in [0, T], \tag{1}$$

where x is an n-dimensional Hölder continuous path with Hölder exponent $\gamma > 0$, $a \in \mathbb{R}^d$ stands for an initial condition, and $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d,n}$ is a smooth enough function.

Motivated by the previous works on Volterra equations driven by a Brownian motion or a semi-martingale [2, 3, 15, 21], often in an anticipative context [1, 4, 5, 19, 18, 20], we have taken up the program of defining and solving Eq. (1) in a pathwise way, allowing for instance a straightforward application to a fractional Brownian motion with Hurst parameter $H > 1/3$. This will be achieved thanks to a variation of the rough path theory due to Gubinelli [11], whose main features are recalled below at Sec. 2 (we refer to [9, 13, 14] for further classical references on rough paths theory). To the best of our knowledge, this is the first occurrence of a paper dealing with Volterra systems driven by a fractional Brownian motion with $H < 1/2$.

More specifically, this paper focuses on the following three cases:

(i) *The Young case*: When x is a γ-Hölder continuous path with $\gamma > 1/2$ (in particular for an n-dimensional fBm with Hurst parameter $H \in (1/2, 1)$), and assuming that $\sigma : [0, T]^2 \times \mathbb{R}^d \to \mathbb{R}^{d,n}$ is regular enough (with respect to its three variables), we shall prove that Eq. (1) can be interpreted and solved in the Young sense (Sec. 3).

(ii) *The Young singular case*: Under the same conditions as in the previous case for x, we are able to handle the case of a coefficient σ admitting a singularity with respect to its first two variables t, u. Namely, if σ can be expressed as $\sigma(t, u, z) = (t - u)^{-\alpha}\psi(z)$, for some $\alpha > 0$ and $\psi : \mathbb{R}^d \to \mathbb{R}^{d,n}$ regular enough, then under some conditions on α, γ, κ (roughly speaking, we ask that $\gamma - \alpha > 1/2$ and $1/2 < \kappa < \gamma$), it is still possible to interpret $\int_0^t \sigma(t, u, y_u)\, dx_u$ as a Young integral when y belongs to a space of κ-Hölder functions, denoted below by $\mathcal{C}_1^\kappa([0, T], \mathbb{R}^d)$. This extension of the Young integral however requires a careful analysis, which will be detailed in Sec. 4. We can then solve Eq. (1) in the space $\mathcal{C}_1^\kappa([0, T], \mathbb{R}^d)$.

(iii) *The rough case*: When x is a γ-Hölder signal with $\gamma \in (1/3, 1/2)$ (this applies obviously to an n-dimensional fBm with Hurst parameter $H \in (1/3, 1/2)$), the integral appearing in Eq. (1) then has to be interpreted in some rough path sense. As mentioned before, we shall resort in this case to the formalism introduced in [11], which allows us to prove the existence and uniqueness of a local solution, defined on a small interval $[0, T_0]$ for some $T_0 \in (0, T]$ (Sec. 5). We will then point out the technical difficulties one must cope with when trying to extend this local solution.

Here is a brief sketch of the strategy we have followed in order to obtain our results: the algebraic integration formalism relies heavily on the notion of increments, which are simply given, in case of a function y of one parameter $t \in [0, T]$, by $(\delta y)_{st} = y_t - y_s$. At a heuristic level, the main difference between classical differential equations driven by rough signals and our Volterra setting lies in the dependence of the increment $(\delta y)_{st}$ of the possible solution on the whole past of the trajectory. Indeed, if y is a solution to Eq. (1), then one has

$$(\delta y)_{st} = \int_s^t \sigma(t, u, y_u)\, dx_u + \int_0^s [\sigma(t, u, y_u) - \sigma(s, u, y_u)]\, dx_u. \tag{2}$$

As one might expect, the first integral in (2) can be dealt with just as the classical diffusion case treated in [11]. In other words, under suitable regularity conditions on σ, the variable t appearing in the integrand does not play a prominent role. The second term on the right-hand side of (2) is the one which is typical of the

Volterra setting, and involves the whole of x. It is still possible to retrieve some $|t-s|$-increments from this term thanks to the regularity of σ with respect to its first variable, in order to solve our equation by a fixed point argument. However, as we shall see in Sec. 5.3, the term $\int_0^s [\sigma(t,u,y_u) - \sigma(s,u,y_u)]\, dx_u$ will eventually induce some severe problems in the classical arguments allowing to get a global solution for our differential system in the rough case. This explains why we have decided to change radically the setting presented here in the companion paper [7]. In this latter reference, by means of what we call generalized convolutional increments, we show how to get a global solution to Eq. (1) in case of a rough driving noise x, for a wide class of coefficients σ. It was however important for us to also include a direct treatment of Volterra systems by the existing rough paths' methods, mainly because (i) it allows to consider a more general driving coefficient σ. (ii) The method presented here works perfectly well for the Young setting, and can be further extended in order to cover the case of a singular coefficient σ.

Here is how our paper is structured: we recall in Sec. 2 the notions of algebraic integration which will be needed later on. Section 3 is devoted to the study of Eq. (1) driven by a γ-Hölder continuous process with $\gamma > 1/2$, when the coefficient σ is regular. Section 4 deals with the same kind of equation, with a singular coefficient σ. Section 5 treats the case of a rough driving signal x, and finally the proof of some technical lemmas can be found in the Appendix.

Let us finish this Introduction by fixing some notations which are used throughout the paper: we call Df the gradient of a function f, defined on $\mathbb{R}^n$, and when we want to stress the fact that we are differentiating f with respect to the jth variable, we denote this by $D_j f$. As far as the regularity of σ is concerned, the following spaces come into play. If E, F are Banach spaces and U an open set of E, denote $\mathcal{C}^{n,\mathbf{b}}(U;F)$ the set of n-times differentiable mappings from U to F with bounded derivatives. For each $\kappa \in (0,1)$, let us also introduce the subset

$$\mathcal{C}^{n,\mathbf{b},\kappa}(U;F) = \left\{ \sigma \in \mathcal{C}^{n,\mathbf{b}}(U;F) : \sup_{x,y \in U} \frac{\|D^{(n)}\sigma(x) - D^{(n)}\sigma(y)\|}{\|x-y\|^\kappa} < \infty \right\}.$$

2. Algebraic Integration

This section is devoted to recall the main concepts of algebraic integration, which will be essential in order to define suitable notions of generalized integrals in our setting. Namely, we shall recall the definition of the spaces of increments $\mathcal{C}_n^\kappa$, of the operator δ, and its inverse called Λ (or sewing map according to the terminology of [8]). We will also recall some elementary but useful algebraic relations on the spaces of increments.

2.1. *Increments*

As mentioned in the Introduction, the extended integral we deal with is based on the notion of increment, together with an elementary operator δ acting on them.

The notion of increment can be introduced in the following way: for two arbitrary real numbers $\ell_2 > \ell_1 \geq 0$, a vector space V, and an integer $k \geq 1$, we denote by $\mathcal{C}_k(V)$ the set of continuous functions $g : [\ell_1, \ell_2]^k \to V$ such that $g_{t_1 \cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \leq k-1$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*(V) = \bigcup_{k\geq 1} \mathcal{C}_k(V)$. The operator δ alluded to above can be seen as an operator acting on k-increments, and is defined as follows on $\mathcal{C}_k(V)$:

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V) \quad (\delta g)_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^i g_{t_1 \cdots \hat{t}_i \cdots t_{k+1}}, \tag{3}$$

where $\hat{t}_i$ means that this particular argument is omitted. Then a fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Ker}\delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Im}\delta$.

Some simple examples of actions of δ, which will be the ones we will use throughout the paper, are obtained by letting $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then, for any $t, u, s \in [\ell_1, \ell_2]$, we have

$$(\delta g)_{st} = g_t - g_s \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}. \tag{4}$$

Furthermore, it is readily checked that the complex $(\mathcal{C}_*, \delta)$ is *acyclic*, i.e. $\mathcal{ZC}_k(V) = \mathcal{BC}_k(V)$ for any $k \geq 1$. In particular, the following basic property, which we label for further use, holds true:

Lemma 2.1. *Let $k \geq 1$ and $h \in \mathcal{ZC}_{k+1}(V)$. Then there exists a (non-unique) $f \in \mathcal{C}_k(V)$ such that $h = \delta f$.*

Observe that Lemma 2.1 implies that all the elements $h \in \mathcal{C}_2(V)$ such that $\delta h = 0$ can be written as $h = \delta f$ for some (non-unique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_2(V)}$: it measures how much a given 1-increment is far from being an *exact* increment of a function (i.e. a finite difference).

Notice that our future discussions will mainly rely on k-increments with $k \leq 23$, for which we will use some analytical assumptions. Namely, we measure the size of these increments by Hölder norms defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_\mu \equiv \sup_{s,t \in [\ell_1, \ell_2]} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \{f \in \mathcal{C}_2(V);\ \|f\|_\mu < \infty\}.$$

With this notation, $\mathcal{C}_1^\mu(V) = \{f \in \mathcal{C}_1(V);\ \|\delta f\|_\mu < \infty\}$. In the same way, for $h \in \mathcal{C}_3(V)$, set

$$\begin{aligned} \|h\|_{\gamma,\rho} &= \sup_{s,u,t \in [\ell_1, \ell_2]} \frac{|h_{sut}|}{|u-s|^\gamma |t-u|^\rho}, \\ \|h\|_\mu &\equiv \inf \left\{ \sum_i \|h_i\|_{\rho_i, \mu-\rho_i};\ h = \sum_i h_i,\ 0 < \rho_i < \mu \right\}, \end{aligned} \tag{5}$$

where the last infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0, \mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \{h \in \mathcal{C}_3(V);\ \|h\|_\mu < \infty\}.$$

Eventually, let $\mathcal{C}_3^{1+}(V) = \bigcup_{\mu>1} \mathcal{C}_3^\mu(V)$, and remark that the same kind of norms can be considered on the spaces $\mathcal{ZC}_3(V)$, leading to the definition of some spaces $\mathcal{ZC}_3^\mu(V)$ and $\mathcal{ZC}_3^{1+}(V)$. In order to avoid ambiguities, we shall denote by $\mathcal{N}[f;\, \mathcal{C}_j^\kappa]$ the κ-Hölder norm on the space $\mathcal{C}_j$, for $j = 1, 2, 3$. For $\zeta \in \mathcal{C}_j(V)$, we also set $\mathcal{N}[\zeta; \mathcal{C}_j^0(V)] = \sup_{s\in[\ell_1;\ell_2]^j} \|\zeta_s\|_V$.

Recall that Lemma 2.1 states that for any $h \in \mathcal{ZC}_3$, there exists a $f \in \mathcal{C}_2$ such that $\delta f = h$. Importantly enough for the construction of our generalized integrals, this increment f is unique under some additional regularity conditions expressed in terms of the Hölder spaces we have just introduced:

Theorem 2.2. (The sewing map) *Let $\mu > 1$. For any $h \in \mathcal{ZC}_3^\mu([0,1];V)$, there exists a unique $\Lambda h \in \mathcal{C}_2^\mu([0,1];V)$ such that $\delta(\Lambda h) = h$. Furthermore,*

$$\|\Lambda h\|_\mu \le c_\mu\, \mathcal{N}[h;\, \mathcal{C}_3^\mu(V)], \tag{6}$$

with $c_\mu = 2 + 2^\mu \sum_{k=1}^\infty k^{-\mu}$. This gives rise to a linear continuous map $\Lambda : \mathcal{ZC}_3^\mu([0,1];V) \to \mathcal{C}_2^\mu([0,1];V)$ such that $\delta\Lambda = \mathrm{Id}_{\mathcal{ZC}_3^\mu([0,1];V)}$.

Proof. The original proof of this result can be found in [11]. We refer to [7, 12] for two simplified versions. □

At this point the connection of the structure we introduced with the problem of integration of irregular functions can still be quite obscure to the non-initiated reader. However, something interesting is already going on and the previous corollary has a very nice consequence which is the subject of the following property.

Corollary 2.3. (Integration of small increments) *For any 1-increment $g \in \mathcal{C}_2(V)$ such that $\delta g \in \mathcal{C}_3^{1+}$, set $h = (\mathrm{Id} - \Lambda\delta)g$. Then there exists $f \in \mathcal{C}_1(V)$ such that $h = \delta f$ and*

$$(\delta f)_{st} = \lim_{|\Pi_{st}|\to 0} \sum_{i=0}^n g_{t_i t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \dots, t_n = t\}$ of $[s,t]$ whose mesh tends to zero. The 1-increment δf is the indefinite integral of the 1-increment g.

Proof. For any partition $\Pi_t = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of $[s,t]$, write

$$(\delta f)_{st} = \sum_{i=0}^n (\delta f)_{t_i t_{i+1}} = \sum_{i=0}^n g_{t_i t_{i+1}} - \sum_{i=0}^n \Lambda_{t_i t_{i+1}}(\delta g).$$

Observe now that for some $\mu > 1$ such that $\delta g \in \mathcal{C}_3^\mu$,

$$\left\| \sum_{i=0}^n \Lambda_{t_i t_{i+1}}(\delta g) \right\|_V \le \sum_{i=0}^n \|\Lambda_{t_i t_{i+1}}(\delta g)\|_V \le \|\Lambda(\delta g)\|_\mu\ |\Pi_{st}|^{\mu-1}\ |t-s|\,,$$

and as a consequence, $\lim_{|\Pi_{st}|\to 0} \sum_{i=0}^n \Lambda_{t_i t_{i+1}}(\delta g) = 0$. □

2.2. Computations in $\mathcal{C}_*$

We gather in this section some elementary but useful algebraic rules for increments. We refer again to [7, 12] for the proof of these statements.

For the sake of simplicity, let us assume for the moment that $V = \mathbb{R}$ (the multidimensional version of the below considerations can be found in [16]), and set $\mathcal{C}_k(\mathbb{R}) = \mathcal{C}_k$. Then the complex $(\mathcal{C}_*, \delta)$ is an (associative, non-commutative) graded algebra once endowed with the following product: for $g \in \mathcal{C}_n$ and $h \in \mathcal{C}_m$ let $gh \in \mathcal{C}_{n+m}$ the element defined by

$$(gh)_{t_1,\dots,t_{m+n-1}} = g_{t_1,\dots,t_n} h_{t_n,\dots,t_{m+n-1}}, \quad t_1,\dots,t_{m+n+1} \in [\ell_1,\ell_2]. \tag{7}$$

In this context, we have the following useful properties.

Proposition 2.4. *The following differentiation rules hold true*:

(1) *Let g, h be two elements of $\mathcal{C}_1$. Then*

$$\delta(gh) = \delta g h + g \delta h. \tag{8}$$

(2) *Let $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$. Then*

$$\delta(gh) = \delta g h + g\delta h, \quad \delta(hg) = \delta h g - h \delta g.$$

The iterated integrals of smooth functions on $[\ell_1, \ell_2]$ are obviously particular cases of elements of $\mathcal{C}$ which will be of interest for us, and let us recall some basic rules for these objects: consider $f, g \in \mathcal{C}_1^\infty$, where $\mathcal{C}_1^\infty$ is the set of smooth functions from $[\ell_1, \ell_2]$ to $\mathbb{R}$. Then the integral $\int dg\, f$, which will be denoted by $\mathcal{J}(dg\, f)$, can be considered as an element of $\mathcal{C}_2^\infty$. That is, for $s, t \in [\ell_1, \ell_2]$, we set

$$\mathcal{J}_{st}(dg\, f) = \left(\int dg f\right)_{st} = \int_s^t dg_u f_u.$$

The multiple integrals can also be defined in the following way: given a smooth element $h \in \mathcal{C}_2^\infty$ and $s, t \in [\ell_1, \ell_2]$, we set

$$\mathcal{J}_{st}(dg\, h) \equiv \left(\int dgh\right)_{st} = \int_s^t dg_u h_{us}.$$

In particular, the double integral $\mathcal{J}_{st}(df^3 df^2\, f^1)$ is defined, for $f^1, f^2, f^3 \in \mathcal{C}_1^\infty$, as

$$\mathcal{J}_{st}(df^3 df^2\, f^1) = \left(\int df^3 df^2\, f^1\right)_{st} = \int_s^t df_u^3\, \mathcal{J}_{us}(df^2\, f^1).$$

Now, suppose that the nth order iterated integral of $df^n \cdots df^2\, f^1$, still denoted by $\mathcal{J}(df^n \cdots df^2\, f^1)$, has been defined for $f^1, f^2, \dots, f^n \in \mathcal{C}_1^\infty$. Then, if $f^{n+1} \in \mathcal{C}_0^\infty$, we set

$$\mathcal{J}_{st}(df^{n+1} df^n \cdots df^2 f^1) \int_s^t df_u^{n+1}\, \mathcal{J}_{us}(df^n \cdots df^2\, f^1), \tag{9}$$

which defines the iterated integrals of smooth functions recursively. Observe that an nth order integral $\mathcal{J}(df^n \cdots df^2 df^1)$ (instead of $\mathcal{J}(df^n \cdots df^2 f^1)$) could be defined along the same lines.

The following relations between multiple integrals and the operator δ will also be useful in the remainder of the paper:

Proposition 2.5. *Let f, g be two elements of $\mathcal{C}_1^\infty$. Then, recalling the convention* (7), *it holds that*

$$\delta f = \mathcal{J}(df), \quad \delta\left(\mathcal{J}(dgf)\right) = 0, \quad \delta\left(\mathcal{J}(dgdf)\right) = (\delta g)(\delta f) = \mathcal{J}(dg)\mathcal{J}(df),$$

and, in general,

$$\delta\left(\mathcal{J}(df^n \cdots df^1)\right) = \sum_{i=1}^{n-1} \mathcal{J}(df^n \cdots df^{i+1})\mathcal{J}(df^i \cdots df^1).$$

3. The Young Case

In this section, we assume that the driving process x of Eq. (1) is a continuous process in $\mathcal{C}_1^\gamma([0,T];\mathbb{R}^n)$, for some $\gamma \in (1/2, 1)$. If $z \in \mathcal{C}_1^\rho([0,T];\mathbb{R}^{d,n})$, the formalism introduced in the previous section enables to give a meaning to the integral $\int_s^t z_u\, dx_u$ when $\rho + \gamma > 1$, in the Young sense. This is the issue of the following proposition, borrowed from [11] :

Proposition 3.1. *If $z \in \mathcal{C}_1^\rho([0,T];\mathbb{R}^{d,n})$ for some $\rho > 0$ such that $\rho + \gamma > 1$, we can define, for any $s, t \in [0,T]$,*

$$\mathcal{J}_{st}(z\, dx) := z_s(\delta x)_{st} - \Lambda_{st}(\delta z\, \delta x). \tag{10}$$

Then $\mathcal{J}(z\, dx) \in \mathcal{C}_2^\gamma([0,T];\mathbb{R}^d)$ and

$$\begin{aligned}\mathcal{N}[\mathcal{J}(z\, dx); \mathcal{C}_2^\gamma([0,T];\mathbb{R}^d)]& \\ \le c_x\{\mathcal{N}[z;\mathcal{C}_1^0([0,T];\mathbb{R}^{d,n})] &+ T^\rho \mathcal{N}[z;\mathcal{C}_1^\rho([0,T];\mathbb{R}^{d,n})]\}.\end{aligned} \tag{11}$$

Remark 3.2. Thanks to Corollary 2.3, $\mathcal{J}_{st}(z\, dx)$ can also be seen as a Young integral, that is

$$\mathcal{J}_{st}(z\, dx) = \lim_{|\Delta| \to 0} \sum_{\Delta} z_{t_i}(\delta x)_{t_i t_{i+1}}. \tag{12}$$

Nevertheless, as we shall see in a moment, the exact expression (10) of the integral is easier to deal with for computational purposes than the limit expression (12), owing to a better knowledge of the remainder $\Lambda(\delta z\, \delta x)$.

With this definition in mind, the Volterra equation (1) will now be interpreted in the Young sense, and is written as:

$$y_t = a + \mathcal{J}_{0t}(\sigma(t, ., y_.)\, dx). \tag{13}$$

The next lemma ensures that the latter integral is well-defined:

Lemma 3.3. *If $y \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^d)$ and $\sigma \in \mathcal{C}^{1,\mathbf{b}}([0,T]^2 \times \mathbb{R}^d;\mathbb{R}^{d,n})$, then for any $t \ge 0$, $\sigma(t, ., y_.) \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^{d,n})$ and*

$$\mathcal{N}[\sigma(t, ., y_.);\mathcal{C}_1^\gamma] \le c_\sigma(T^{1-\gamma} + \mathcal{N}[y;\mathcal{C}_1^\gamma]). \tag{14}$$

Proof. This is obvious: recall that we denote by $D\sigma$ the gradient of σ. Then, if $0 \le u < v \le T$ we get:

$$\|\sigma(t,v,y_v) - \sigma(t,u,y_u)\| \le \|D\sigma\|_\infty(|v-u| + \mathcal{N}[y;\mathcal{C}_1^\gamma]|v-u|^\gamma).$$

Hence $\mathcal{N}[\sigma(t,.,y_.);\mathcal{C}_1^\gamma] \le \|D\sigma\|_\infty(T^{1-\gamma} + \mathcal{N}[y;\mathcal{C}_1^\gamma])$. □

We are now in a position to prove the existence and uniqueness result for the Volterra equation in the Young case:

Theorem 3.4. *Assume that the driving process x is an element of $\mathcal{C}_1^\gamma([0,T];\mathbb{R}^n)$ with $\gamma > 1/2$. Let $\kappa \in (0,1)$ such that $\kappa(1+\gamma) > 1$, $a \in \mathbb{R}^d$, $\sigma \in \mathcal{C}^{2,\mathbf{b},\kappa}([0,T]^2 \times \mathbb{R}^d;\mathbb{R}^{d,n})$. Then Eq.* (13) *admits a unique solution in $\mathcal{C}_1^\gamma([0,T];\mathbb{R}^d)$.*

This theorem can obviously be applied to the fractional Brownian motion, in the following sense:

Corollary 3.5. *Let B be an n-dimensional fractional Brownian motion with Hurst parameter $H > 1/2$, defined on a complete probability space $(\Omega,\mathcal{F},P)$. Then almost surely, B fulfills the hypotheses of Theorem* 3.4.

We divide the proof of Theorem 3.4 into two propositions: first, we will look for a local solution defined on some interval $[0,T_0]$ with $0 < T_0 \le T$, and then we will settle a patching argument to extend it onto the whole interval $[0,T]$.

Notations. Before going into the details of the proof, let us mention a few conventions that will be used in the sequel. We assume that we always work with a fixed (finite) horizon T to be distinguished from the intermediate times $T_1, T_0, \ldots$. In particular, this means that the constants that will appear in the following calculations may depend on T without explicit note.

For the sake of conciseness, let us denote $\mathcal{Y}_u = (u,y_u) \in [0,T]\times\mathbb{R}^d$ and $\sigma^t(\mathcal{Y}_u) = \sigma(t,\mathcal{Y}_u)$.

The local existence and uniqueness result for our Volterra equation is as follows:

Proposition 3.6. *Under the hypothesis of Theorem* 3.4, *there exists $T_0 \in (0,T]$ such that Eq.* (13) *admits a unique solution in $\mathcal{C}_1^\gamma([0,T_0];\mathbb{R}^d)$.*

Proof. We are going to resort to a fixed point argument. To this end, let us associate to each $y \in \mathcal{C}_1^\gamma([0,T_0])$ the element $z = \Gamma(y)$ defined by

$$z_t = \Gamma(y)_t = y_0 + \mathcal{J}_{0t}(\sigma^t(\mathcal{Y}_.)\,dx).$$

The solution we are looking for will then be constructed as a fixed point of Γ.

Step 1. *Invariance of a ball.* Fix a time $T_1 \in (0,T]$ (T_1 will be chosen retrospectively). Let $y \in \mathcal{C}_1^\gamma([0,T_1])$ such that $y_0 = a$ and set $z = \Gamma(y)$, where, of course, the application Γ has been adapted to $[0,T_1]$.

At this point, let us remind the reader of some specification of the Volterra setting that we evoked in the Introduction. As in (2), the increment $(\delta z)_{ts}$ can be decomposed as a sum of two terms that will receive a distinct treatment: $I^1_{st} = \mathcal{J}_{st}(\sigma^t(\mathcal{Y})\,dx)$ and $I^2_{st} = \mathcal{J}_{os}([\sigma^t - \sigma^s](\mathcal{Y})\,dx)$. In order to estimate those two integrals, we shall of course resort to inequality (11). However, as far as I^2_{st} is concerned, it is clear that the latter inequality will not be sufficient so as to retrieve $|t-s|$-increments (remember that we are looking for an estimation of $\mathcal{N}[z;\mathcal{C}_1^\gamma]$, hence a relation of the form $\|I^2_{st}\| \le |t-s|^\gamma f(y)$). This is where the following lemma, which also anticipates the contraction argument, will come into play. □

Lemma 3.7. *Let $I=[a,b]\subset[0,T]$ and $y,\tilde{y}\in\mathcal{C}_1^\gamma(I;\mathbb{R}^d)$ such that $y_a=\tilde{y}_a$. Then, under the hypothesis of Theorem* 3.4, *for any $s,t\in I$,*

$$\mathcal{N}[[\sigma^t-\sigma^s](\mathcal{Y});\mathcal{C}_1^\gamma(I)] \le c_\sigma\,|t-s|\,\{1+\mathcal{N}[y;\mathcal{C}_1^\gamma(I)]\}, \tag{15}$$

$$\mathcal{N}[\sigma^t(\mathcal{Y})-\sigma^t(\tilde{\mathcal{Y}});\mathcal{C}_1^\gamma(I)] \le c_\sigma\{1+\mathcal{N}[y;\mathcal{C}_1^\gamma(I)]+\mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma(I)]\}\,\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma(I)], \tag{16}$$

$$\begin{aligned}\mathcal{N}[[\sigma^t-\sigma^s](\mathcal{Y})-[\sigma^t-\sigma^s](\tilde{\mathcal{Y}});\mathcal{C}_1^{\kappa\gamma}(I)]&\\ \le c_\sigma\,|t-s|\,\{1+\mathcal{N}[y;\mathcal{C}_1^\gamma(I)]^\kappa+\mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma(I)]^\kappa\}\,\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma(I)].&\end{aligned} \tag{17}$$

Proof. See the Appendix. □

Now, let us go into the details. To deal with I^1, use (11) to get

$$\begin{aligned}\|I^1_{st}\| &\le c_x\,|t-s|^\gamma\,\{\mathcal{N}[\sigma^t(\mathcal{Y});\mathcal{C}_1^0]+T_1^\gamma\mathcal{N}[\sigma^t(\mathcal{Y});\mathcal{C}_1^\gamma]\}\\ &\le c_{x,\sigma}\,|t-s|^\gamma\,\{1+T_1^\gamma\mathcal{N}[\sigma^t(\mathcal{Y});\mathcal{C}_1^\gamma]\},\end{aligned}$$

and thus, thanks to Lemma 3.3, $\mathcal{N}[I^1;\mathcal{C}_2^\gamma]\le c_{x,\sigma}\{1+T_1^\gamma\mathcal{N}[y;\mathcal{C}_1^\gamma]\}$.

Split I^2 into $I^2=I^{2,1}+I^{2,2}$, with

$$I^{2,1}_{st}=[\sigma^t-\sigma^s](\mathcal{Y}_0)\,(\delta x)_{0s} \quad\text{and}\quad I^{2,2}_{st}=\Lambda_{0s}(\delta([\sigma^t-\sigma^s](\mathcal{Y}))\,\delta x).$$

First, notice that $\|I^{2,1}_{st}\|\le\|D\sigma\|_\infty\,|t-s|\,\mathcal{N}[x;\mathcal{C}_1^\gamma]T_1^\gamma$, which gives $\mathcal{N}[I^{2,1};\mathcal{C}_2^\gamma]\le c_{x,\sigma}T_1$. As for $I^{2,2}$, use the contraction property (6) and the estimate (15) to deduce

$$\begin{aligned}\|I^{2,2}_{st}\| &\le c_x\,\mathcal{N}[[\sigma^t-\sigma^s](\mathcal{Y});\mathcal{C}_1^\gamma]\,T_1^\gamma\\ &\le c_{x,\sigma}\;|t-s|\,\{1+\mathcal{N}[y;\mathcal{C}_1^\gamma]\}T_1^{2\gamma},\end{aligned}$$

so that $\mathcal{N}[I^{2,2};\mathcal{C}_2^\gamma]\le c_{x,\sigma}\,T_1^{1+\gamma}\,(1+\mathcal{N}[y;\mathcal{C}_1^\gamma])$.

Therefore, putting together our bounds on I^1 and I^2, we have obtained $\mathcal{N}[z;\mathcal{C}_1^\gamma]\le c_{x,\sigma}\{1+T_1^\gamma\mathcal{N}[y;\mathcal{C}_1^\gamma]\}$. We can thus pick $T_1\in(0,T]$ such that for each $0<T_0\le T_1$, there exists a radius A_{T_0} for which the ball

$$B^{A_{T_0}}_{T_0,a}=\{y\in\mathcal{C}_1^\gamma([0,T_0]):\ y_0=a,\ \mathcal{N}[y;\mathcal{C}_1^\gamma([0,T_0])]\le A_{T_0}\}$$

is invariant by Γ. Notice that the radius A_{T_0} is an increasing function of T_0, a fact which will be used in the second step.

Step 2. *Contraction property.* Fix a time $T_0 \in (0,T_1]$ and let $y,\tilde{y} \in B_{T_0,a}^{A_{T_0}}$. Set $z=\Gamma(y)$, $\tilde{z}=\Gamma(\tilde{y})$ and decompose again $\delta(z-\tilde{z})$ into $\delta(z-\tilde{z}) = J^{1,1}+J^{1,2}+J^2$, with

$$J_{st}^{1,1} = (\sigma^t(\mathcal{Y}_s)-\sigma^t(\tilde{\mathcal{Y}}_s))\,(\delta x)_{st}, \quad J_{st}^{1,2} = \Lambda_{st}(\delta(\sigma^t(\mathcal{Y})-\sigma^t(\tilde{\mathcal{Y}}))\,\delta x),$$
$$J_{st}^2 = \Lambda_{0s}(\delta([\sigma^t-\sigma^s](\mathcal{Y})-[\sigma^t-\sigma^s](\tilde{\mathcal{Y}}))\,\delta x).$$

Let us now estimate the γ-Hölder norm of each of these three terms.

Case of $J^{1,1}$. We have $\mathcal{N}[J^{1,1};\mathcal{C}_2^\gamma] \le \|D\sigma\|_\infty \mathcal{N}[y-\tilde{y};\mathcal{C}_1^0]\,\mathcal{N}[x;\mathcal{C}_1^\gamma]$. However, since $y_0=\tilde{y}_0=a$, we have $y_s-\tilde{y}_s = y_s-\tilde{y}_s-(y_0-\tilde{y}_0)$, $\mathcal{N}[y-\tilde{y};\mathcal{C}_1^0] \le \mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]\,T_0^\gamma$, so that

$$\mathcal{N}[J^{1,1};\mathcal{C}_2^\gamma] \le c_{x,\sigma}\,\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]\,T_0^\gamma.$$

Case of $J^{1,2}$. Inequalities (6) and (16) yield:

$$\begin{aligned}\|J_{st}^{1,2}\| &\le c\,\mathcal{N}[\sigma^t(\mathcal{Y})-\sigma^t(\tilde{\mathcal{Y}});\mathcal{C}_1^\gamma]\,\mathcal{N}[x;\mathcal{C}_1^\gamma]\;|t-s|^{2\gamma}\\ &\le c_{x,\sigma}\,|t-s|^\gamma\,(1+\mathcal{N}[y;\mathcal{C}_1^\gamma]+\mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma])\,\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]T_0^\gamma,\end{aligned}$$

which gives $\mathcal{N}[J^{1,2};\mathcal{C}_2^\gamma] \le c_{x,\sigma}\,(1+\mathcal{N}[y;\mathcal{C}_1^\gamma]+\mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma])\,\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]T_0^\gamma$.

Case of J^2. By (6) and (17),

$$\begin{aligned}\|J_{st}^2\| &\le c\,\mathcal{N}[[\sigma^t-\sigma^s](\mathcal{Y})-[\sigma^t-\sigma^s](\tilde{\mathcal{Y}});\mathcal{C}_1^{\kappa\gamma}]\,\mathcal{N}[x;\mathcal{C}_1^\gamma]\,T_0^{\gamma(1+\kappa)}\\ &\le c_{\sigma,x}\,|t-s|^\gamma\,T_0^{1+\gamma\kappa}\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]\{1+\mathcal{N}[y;\mathcal{C}_1^\gamma]^\kappa+\mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma]^\kappa\},\end{aligned}$$

or in other words, $\mathcal{N}[J^2;\mathcal{C}_2^\gamma] \le c_{\sigma,x}T_0^{1+\gamma\kappa}\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]\{1+\mathcal{N}[y;\mathcal{C}_1^\gamma]^\kappa+\mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma]^\kappa\}$.

Therefore, $\mathcal{N}[z-\tilde{z};\mathcal{C}_1^\gamma] \le c_{\sigma,x}T_0^\gamma\,\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]\{1+A_{T_0}\}$. Since the radius A_{T_0} decreases as T_0 tends to 0, we can choose a sufficiently small time $T_0 \in (0,T_1]$ such that the application Γ, restricted to the (stable) ball $B_{T_0,a}^{A_{T_0}}$, is a strict contraction. Hence the existence and uniqueness of a fixed point in this set.

The next proposition summarizes our considerations in order to get the global existence and uniqueness for solution to Eq. (13):

Proposition 3.8. *Under the hypothesis of Theorem* 3.4, *the local solution* $y^{(1)}$ *defined by the previous proposition can be extended to a global and unique solution in* $\mathcal{C}_1^\gamma([0,T];\mathbb{R}^d)$.

Proof. In fact, we are going to show the existence of a small $\varepsilon>0$, which *shall not depend on* $y^{(1)}$, such that $y^{(1)}$ can be extended to a solution on $[0,T_0+\varepsilon]$. The conclusion then follows by a simple iteration argument.

To this end, let us introduce the application Γ defined for any $z\in\mathcal{C}_1^\gamma([0,T_0+\varepsilon])$ such that $z_{|[0,T_0]} = y^{(1)}$ as

$$\hat{z}_t = \Gamma(z)_t = \begin{cases} y_t^{(1)} & \text{if } t\in[0,T_0],\\ a+\mathcal{J}_{0t}(\sigma^t(\mathcal{Z})\,dx) & \text{if } t\in[T_0,T_0+\varepsilon].\end{cases}$$

Just as in the previous proof, we are looking for a fixed point of Γ.

Step 1. *Invariance of a ball.* In order to estimate $\mathcal{N}[\hat{z};\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]$, let us consider the three cases $(s,t\in[0,T_0])$, $(s,t\in[T_0,T_0+\varepsilon])$ and $(s\le T_0\le t\le T_0+\varepsilon)$.

In the first case, we simply have $\mathcal{N}[\hat{z};\mathcal{C}_1^\gamma([0,T_0])]\le\mathcal{N}[y^{(1)};\mathcal{C}_1^\gamma([0,T_0])]$. Consider the second case $s,t\in[T_0,T_0+\varepsilon]$, and decompose $(\delta\hat{z})_{st}$ as above, that is $(\delta\hat{z})_{st}=I_{st}^{1,1}+I_{st}^{1,2}+I_{st}^{2,1}+I_{st}^{2,2}$, with

$$I_{st}^{1,1}=\sigma^t(\mathcal{Z}_s)\,(\delta x)_{st},\quad I_{st}^{1,2}=\Lambda_{st}(\delta(\sigma^t(\mathcal{Z}))\,\delta x),$$
$$I_{st}^{2,1}=[\sigma^t-\sigma^s](\mathcal{Z}_0)\,(\delta x)_{0s},\quad I_{st}^{2,2}=\Lambda_{0s}(\delta([\sigma^t-\sigma^s](\mathcal{Z}))\,\delta x).$$

Let us now bound each of these terms: first, owing to (6) and (14), $I_{st}^{1,2}$ can be estimated as follows:

$$\begin{aligned}\|I_{st}^{1,2}\|&\le c\,\mathcal{N}[\sigma^t(\mathcal{Z});\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\,\mathcal{N}[x;\mathcal{C}_1^\gamma]\,|t-s|^{2\gamma}\\&\le c_{\sigma,x}\{1+\mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\}\,|t-s|^{2\gamma}\,.\end{aligned}$$

It is thus readily checked that $\mathcal{N}[I^{1,2};\mathcal{C}_2^\gamma([T_0,T_0+\varepsilon])]\le c_{\sigma,x}\,\varepsilon^\gamma\{1+\mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\}$. Thanks to (6) and (15), we also have the following bound for $I_{st}^{2,2}$:

$$\begin{aligned}\|I_{st}^{2,2}\|&\le c\,\mathcal{N}[[\sigma^t-\sigma^s](\mathcal{Z});\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\,\mathcal{N}[x;\mathcal{C}_1^\gamma]\,T^{2\gamma}\\&\le c_{\sigma,x}\,|t-s|\,\{1+\mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\},\end{aligned}$$

which gives $\mathcal{N}[I^{2,2};\mathcal{C}_2^\gamma([T_0,T_0+\varepsilon])]\le c_{\sigma,x}\,\varepsilon^{1-\gamma}\{1+\mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\}$. Since trivially $\mathcal{N}[I^{i,1};\mathcal{C}_2^\gamma([T_0,T_0+\varepsilon])]\le c_{\sigma,x}$ for $i=1,2$, we get

$$\mathcal{N}[\hat{z};\mathcal{C}_1^\gamma([T_0,T_0+\varepsilon])]\le c_{\sigma,x}\{1+\varepsilon^{1-\gamma}\mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\}.$$

Finally, let us treat the third case $0\le s\le T_0\le t\le T_0+\varepsilon$: write

$$\begin{aligned}\|(\delta\hat{z})_{st}\|&=\|(\delta\hat{z})_{sT_0}+(\delta\hat{z})_{T_0t}\|\\&\le\mathcal{N}[y^{(1)};\mathcal{C}_1^\gamma([0,T_0])]\,|T_0-s|^\gamma+\mathcal{N}[\hat{z};\mathcal{C}_1^\gamma([T_0,T_0+\varepsilon])]\,|t-T_0|^\gamma\\&\le\{\mathcal{N}[y^{(1)};\mathcal{C}_1^\gamma([0,T_0])]+\mathcal{N}[\hat{z};\mathcal{C}_1^\gamma([T_0,T_0+\varepsilon])]\}\,|t-s|^\gamma\,.\end{aligned}$$

Putting together the three cases we have just studied, the following bound is obtained for $\hat{z}$ on the whole interval $[0,T_0+\varepsilon])]$:

$$\mathcal{N}[\hat{z};\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\le c_{\sigma,x}^1\{1+\mathcal{N}[y^{(1)};\mathcal{C}_1^\gamma([0,T_0])]+\varepsilon^{1-\gamma}\mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\}.$$

Therefore, set

$$\varepsilon=(2c_{\sigma,x}^1)^{-1/(1-\gamma)}\ (\varepsilon\text{ does not depend on }y^{(1)})\text{ and}$$
$$N_1=2c_{\sigma,x}^1\{1+\mathcal{N}[y^{(1)};\mathcal{C}_1^\gamma([0,T_0])]\},$$

so that if $\mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\le N_1$, then $\mathcal{N}[\hat{z};\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\le\frac{N_1}{2}+\frac{N_1}{2}=N_1$. In other words, we have found that the ball

$$B_{y^{(1)},T_0,\varepsilon}^{N_1}=\{z\in\mathcal{C}_1^\gamma([0,T_0+\varepsilon]):\ z_{|[0,T_0]}=y^{(1)},\ \mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\le N_1\}$$

is invariant by Γ.

Step 2. *Contraction property.* This second step consists of finding a small $\eta \in (0,\varepsilon]$ such that the previous application Γ (adapted to $[0,T_0+\eta]$) satisfies a contraction property when restricted to some (invariant) ball.

Let $z^{(1)}, z^{(2)} \in B^{N_1}_{y^{(1)},T_0,\eta}$ and set $\hat{z}^{(1)} = \Gamma(z^{(1)})$, $\hat{z}^{(2)} = \Gamma(z^{(2)})$. Of course, since $\hat{z}^{(1)}$ and $\hat{z}^{(2)}$ share the same initial condition on $[0,T_0]$, we have $\mathcal{N}[\hat{z}^{(1)} - \hat{z}^{(2)}; \mathcal{C}_1^\gamma([0,T_0+\eta])] = \mathcal{N}[\hat{z}^{(1)} - \hat{z}^{(2)}; \mathcal{C}_1^\gamma([T_0,T_0+\eta])]$. Let then $T_0 \le s < t \le T_0+\eta$ and as in the proof of Proposition 3.6, use the decomposition $\delta(\hat{z}^{(1)} - \hat{z}^{(2)})_{st} = J^{1,1}_{st} + J^{1,2}_{st} + J^2_{st}$, where

$$J^{1,1}_{st} = (\sigma^t(\mathcal{Z}^{(1)}_s) - \sigma^t(\mathcal{Z}^{(2)}_s))\,(\delta x)_{st}, \quad J^{1,2}_{st} = \Lambda_{st}(\delta(\sigma^t(\mathcal{Z}^{(1)}) - \sigma^t(\mathcal{Z}^{(2)}))\,\delta x),$$
$$J^2_{st} = \Lambda_{0s}(\delta([\sigma^t - \sigma^s](\mathcal{Z}^{(1)}) - [\sigma^t - \sigma^s](\mathcal{Z}^{(2)}))\,\delta x).$$

We will bound again each of these terms separately: for $J^{1,1}$, we have

$$\|J^{1,1}_{st}\| \le \|D\sigma\|_\infty \|z^{(1)}_s - z^{(2)}_s\| \mathcal{N}[x; \mathcal{C}_1^\gamma]\,|t-s|^\gamma\,.$$

But

$$\|z^{(1)}_s - z^{(2)}_s\| = \|[z^{(1)}_s - z^{(2)}_s] - [z^{(1)}_{T_0} - z^{(2)}_{T_0}]\| \le \mathcal{N}[z^{(1)} - z^{(2)}; \mathcal{C}_1^\gamma([0,T_0+\eta])]\,\eta^\gamma,$$

and so

$$\mathcal{N}[J^{1,1}; \mathcal{C}_2^\gamma([T_0,T_0+\eta])] \le c_{x,\sigma}\,\eta^\gamma \mathcal{N}[z^{(1)} - z^{(2)}; \mathcal{C}_1^\gamma([0,T_0+\eta])]. \tag{18}$$

The term $J^{1,2}_{st}$ can be estimated as follows: by (6) and (16),

$$\begin{aligned}\|J^{1,2}_{st}\| &\le c\,\mathcal{N}[\sigma^t(\mathcal{Z}^{(1)}) - \sigma^t(\mathcal{Z}^{(2)}); \mathcal{C}_1^\gamma([0,T_0+\eta])] \mathcal{N}[x; \mathcal{C}_1^\gamma]\,|t-s|^{2\gamma} \\ &\le c_{\sigma,x}\,|t-s|^\gamma\,\eta^\gamma \{1 + 2N_1\} \mathcal{N}[z^{(1)} - z^{(2)}; \mathcal{C}_1^\gamma([0,T_0+\eta])].\end{aligned}$$

Finally, according to (6) and (17), we have:

$$\begin{aligned}\|J^2_{st}\| &\le c\,\mathcal{N}[[\sigma^t - \sigma^s](\mathcal{Z}^{(1)}) - [\sigma^t - \sigma^s](\mathcal{Z}^{(2)}); \mathcal{C}_1^{\kappa\gamma}([0,T_0+\eta])] \mathcal{N}[x; \mathcal{C}_1^\gamma]\,T^{\gamma(1+\kappa)} \\ &\le c_{\sigma,x}\,|t-s|^\gamma\,\eta^{1-\gamma} \{1 + 2N_1^\kappa\} \mathcal{N}[z^{(1)} - z^{(2)}; \mathcal{C}_1^\gamma([0,T_0+\eta])].\end{aligned}$$

As a result, putting together the bounds on $J^{1,1}_{st}$, $J^{1,2}_{st}$ and J^2_{st}, we end up with:

$$\mathcal{N}[\hat{z}^{(1)} - \hat{z}^{(2)}; \mathcal{C}_1^\gamma([0,T_0+\eta])] \le c^1_{\sigma,x} \eta^{1-\gamma} \{1 + N_1^\kappa + N_1\} \mathcal{N}[z^{(1)} - z^{(2)}; \mathcal{C}_1^\gamma([0,T_0+\eta])].$$

We can now pick $\eta \in (0,\varepsilon]$ such that $c^1_{\sigma,x}\eta^{1-\gamma}\{1 + N_1^\kappa + N_1\} \le \frac{1}{2}$, and the application Γ becomes a strict contraction on $B^{N_1}_{y^{(1)},T_0,\eta}$. It is easy to check (see Lemma 3.9 below) that $B^{N_1}_{y^{(1)},T_0,\eta}$ is also invariant by Γ, hence the existence and uniqueness of a fixed point in this set, denoted by $y^{(1),\eta}$.

Notice now that the arguments leading to uniqueness remain true on the (stable) ball

$$\{z \in \mathcal{C}_1^\gamma([0,T_0+2\eta]) : \ z_{|[0,T_0+\eta]} = y^{(1),\eta}, \ \mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+2\eta])] \leq N_1\}.$$

For instance, to establish the equivalent of relation (18) on this extended interval, notice that if $s \in [T_0+\eta, T_0+2\eta]$,

$$\|z_s^{(1)} - z_s^{(2)}\| = \|[z_s^{(1)} - z_s^{(2)}] - [z_{T_0+\eta}^{(1)} - z_{T_0+\eta}^{(2)}]\| \leq \mathcal{N}[z^{(1)} - z^{(2)}; \mathcal{C}_1^\gamma([0,T_0+2\eta])]\, \eta^\gamma.$$

This enables to extend $y^{(1),\eta}$ into a solution $y^{(1),2\eta}$ on $[0,T_0+2\eta]$, and then $y^{(1),3\eta}$ on $[0,T_0+3\eta], \ldots$ until $[0,T_0+\varepsilon]$ is covered, as we wished. □

Lemma 3.9. *With the notations of the previous proof, the sets*

$$\{z \in \mathcal{C}_1^\gamma([0,T_0+l\eta]) : \ z_{|[0,T_0+(l-1)\eta]} = y^{(1),(l-1)\eta}, \ \mathcal{N}[z;\mathcal{C}_1^\gamma([0,T_0+l\eta])] \leq N_1\}$$

are invariant by Γ.

Proof. If z belongs to such a ball, set

$$\tilde{z}_t = \begin{cases} z_t & \text{if } t \in [0,T_0+l\eta], \\ z_{T_0+l\eta} & \text{if } t \in [T_0+l\eta, T_0+\varepsilon]. \end{cases}$$

Clearly, $\tilde{z} \in B^{N_1}_{y^{(1)},T_0,\varepsilon}$, so that, thanks to the first step of the previous proof, $\Gamma(\tilde{z}) \in B^{N_1}_{y^{(1)},T_0,\varepsilon}$. Now, since $y^{(1),(l-1)\eta}$ is a solution on $[0,T_0+(l-1)\eta]$, we have $\Gamma(\tilde{z})_{|[0,T_0+(l-1)\eta]} = y^{(1),(l-1)\eta}$, which means that $\Gamma(\tilde{z})$ is an extension of $\Gamma(z)$ and as a result

$$\mathcal{N}[\Gamma(z);\mathcal{C}_1^\gamma([0,T_0+l\eta])] \leq \mathcal{N}[\Gamma(\tilde{z});\mathcal{C}_1^\gamma([0,T_0+\varepsilon])] \leq N_1.$$ □

A classical and fundamental result of Rough Path Theory is the continuity of the Itô map, which associates to any initial condition a and any driving signal x the unique solution to the (standard) differential system (see [11] for further details)

$$(\delta y)_{st} = \mathcal{J}_{st}(\sigma(y)\, dx), \quad y_0 = a.$$

In our Volterra context, this continuity result still holds true. It is contained in the following proposition.

Proposition 3.10. *Define the Itô map F by $F(a,x) = y$, where y is the unique solution* (*given by Theorem* 3.4) *to the Volterra system* (13). *Then F is locally Lipschitz in the following sense*: *there exists an application $C : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ bounded on compact sets such that for any $a, \tilde{a} \in \mathbb{R}^d$, $x, \tilde{x} \in \mathcal{C}_1^\gamma([0,T])$,*

$$\mathcal{N}[F(a,x) - F(\tilde{a},\tilde{x});\mathcal{C}_1^\gamma([0,T])] \leq C\big(\mathcal{N}[x;\mathcal{C}_1^\gamma([0,T])], \mathcal{N}[\tilde{x};\mathcal{C}_1^\gamma([0,T])]\big) \{\|a-\tilde{a}\| + \mathcal{N}[x-\tilde{x};\mathcal{C}_1^\gamma([0,T])]\}. \tag{19}$$

Proof. In fact, (19) is easily obtained by combining the estimations we established in the proofs of Propositions 3.6 and 3.8. We only outline here the main steps of the reasoning, leaving the details to the reader.

Fix two elements $(a,x),(\tilde{a},\tilde{x}) \in \mathbb{R}^d \times \mathcal{C}_1^\gamma([0,T])$ and denote $y = F(a,x)$, $\tilde{y} = F(\tilde{a},\tilde{x})$.

Step 1. *Local inequality.* Consider a time $T_0 \le T$ that will be fixed at the end of this first step. For the sake of conciseness, we shall write $\mathcal{N}[.;\mathcal{C}_1^\gamma] = \mathcal{N}[.;\mathcal{C}_1^\gamma([0,T])]$ and we introduce the notation $R = \{1+\mathcal{N}[y;\mathcal{C}_1^\gamma]+\mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma]\}\{1+\mathcal{N}[x;\mathcal{C}_1^\gamma]+\mathcal{N}[\tilde{x};\mathcal{C}_1^\gamma]\}$. By definition of $y,\tilde{y}$, one has $y_t = a + \mathcal{J}_{0t}(\sigma^t(\mathcal{Y})\,dx)$ and $\tilde{y}_t = \tilde{a} + \mathcal{J}_{0t}(\sigma^t(\tilde{\mathcal{Y}})\,d\tilde{x})$, hence, for any $s,t \in [0,T_0]$, $\delta(y-\tilde{y})_{st} = I_{st}^{1,1,\Delta} + I_{st}^{1,2,\Delta} + I_{st}^{2,1,\Delta} + I_{st}^{2,2,\Delta}$, with

$$I_{st}^{1,1,\Delta} = \sigma^t(\mathcal{Y}_s)(\delta x)_{st} - \sigma^t(\tilde{\mathcal{Y}}_s)(\delta\tilde{x})_{st}, \quad I_{st}^{1,2,\Delta} = \Lambda_{st}(\delta(\sigma^t(\mathcal{Y}))\delta xs - \delta(\sigma^t(\tilde{\mathcal{Y}}))\delta\tilde{x}),$$

$$I_{st}^{2,1,\Delta} = [\sigma^t-\sigma^s](\mathcal{Y}_0)(\delta x)_{0s} - [\sigma^t-\sigma^s](\tilde{\mathcal{Y}}_0)(\delta\tilde{x})_{0s},$$

$$I_{st}^{2,2} = \Lambda_{0s}(\delta([\sigma^t-\sigma^s](\mathcal{Y}))\delta x - \sigma^t-\sigma^s](\tilde{\mathcal{Y}}))\delta\tilde{x}).$$

Then write for instance

$$I_{st}^{1,1,\Delta} = \sigma^t(\mathcal{Y}_s)\delta(x-\tilde{x})_{st} + [\sigma^t(\mathcal{Y}_s)-\sigma^t(\tilde{\mathcal{Y}}_s)](\delta\tilde{x})_{st}, \tag{20}$$

so that, as in the proof of (3.6),

$$\mathcal{N}[I^{1,1,\Delta};\mathcal{C}_2^\gamma([0,T_0])] \le c_\sigma R\{T_0^\gamma\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T_0])] + \|a-\tilde{a}\| + \mathcal{N}[x-\tilde{x};\mathcal{C}_1^\gamma]\}.$$

Proceed in the same way for $I^{1,2,\Delta}, I^{2,1,\Delta}, I^{2,2,\Delta}$ to get

$$\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T_0])] \le c_\sigma^1 R\{T_0^\gamma\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T_0])] + \|a-\tilde{a}\| + \mathcal{N}[x-\tilde{x};\mathcal{C}_1^\gamma]\}.$$

Choose now $T_0 = (2c_\sigma^1 R)^{-1/\gamma}$ and the previous inequality gives

$$\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T_0])] \le 2c_\sigma^1 R\{\|a-\tilde{a}\| + \mathcal{N}[x-\tilde{x};\mathcal{C}_1^\gamma]\}.$$

Step 2. *Extending the inequality.* Consider a small $\varepsilon > 0$ that we shall fix retrospectively. Following the same lines as in the proof of Proposition 3.8, together with decompositions such that (20), it is not hard to establish that

$$\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T_0+\varepsilon])] \le c_\sigma^2 R\{\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T_0])] + \|a-\tilde{a}\|$$
$$+\mathcal{N}[x-\tilde{x};\mathcal{C}_1^\gamma] + \varepsilon^{1-\gamma}\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T_0+\varepsilon])]\}.$$

As a result, take $\varepsilon = (2c_\sigma^2 R)^{-1/(1-\gamma)}$ to obtain

$$\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T_0+\varepsilon])] \le 2c_\sigma^2 R(2c_\sigma^1 R+1)\{\|a-\tilde{a}\| + \mathcal{N}[x-\tilde{x};\mathcal{C}_1^\gamma]\}.$$

We can repeat this procedure on $[0,T_0+2\varepsilon],[0,T_0+3\varepsilon],\ldots,[0,T_0+l(R)\varepsilon]$ until $T_0+l(R)\varepsilon=T$, and finally $\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma([0,T])]\le D(R)\{\|a-\tilde{a}\|+\mathcal{N}[x-\tilde{x};\mathcal{C}_1^\gamma]\}$ for some growing function $D:[1,\infty[\to\mathbb{R}^+$.

Step 3. *Conclusion.* It only remains to notice that the same kind of reasoning easily leads to $\mathcal{N}[y;\mathcal{C}_1^\gamma]\le G(\mathcal{N}[x;\mathcal{C}_1^\gamma])$ and $\mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma]\le G(\mathcal{N}[\tilde{x};\mathcal{C}_1^\gamma])$ for some function $G:\mathbb{R}^+\to\mathbb{R}^+$ bounded on compact sets. Thus

$$R\le\{1+G(\mathcal{N}[x;\mathcal{C}_1^\gamma])+G(\mathcal{N}[\tilde{x};\mathcal{C}_1^\gamma])\}\{1+\mathcal{N}[x;\mathcal{C}_1^\gamma]+\mathcal{N}[\tilde{x};\mathcal{C}_1^\gamma]\}$$

and inequality (19) holds with $C(a,b)=D(\{1+G(a)+G(b)\}\{1+a+b\})$. □

4. The Young Singular Case

This section is devoted to the study of a particular case of Eq. (1), when the coefficient σ admits a singularity in (t,u) on the diagonal. Namely, we shall consider an equation of the form

$$y_t=a+\int_0^t(t-u)^{-\alpha}\psi(y_u)\,dx_u, \tag{21}$$

with $\psi:\mathbb{R}^d\to\mathbb{R}^{d,n}$ a sufficiently regular mapping and $x\in\mathcal{C}_1^\gamma([0,T];\mathbb{R}^n)$, for some γ and α to be precised. Thus, the application σ appearing in (1) tends to explode when approaching the diagonal

$$D\times\mathbb{R}^d=\{(t,t,y),\ t\in[0,T],y\in\mathbb{R}^d\}.$$

This singularity prevents us from directly applying the algebraic formalism introduced in Sec. 2 in order to define the integral $\int_0^t(t-u)^{-\alpha}\psi(y_u)\,dx_u$ above. However, as in Sec. 3, we shall see that this latter integral can still be defined thanks to a slight extension of Young's interpretation, insofar as the integral will simply be seen as the limit of the associated Riemann sums. In other words, we will be able to set

$$\int_s^t(t-u)^{-\alpha}\psi(y_u)\,dx_u=\lim_{k\to\infty}\sum_{\Delta_k([s,t))}(t-t_i)^{-\alpha}\psi(y_{t_i})\,(\delta x)_{t_it_{i+1}}, \tag{22}$$

where $\Delta_k([s,t))=\{s=t_0<t_1<\cdots<t_k<t\}$ is any sequence of partitions whose meshes tend to 0, and where $t_k\to t$. In this context, Theorem 4.6 is quite close to Theorem 3.4.

Remark 4.1. The tedious calculations to come will give us an idea of how the Λ-formalism used in the previous sections makes the writing more fluent (when it can be applied), by avoiding the often cumbersome study of Riemann sums. One may then be tempted to resort to a regularization argument so as to reduce the problem to the regular case we dealt with in the previous section. And yet, as explained in Remark 4.9, this procedure also requires estimations of Riemann sums similar to those we are about to set.

4.1. *Young singular integrals*

This section deals with a rigorous definition of integrals like (22). A first technical lemma in this direction is then the following:

Lemma 4.2. *Let $a < b$, $f \in \mathcal{C}^{1,\mathbf{b}}([a,b];\mathbb{R}), g \in \mathcal{C}_1^{\lambda_1}([a,b];\mathbb{R}^{d,n}), h \in \mathcal{C}_1^{\lambda_2}([a,b];\mathbb{R}^n)$ with $\lambda_1 + \lambda_2 > 1$. Then*

$$\int_a^b d(fg)_u \, h_u = \int_a^b df_u \, g_u h_u + \int_a^b dg_u \, f_u h_u,$$

the three integrals being understood in the Young sense.

Proof. Consider a partition $\Delta = \{a = t_0 < \cdots < t_n = b\}$, with mesh $|\Delta|$, and use the decomposition

$$\sum_i \delta(fg)_{t_i t_{i+1}} h_{t_i} = \sum_i (\delta f)_{t_i t_{i+1}} g_{t_i} h_{t_i} + \sum_i (\delta g)_{t_i t_{i+1}} f_{t_i} h_{t_i} + \sum_i (\delta f)_{t_i t_{i+1}} (\delta g)_{t_i t_{i+1}} h_{t_i}.$$

Notice then that

$$\left\| \sum_i (\delta f)_{t_i t_{i+1}} (\delta g)_{t_i t_{i+1}} h_{t_i} \right\| \leq \mathcal{N}[f;\mathcal{C}^{1,\mathbf{b}}] \mathcal{N}[g;\mathcal{C}_1^{\lambda_1}] \, |\Delta|^{\lambda_1} \, \mathcal{N}[h;\mathcal{C}_1^0] \, |b-a|,$$

which tends to 0 as $|\Delta| \to 0$. The proof is thus completed. □

Note that if $x \in \mathcal{C}_1^\gamma$ and $y \in \mathcal{C}_1^\kappa$, the Young (regular) integral $\mathcal{J}(y\,dx)$ is well-defined if $\gamma + \kappa > 1$. The latter hypothesis ensures that the Riemann sums actually converge. In the Young singular case (22), this condition extends to $(\gamma - \alpha) + \kappa > 1$. This fact can be easily understood by the following heuristic argument: assume that the Riemann sums in (22) are considered along a dyadic partition of the form $s_n^i = s + [i(t-s)]/2^n$, for $n \geq 1$ and $i \leq 2^n$ (as will be done in Proposition 4.5). It is then easily seen that a necessary condition for a convergence of those Riemann sums is that, when $(t - s_n^i)$ is of order 2^{-n}, then

$$(t - s_{n+1}^{2i})^{-\alpha} [\psi(y_{s_{n+1}^{2i+1}}) - \psi(y_{s_{n+1}^{2i}})] (\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2i+2}}$$

is of order $2^{-\mu n}$ with $\mu > 1$. But provided ψ is a Lipschitz function, the latter condition is obviously ensured by the assumption $(\gamma - \alpha) + \kappa > 1$. This simple idea is formalized in the following lemma:

Lemma 4.3. *Let $x \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^n)$, $\psi \in \mathcal{C}^{1,\mathbf{b}}(\mathbb{R}^d;\mathbb{R}^{d,n})$ and assume that $0 < \alpha < \gamma$. Then for any κ such that $(\gamma - \alpha) + \kappa > 1$ and any $y \in \mathcal{C}_1^\kappa([0,T];\mathbb{R}^d)$, the integral $I_{st} := \int_s^t (t-u)^{-\alpha} \psi(y_u)\, dx_u$ exists in the Young sense. More specifically, for any $0 \leq s < t \leq T$ and $0 < \varepsilon < t - s$, set $I_{st}^\varepsilon := \int_s^{t-\varepsilon} (t-u)^{-\alpha} \psi(y_u)\, dx_u$, defined in the Young sense of Proposition* 3.1. *Then I_{st}^ε converges to a quantity, which is denoted again by $\int_s^t (t-u)^{-\alpha} \psi(y_u)\, dx_u$.*

Proof. Let $\varepsilon > 0$. If $u, v \in [s, t-\varepsilon]$,

$$\left\| \frac{\psi(y_v)}{(t-v)^\alpha} - \frac{\psi(y_u)}{(t-u)^\alpha} \right\| \leq \|\psi\|_\infty \left| \frac{1}{(t-v)^\alpha} - \frac{1}{(t-u)^\alpha} \right| + \left| \frac{1}{(t-u)^\alpha} \right| \|\psi(y_v) - \psi(y_u)\|$$
$$\leq \|\psi\|_\infty \frac{\alpha}{\varepsilon^{\alpha+1}} |v-u| + \frac{1}{\varepsilon^\alpha} \|\psi'\|_\infty \mathcal{N}[y; \mathcal{C}_1^\kappa([0,T])] \, |v-u|^\kappa ,$$

hence $u \mapsto \frac{\psi(y_u)}{(t-u)^\alpha} \in \mathcal{C}^\kappa([s, t-\varepsilon])$ and since $\kappa + \gamma > 1$, the integral I_{st}^ε is well-defined in the Young sense of Proposition 3.1. We will now study the convergence of I_{st}^ε when $\varepsilon \to 0$.

It is easily checked from relation (10) that one is allowed to perform an integration by parts in I_{st}^ε, in order to deduce

$$\begin{aligned} I_{st}^\varepsilon &= \int_s^{t-\varepsilon} (t-u)^{-\alpha} \psi(y_u) \, dx_u \\ &= \int_s^{t-\varepsilon} (t-u)^{-\alpha} \psi(y_u) \, d(x_u - x_t) \\ &= \frac{\psi(y_{t-\varepsilon})}{\varepsilon^\alpha} (x_{t-\varepsilon} - x_t) + \frac{\psi(y_s)}{(t-s)^\alpha} (x_t - x_s) + \int_s^{t-\varepsilon} d\left(\frac{\psi(y_u)}{(t-u)^\alpha} \right) (x_t - x_u) \\ &:= I_{st}^{\varepsilon,1} + I_{st}^{\varepsilon,2} + I_{st}^{\varepsilon,3}. \end{aligned}$$

Let us analyze now the three terms we have obtained: since

$$\left\| \frac{\psi(y_{t-\varepsilon})}{\varepsilon^\alpha} (x_{t-\varepsilon} - x_t) \right\| \leq \|\psi\|_\infty \mathcal{N}[x; \mathcal{C}_1^\gamma] \varepsilon^{\gamma - \alpha},$$

it is readily checked that $I_{st}^{\varepsilon,1} \to 0$ as $\varepsilon \to 0$. In order to treat the term $I_{st}^{\varepsilon,3}$ observe that, according to Lemma 4.2, we have

$$\begin{aligned} I_{st}^{\varepsilon,3} &= \int_s^{t-\varepsilon} d\left(\frac{\psi(y_u)}{(t-u)^\alpha} \right) (x_t - x_u) \\ &= \int_s^{t-\varepsilon} d(\psi(y_u)) \frac{(x_t - x_u)}{(t-u)^\alpha} + \alpha \int_s^{t-\varepsilon} \frac{du}{(t-u)^{\alpha+1}} \psi(y_u)(x_t - x_u) \\ &:= I_{st}^{\varepsilon,3,1} + I_{st}^{\varepsilon,3,2}. \end{aligned} \tag{23}$$

Notice then that

$$\left\| \frac{\psi(y_u)}{(t-u)^{\alpha+1}} (x_t - x_u) \right\| \leq \|\psi\|_\infty \mathcal{N}[x; \mathcal{C}_1^\gamma] \frac{1}{|t-u|^{1-(\gamma-\alpha)}},$$

and thus $u \mapsto \frac{\psi(y_u)}{(t-u)^{\alpha+1}} (x_t - x_u)$ is (Lebesgue-)integrable in t. This trivially yields the convergence of $I_{st}^{\varepsilon,3,2}$ as $\varepsilon \to 0$. As for the first term $I_{st}^{\varepsilon,3,1}$ in (23), we know that $u \mapsto \psi(y_u) \in \mathcal{C}_1^\kappa$. In order to study the convergence of $I_{st}^{\varepsilon,3,1}$, it only remains to prove that the application $\varphi : [s,t) \to \mathbb{R}^n$, $u \mapsto \frac{(x_t - x_u)}{(t-u)^\alpha}$, continuously extended by 0 in t, belongs to $\mathcal{C}_1^\rho([s,t])$, for some $\rho > 0$ satisfying $\rho + \kappa > 1$.

However, if $0 < u < v < t$,

$$
\begin{aligned}
&\|\varphi_v - \varphi_u\| \\
&\quad \le \|x_t - x_v\| |(t-v)^{-\alpha} - (t-u)^{-\alpha}| + |(t-u)^{-\alpha}| \, \|x_t - x_v - (x_t - x_u)\| \\
&\quad \le \mathcal{N}[x; \mathcal{C}_1^\gamma] |t-v|^\gamma \left(\frac{1}{|t-v|^\alpha} \right)^{1-(\gamma-\alpha)} \left(\alpha \frac{|v-u|}{|t-v|^{\alpha+1}} \right)^{\gamma-\alpha} \\
&\qquad + \frac{1}{|v-u|^\alpha} \mathcal{N}[x; \mathcal{C}_1^\gamma] |v-u|^\gamma \\
&\quad \le c\,\mathcal{N}[x; \mathcal{C}_1^\gamma] |v-u|^{\gamma-\alpha} + \mathcal{N}[x; \mathcal{C}_1^\gamma] |v-u|^{\gamma-\alpha},
\end{aligned}
$$

while if $u < v = t$, as $\varphi_t = 0$,

$$
\|\varphi_v - \varphi_u\| = \frac{\|x_v - x_u\|}{|(v-u)^\alpha|} \le \mathcal{N}[x; \mathcal{C}_1^\gamma] \, |v-u|^{\gamma-\alpha} .
$$

Thus, $\varphi \in \mathcal{C}_1^{\gamma-\alpha}([0,t])$, which achieves the proof since, by hypothesis, $(\gamma-\alpha)+\kappa > 1$. □

It is also important to control the Hölder continuity of the singular Young integral defined above, just as in Proposition 3.1. Before we turn to this task, let us quote an elementary estimate for further use:

Lemma 4.4. *Let* $0 < s < t \le T$. *For any* $\beta \in [0,1]$, *there exists a constant* c_β *such that for any* $u \in (0,s)$,

$$
\left| (t-u)^{-\alpha} - (s-u)^{-\alpha} \right| \le c_\beta \, |s-u|^{-\alpha-\beta} \, |t-s|^\beta . \tag{24}
$$

Then our regularity result is the following:

Proposition 4.5. *Under the same hypotheses as in Lemma 4.3, and assuming in addition that* $\kappa < \gamma - \alpha$, *set* $z_t = I_{0t}$ *for all* $t \in [0,T]$. *Then, for any* $T_0 \le T$, *the path* z *is an element of* $\mathcal{C}_1^\kappa([0,T_0])$, *and the following estimate holds true*:

$$
\mathcal{N}[z; \mathcal{C}_1^\kappa([0,T_0])] \le c_{\psi,x} T_0^{\gamma-\alpha-\kappa} \{1 + \mathcal{N}[y; \mathcal{C}_1^\kappa([0,T_0])]\}. \tag{25}
$$

Proof. We rely on the decomposition $(\delta z)_{st} = I_{st} + II_{st}$, with

$$
\begin{aligned}
I_{st} &= \int_s^t (t-u)^{-\alpha} \psi(y_u) dx_u \quad \text{and} \\
II_{st} &= \int_0^s [(t-u)^{-\alpha} - (s-u)^{-\alpha}] \psi(y_u) \, dx_u.
\end{aligned} \tag{26}
$$

Notice that the term I is exactly the one introduced in Lemma 4.3. Let us now bound each of these terms.

Case I. It is easily seen that I can also be obtained thanks to the following approximation sequence: for $n \geq 1$, set

$$J_n = \sum_{i=0}^{2^n-1} (t - s_n^i)^{-\alpha} \psi(y_{s_n^i})(\delta x)_{s_n^i, s_n^{i+1}}, \quad \text{where } s_n^i = s + \frac{i(t-s)}{2^n}.$$

Then I_{st} is obtained as $\lim_{n\to\infty} J_n$. Moreover, it is readily checked that

$$\begin{aligned} J_{n+1} - J_n &= \sum_{i=0}^{2^n-1} [(t - s_{n+1}^{2i+1})^{-\alpha} \psi(y_{s_{n+1}^{2i+1}}) - (t - s_{n+1}^{2i})^{-\alpha} \psi(y_{s_{n+1}^{2i}})](\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2+2}} \\ &= \sum_{i=0}^{2^n-1} [(t - s_{n+1}^{2i+1})^{-\alpha} - (t - s_{n+1}^{2i})^{-\alpha}] \psi(y_{s_{n+1}^{2i+1}})(\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2i+2}} \\ &\quad + \sum_{i=0}^{2^n-1} (t - s_{n+1}^{2i})^{-\alpha} [\psi(y_{s_{n+1}^{2i+1}}) - \psi(y_{s_{n+1}^{2i}})](\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2i+2}} \\ &:= A + B. \end{aligned} \tag{27}$$

But

$$\|A\| \leq \|\psi\|_\infty \mathcal{N}[x; \mathcal{C}_1^\gamma] \frac{|t-s|^\gamma}{(2^{n+1})^\gamma} \sum_{i=0}^{2^n-1} \left| (t - s_{n+1}^{2i+1})^{-\alpha} - (t - s_{n+1}^{2i})^{-\alpha} \right|,$$

and we can show that

$$\begin{aligned} &\sum_{i=0}^{2^n-1} \left| (t - s_{n+1}^{2i+1})^{-\alpha} - (t - s_{n+1}^{2i})^{-\alpha} \right| \\ &\quad = (t-s)^{-\alpha} \sum_{i=0}^{2^n-1} \left\{ \left(1 - \frac{2i+1}{2^{n+1}}\right)^{-\alpha} - \left(1 - \frac{2i}{2^{n+1}}\right)^{-\alpha} \right\} \\ &\quad \leq (t-s)^{-\alpha} \left(1 - \frac{2^{n+1}-1}{2^{n+1}}\right)^{-\alpha} \\ &\quad \leq (t-s)^{-\alpha} (2^{n+1})^\alpha. \end{aligned} \tag{28}$$

Hence

$$\|A\| \leq c_{\psi,x} \, |t-s|^{\gamma-\alpha} \left(\frac{1}{2^{\gamma-\alpha}}\right)^{n+1} \leq c_{\psi,x} \, |t-s|^\kappa \, T_0^{\gamma-\alpha-\kappa} \left(\frac{1}{2^{\gamma-\alpha}}\right)^{n+1}. \tag{29}$$

As for B, the following bound holds true:

$$\|B\| \leq \left(\sum_{i=0}^{2^n-1} (t - s_{n+1}^{2i})^{-\alpha} \right) \|\psi'\|_\infty \mathcal{N}[y; \mathcal{C}_1^\kappa] \frac{|t-s|^\kappa}{(2^{n+1})^\kappa} \mathcal{N}[x; \mathcal{C}_1^\gamma] \frac{|t-s|^\gamma}{(2^{n+1})^\gamma},$$

with

$$\sum_{i=0}^{2^n-1} (t - s_{n+1}^{2i})^{-\alpha} = (t-s)^{-\alpha} \sum_{i=0}^{2^n-1} \left(1 - \frac{2i}{2^{n+1}}\right)^{-\alpha}$$
$$\leq \frac{2^{n+1}}{(t-s)^\alpha} \int_0^1 \frac{du}{(1-u)^\alpha}$$
$$\leq \frac{2^{n+1}}{1-\alpha}(t-s)^{-\alpha},$$

and accordingly

$$\|B\| \leq c_{\psi,x} \mathcal{N}[y; \mathcal{C}_1^\kappa] \, |t-s|^{\kappa+\gamma-\alpha} \left(\frac{1}{2^{\kappa+\gamma-1}}\right)^{n+1}$$
$$\leq c_{\psi,x} \, |t-s|^\kappa \, \mathcal{N}[y; \mathcal{C}_1^\kappa] T_0^{\gamma-\alpha} \left(\frac{1}{2^{\kappa+\gamma-1}}\right)^{n+1}. \tag{30}$$

Going back to (27) and putting together our estimates for A and B, we get

$$\|J_{n+1} - J_n\| \leq T_0^{\gamma-\alpha-\kappa} \, |t-s|^\kappa \, \{1 + \mathcal{N}[y; \mathcal{C}_1^\kappa]\} \, v_n,$$

where v_n is the general term of a converging series. Now, write $J_N = J_0 + \sum_{n=0}^{N-1}(J_{n+1} - J_n)$, so that, by letting N tend to infinity, we obtain

$$\left\| \int_s^t (t-u)^{-\alpha} \psi(y_u) dx_u \right\| \leq \|J_0\| + T_0^{\gamma-\alpha-\kappa} \, |t-s|^\kappa \, \{1 + \mathcal{N}[y; \mathcal{C}_1^\kappa]\}.$$

It only remains to note that

$$\|J_0\| = \|(t-s)^{-\alpha} \psi(y_s)(\delta x)_{st}\| \leq \|\psi\|_\infty \mathcal{N}[x; \mathcal{C}_1^\gamma] \, |t-s|^{\gamma-\alpha} \leq c_{\psi,x} \, |t-s|^\kappa \, T_0^{\gamma-\alpha-\kappa} \tag{31}$$

to conclude

$$\|I_{st}\| \leq T_0^{\gamma-\alpha-\kappa} \, |t-s|^\kappa \, \{1 + \mathcal{N}[y; \mathcal{C}_1^\kappa]\}.$$

Case II. We use the same strategy as for I, with this time $s_n^i = \frac{is}{2^n}$ and

$$J_n = \sum_{i=0}^{2^n-1} f_{s,t}(s_n^i) \psi(y_{s_n^i}) (\delta x)_{s_n^i, s_n^{i+1}}, \quad \text{where } f_{s,t}(u) = (t-u)^{-\alpha} - (s-u)^{-\alpha}.$$

Then

$$J_{n+1} - J_n = \sum_{i=0}^{2^n-1} \{f_{s,t}(s_{n+1}^{2i+1}) \psi(y_{s_{n+1}^{2i+1}}) - f_{s,t}(s_{n+1}^{2i}) \psi(y_{s_{n+1}^{2i}})\} (\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2i+2}}$$
$$= \sum_{i=0}^{2^n-1} \{f_{s,t}(s_{n+1}^{2i+1}) - f_{s,t}(s_{n+1}^{2i})\} \psi(y_{s_{n+1}^{2i+1}}) (\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2i+2}}$$
$$+ \sum_{i=0}^{2^n-1} f_{s,t}(s_{n+1}^{2i}) \{\psi(y_{s_{n+1}^{2i+1}}) - \psi(y_{s_{n+1}^{2i}})\} (\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2i+2}}$$
$$:= D + E. \tag{32}$$

To deal with D, note that $u \mapsto f_{s,t}(u)$ is a decreasing function on $[0,s]$, and hence

$$\sum_{i=0}^{2^n-1} \left|f_{s,t}(s_{n+1}^{2i+1}) - f_{s,t}(s_{n+1}^{2i})\right| \leq \sum_{i=0}^{2^{n+1}-1} \left|f_{s,t}(s_{n+1}^{i+1}) - f_{s,t}(s_{n+1}^{i})\right|$$

$$\leq \left| f_{s,t}\left(\frac{2^{n+1}-1}{2^{n+1}} s\right)\right|. \tag{33}$$

Furthermore, according to our elementary bound (24) applied with $\beta = \kappa$, we have $|f_{s,t}\left(\frac{2^{n+1}-1}{2^{n+1}}s\right)| \leq \frac{c}{s^{\alpha+\kappa}} |t-s|^{\kappa} (2^{\alpha+\kappa})^{n+1}$, so that

$$\|D\| \leq c\|\psi\|_\infty \mathcal{N}[x;\mathcal{C}_1^\gamma] s^{\gamma-\alpha-\kappa} |t-s|^\kappa \left(\frac{1}{2^{\gamma-\alpha-\kappa}}\right)^{n+1}$$

$$\leq c_{\psi,x} T_0^{\gamma-\alpha-\kappa} |t-s|^\kappa \left(\frac{1}{2^{\gamma-\alpha-\kappa}}\right)^{n+1}. \tag{34}$$

As far as E is concerned, use (24) with $\beta = \gamma - \alpha$ to deduce

$$\|E\| \leq c\,\|\psi'\|_\infty \mathcal{N}[y;\mathcal{C}_1^\kappa]\mathcal{N}[x;\mathcal{C}_1^\gamma] s^\kappa |t-s|^{\gamma-\alpha} \left(\frac{1}{2^{\kappa+\gamma}}\right)^{n+1} \sum_{i=0}^{2^n-1} \left(1 - \frac{2i}{2^{n+1}}\right)^{-\gamma}$$

$$\leq c_{\psi,x}\mathcal{N}[y;\mathcal{C}_1^\kappa] s^\kappa |t-s|^{\gamma-\alpha} \left(\frac{1}{2^{\kappa+\gamma-1}}\right)^{n+1} \int_0^1 \frac{dx}{(1-x)^\gamma}$$

$$\leq c_{\psi,x}\mathcal{N}[y;\mathcal{C}_1^\kappa] |t-s|^\kappa |t-s|^{\gamma-\alpha-\kappa} T_0^\kappa \left(\frac{1}{2^{\kappa+\gamma-1}}\right)^{n+1}, \tag{35}$$

hence

$$\|E\| \leq c_{\psi,x}\mathcal{N}[y;\mathcal{C}_1^\kappa] T_0^{\gamma-\alpha} |t-s|^\kappa \left(\frac{1}{2^{\kappa+\gamma-1}}\right)^{n+1}.$$

Just as for I, gathering our bounds on D and E, we can then assert that

$$\left\|\int_0^s [(t-u)^{-\alpha} - (s-u)^{-\alpha}]\psi(y_u)\, dx_u\right\| \leq \|J_0\| + c_{\psi,x} T_0^{\gamma-\alpha-\kappa} |t-s|^\kappa \{1 + \mathcal{N}[y;\mathcal{C}_1^\kappa]\}.$$

Since $|t^{-\alpha} - s^{-\alpha}| \leq c\, s^{-\alpha-\kappa} |t-s|^\kappa$, the term J_0 above can be estimated as:

$$\|J_0\| = \|\{t^{-\alpha} - s^{-\alpha}\}(\delta x)_{0s}\| \leq \mathcal{N}[x;\mathcal{C}_1^\gamma] s^{\gamma-\alpha-\kappa} |t-s|^\kappa, \tag{36}$$

so that

$$\|II_{st}\| = \left\|\int_0^s (t-u)^{-\alpha} - (s-u)^{-\alpha}]\psi(y_u)\, dx_u\right\|$$

$$\leq c_{\psi,x} T_0^{\gamma-\alpha-\kappa} |t-s|^\kappa \{1 + \mathcal{N}[y;\mathcal{C}_1^\kappa]\}.$$

Finally, going back to decomposition (26), our bounds on I and II yield

$$\mathcal{N}[z;\mathcal{C}_1^\kappa] \leq c_{\psi,x} T_0^{\gamma-\alpha-\kappa}(1 + \mathcal{N}[y;\mathcal{C}_1^\kappa]),$$

which was the announced result. □

4.2. *Solving Volterra equations*

Thanks to the considerations of the last section, we can now interpret Eq. (21), and especially its integral term, in the sense given by Lemma 4.3 and Proposition 4.5. We are now in a position to state the main result of this section:

Theorem 4.6. *Assume that $x \in \mathcal{C}_1^{\gamma}([0,T];\mathbb{R}^n)$ for some $\gamma \in (1/2,1)$, let ψ be a function in $\mathcal{C}^{1,\mathbf{b}}(\mathbb{R}^d;\mathbb{R}^{d,n})$, and $\alpha \in (0,1/2)$ such that $\gamma-\alpha > 1/2$. Then, for any $\kappa \in (1-(\gamma-\alpha);\gamma-\alpha)$, Eq.* (21) *admits a unique solution in $\mathcal{C}_1^{\kappa}([0,T];\mathbb{R}^d)$.*

Fix $\kappa \in (1-(\gamma-\alpha),\gamma-\alpha)$. As in Sec. 3, we shall solve our equation by identifying its solution with the fixed point of the map Γ defined, for any $y \in \mathcal{C}_1^{\kappa}([0,T];\mathbb{R}^d)$, by

$$z_t = \Gamma(y)_t = a + \int_0^t (t-u)^{-\alpha}\psi(y_u)\,dx_u. \tag{37}$$

We divide again our proof into two propositions, dealing respectively with local and global existence and uniqueness for the solution.

Proposition 4.7. (Local existence) *Under the hypothesis of Theorem* 4.6, *there exists $T_0 \in (0,T]$ such that Eq.* (21) *admits a unique solution $y^{(1)}$ in $\mathcal{C}_1^{\kappa}([0,T_0];\mathbb{R}^d)$.*

Proof. Fix a time $T_0 \in (0,T]$ and let $y \in \mathcal{C}^{\kappa}([0,T_0])$. Define then $z = \Gamma(y)$ as in Eq. (37).

Step 1. *Invariance of a ball.* A simple application of Proposition 4.5 allows one to conclude the existence of a stable ball

$$\mathcal{B}_{a,T_0} = \{y \in \mathcal{C}^{\kappa}([0,T_0]),\ y_0 = a,\ \mathcal{N}[y;\mathcal{C}_1^{\kappa}] \le A_{T_0}\}$$

for any T_0 small enough and A_{T_0} large enough.

Step 2. *Contraction property.* Let $y,\tilde{y} \in \mathcal{B}_{a,T_0}$, and set $z=\Gamma(y)$, $\tilde{z}=\Gamma(\tilde{y})$. Thus, $\delta(z-\tilde{z})_{st} = III_{st} + IV_{st}$, with

$$\begin{aligned} III_{st} &= \int_s^t (t-u)^{-\alpha}[\psi(y_u)-\psi(\tilde{y}_u)]dx_u, \\ IV_{st} &= \int_0^s [(t-u)^{-\alpha}-(s-u)^{-\alpha}]\psi(y_u)-\psi(\tilde{y}_u)]dx_u. \end{aligned} \tag{38}$$

We will now estimate these two terms, according to the same strategy as for Proposition 4.5, i.e. invoking approximations by dyadic partitions.

Case III. Denote

$$s_n^i = s + \frac{i(t-s)}{2^n}, \quad J_n = \sum_{i=0}^{2^n-1} (t - s_n^i)^{-\alpha} [\psi(y_{s_n^i}) - \psi(\tilde{y}_{s_n^i})](\delta x)_{s_n^i, s_n^{i+1}}.$$

Then

$$\begin{aligned} J_{n+1} - J_n \ = \ & \sum_{i=0}^{2^n-1} \{[(t - s_{n+1}^{2i+1})^{-\alpha} - (t - s_{n+1}^{2i})^{-\alpha}][\psi(y_{s_{n+1}^{2i+1}}) \\ & - \psi(\tilde{y}_{s_{n+1}^{2i+1}})]\}(\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2i+2}} \\ & + \sum_{i=0}^{2^n-1} \{(t - s_{n+1}^{2i})^{-\alpha}[\psi(y_{s_{n+1}^{2i+1}}) - \psi(\tilde{y}_{s_{n+1}^{2i+1}}) - \psi(y_{s_{n+1}^{2i}}) \\ & + \psi(\tilde{y}_{s_{n+1}^{2i}})]\}(\delta x)_{s_{n+1}^{2i+1}, s_{n+1}^{2i+2}} \\ := \ & F + G. \end{aligned} \tag{39}$$

For F, we have, since $(y - \tilde{y})_0 = 0$,

$$\|F\| \le \mathcal{N}[x; \mathcal{C}_1^\gamma] \frac{|t-s|^\gamma}{(2^{n+1})^\gamma} \|\psi'\|_\infty \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\kappa] T_0^\kappa \sum_{i=0}^{2^n-1} |(t - s_{n+1}^{2i+1})^{-\alpha} - (t - s_{n+1}^{2i})^{-\alpha}|,$$

which, thanks to (28), gives

$$\|F\| \le c_{\psi,x} \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\kappa] \, |t-s|^{\gamma-\alpha-\kappa} \, |t-s|^\kappa \left(\frac{1}{2^{\gamma-\alpha}} \right)^{n+1} T_0^\kappa. \tag{40}$$

As far as G is concerned, use (16) to assert that

$$\begin{aligned} & \|\psi(y_{s_{n+1}^{2i+1}}) - \psi(\tilde{y}_{s_{n+1}^{2i+1}}) - \psi(y_{s_{n+1}^{2i}}) + \psi(\tilde{y}_{s_{n+1}^{2i}})\| \\ & \qquad \le c_\psi \{1 + \mathcal{N}[y; \mathcal{C}_1^\kappa] + \mathcal{N}[\tilde{y}; \mathcal{C}_1^\kappa]\} \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\kappa] \frac{|t-s|^\kappa}{(2^{n+1})^\kappa}. \end{aligned}$$

Besides,

$$\sum_{i=0}^{2^n-1} (t - s_{n+1}^{2i})^{-\alpha} \le \frac{2^{n+1}}{(t-s)^\alpha} \int_0^1 \frac{du}{(1-u)^\alpha},$$

so that

$$\|G\| \le c_{\psi,x} \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\kappa] \{1 + 2A_{T_0}\} \, |t-s|^\kappa \left(\frac{1}{2^{\gamma+\kappa-1}} \right)^{n+1} |t-s|^{\gamma-\kappa}. \tag{41}$$

Now, relations (40) and (41) entail

$$\|III_{st}\| \le \|J_0\| + \sum_{i=0}^\infty \|J_{n+1} - J_n\| \le \|J_0\| + c_{\psi,x} T_0^{\gamma-\alpha} \{1 + 2A_{T_0}\} \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\kappa] \, |t-s|^\kappa.$$

Furthermore, we have

$$
\begin{aligned}
\|J_0\| &= \|(t-s)^{-\alpha}[\psi(y_s)-\psi(\tilde{y}_s)](\delta x)_{st}\| \\
&\le |t-s|^{\kappa}\,|t-s|^{\gamma-\alpha-\kappa}\,\mathcal{N}[x;\mathcal{C}_1^{\gamma}]\|D\psi\|_\infty \mathcal{N}[y-\tilde{y};\mathcal{C}_1^{\kappa}]s^{\kappa}, \\
&\le c_{\psi,x}T_0^{\gamma-\alpha-\kappa}\,|t-s|^{\kappa}\,\mathcal{N}[y-\tilde{y};\mathcal{C}_1^{\kappa}] \qquad (42)
\end{aligned}
$$

which finally yields

$$
\|III_{st}\| \le c_{\psi,x}T_0^{\gamma-\alpha-\kappa}\{1+2A_{T_0}\}\mathcal{N}[y-\tilde{y};\mathcal{C}_1^{\kappa}]\,|t-s|^{\kappa}.
$$

Case IV. In this case, the approximating sequence is defined by:

$$
s_n^i=\frac{is}{2^n},\quad J_n=\sum_{i=0}^{2^n-1} f_{s,t}(s_n^i)[\psi(y_{s_n^i})-\psi(\tilde{y}_{s_n^i})](\delta x)_{s_n^i,s_n^{i+1}}.
$$

Hence, the difference $J_{n+1}-J_n$ can be decomposed into:

$$
\begin{aligned}
J_{n+1}-J_n \;=\; &\sum_{i=0}^{2^n-1}\{[f_{s,t}(s_{n+1}^{2i+1})-f_{s,t}(s_{n+1}^{2i})][\psi(y_{s_{n+1}^{2i+1}})-\psi(\tilde{y}_{s_{n+1}^{2i+1}})]\}(\delta x)_{s_{n+1}^{2i+1},s_{n+1}^{2i+2}} \\
&+\sum_{i=0}^{2^n-1}\{f_{s,t}(s_{n+1}^{2i})[\psi(y_{s_{n+1}^{2i+1}})-\psi(\tilde{y}_{s_{n+1}^{2i+1}})-\psi(y_{s_{n+1}^{2i}}) \\
&+\psi(\tilde{y}_{s_{n+1}^{2i}})]\}(\delta x)_{s_{n+1}^{2i+1},s_{n+1}^{2i+2}} \\
:=\; &H+K. \qquad (43)
\end{aligned}
$$

In order to bound these two terms, let us introduce first some $\lambda\in(\kappa,\gamma-\alpha)$. From (33), and invoking (24) with $\beta=\lambda$, we obtain

$$
\sum_{i=0}^{2^n-1}\left|f_{s,t}(s_{n+1}^{2i+1})-f_{s,t}(s_{n+1}^{2i})\right| \le c\,|t-s|^{\lambda}\,\frac{(2^{\alpha+\lambda})^{n+1}}{s^{\alpha+\lambda}},
$$

while $\|\psi(y_{s_{n+1}^{2i+1}})-\psi(\tilde{y}_{s_{n+1}^{2i+1}})\|\le\|\psi'\|_\infty\mathcal{N}[y-\tilde{y};\mathcal{C}_1^{\kappa}]\,s^{\kappa}$, and so

$$
\begin{aligned}
\|H\| &\le c_{\psi,x}\,|t-s|^{\kappa}\,|t-s|^{\lambda-\kappa}\,\mathcal{N}[y-\tilde{y};\mathcal{C}_1^{\kappa}]s^{\gamma+\kappa-\alpha-\lambda}\left(\frac{1}{2^{\gamma-\alpha-\lambda}}\right)^{n+1} \\
&\le c_{\psi,x}\,|t-s|^{\kappa}\,T_0^{\gamma-\kappa}\mathcal{N}[y-\tilde{y};\mathcal{C}_1^{\kappa}]\left(\frac{1}{2^{\gamma-\alpha-\lambda}}\right)^{n+1}. \qquad (44)
\end{aligned}
$$

To estimate $\|K\|$, remember that

$$
\begin{aligned}
&\|\psi(y_{s_{n+1}^{2i+1}})-\psi(\tilde{y}_{s_{n+1}^{2i+1}})-\psi(y_{s_{n+1}^{2i}})+\psi(\tilde{y}_{s_{n+1}^{2i}})\| \\
&\qquad\le\; c_\psi\{1+\mathcal{N}[y;\mathcal{C}_1^{\kappa}]+\mathcal{N}[\tilde{y};\mathcal{C}_1^{\kappa}]\}\mathcal{N}[y-\tilde{y};\mathcal{C}_1^{\kappa}]\frac{s^{\kappa}}{(2^{n+1})^{\kappa}},
\end{aligned}
$$

which, together with (24) applied with $\beta = \gamma - \alpha$, gives

$$\begin{aligned}\|K\| &\le c_{\psi,x} |t-s|^{\gamma-\alpha} \{1+2A_{T_0}\} \mathcal{N}[y-\tilde{y}; \mathcal{C}_1^\kappa] s^\kappa \left(\frac{1}{2^{\kappa+\gamma}}\right)^{n+1} \sum_{i=0}^{2^n-1} \left(1 - \frac{2i}{2^{n+1}}\right)^{-\gamma} \\ &\le c_{\psi,x} |t-s|^{\kappa} |t-s|^{\gamma-\alpha-\kappa} \{1+2A_{T_0}\} \\ &\quad \times \mathcal{N}[y-\tilde{y}; \mathcal{C}_1^\kappa] T_0^\kappa \left(\frac{1}{2^{\kappa+\gamma-1}}\right)^{n+1} \int_0^1 \frac{du}{(1-u)^\gamma} \\ &\le c_{\psi,x} |t-s|^{\kappa} T_0^{\gamma-\alpha} \{1+2A_{T_0}\} \mathcal{N}[y-\tilde{y}; \mathcal{C}_1^\kappa] \left(\frac{1}{2^{\kappa+\gamma-1}}\right)^{n+1}. \end{aligned} \tag{45}$$

As a result, combining the estimates for H and K along the same lines as for the term III_{st}, we end up with:

$$\|IV_{st}\| \le \|J_0\| + c_{\psi,x}\{1+2A_{T_0}\} \mathcal{N}[y-\tilde{y}; \mathcal{C}_1^\kappa] |t-s|^\kappa T_0^{\gamma-\alpha}.$$

But $J_0 = [t^{-\alpha} - s^{-\alpha}][\psi(y_0) - \psi(\tilde{y}_0)](\delta x)_{0s} = 0$, so that finally

$$\|IV_{st}\| \le c_{\psi,x} T_0^{\gamma-\alpha} \{1+2A_{T_0}\} \mathcal{N}[y-\tilde{y}; \mathcal{C}_1^\kappa] |t-s|^\kappa .$$

We have thus proved that

$$\mathcal{N}[z-\tilde{z}; \mathcal{C}_1^\kappa] \le c_{\psi,x} T_0^{\gamma-\alpha-\kappa} \{1+2A_{T_0}\} \mathcal{N}[y-\tilde{y}; \mathcal{C}_1^\kappa].$$

The contraction property then clearly holds when Γ is restricted to a stable ball $\mathcal{B}_{a,T_0}$, for T_0 small enough. This easily yields the existence and uniqueness of a solution to (21) on $[0, T_0]$. □

The following proposition summarizes the extension of the unique solution to (21) to an arbitrary interval.

Proposition 4.8. (Global existence) *Under the same hypothesis as for Theorem* 4.6, *the local solution* $y^{(1)} \in \mathcal{C}_1^\kappa([0, T_0])$ *can be extended in a unique way into a global solution in* $\mathcal{C}_1^\kappa([0, T])$.

Proof. We resort to the same scheme as in Proposition 3.8, in which we try to exploit the estimations of the previous proof.

Step 1. *Invariance of a ball.* Let $\varepsilon > 0$ and $y \in \mathcal{C}^\kappa([0, T_0+\varepsilon])$ such that $y_{|[0,T_0]} = y^{(1)}$. Set

$$z_t = \Gamma(y)_t = \begin{cases} y_t^{(1)} & \text{if } t \in [0, T_0], \\ a + \int_0^t (t-u)^{-\alpha} \psi(y_u)\, dx_u & \text{if } t \in [T_0, T_0+\varepsilon]. \end{cases}$$

Let $s,t \in [T_0, T_0+\varepsilon]$ and consider the decomposition (26) of $(\delta z)_{st}$. For I, use (27), together with the estimations (29)–(31), to deduce

$$\left\| \int_s^t (t-u)^{-\alpha} \psi(y_u)\, dx_u \right\| \leq c_{\psi,x} \left| t-s \right|^{\kappa} \{1+\varepsilon^{\gamma-\alpha} \mathcal{N}[y;\mathcal{C}_1^\kappa]\}.$$

As for II, use (32), together with (34), (35) and (36) to assert

$$\left\| \int_0^s [(t-u)^{-\alpha} - (s-u)^{-\alpha}] \psi(y_u)\, dx_u \right\| \leq c_{\psi,x} \left| t-s \right|^{\kappa} \{1+\varepsilon^{\gamma-\alpha-\kappa} \mathcal{N}[y;\mathcal{C}_1^\kappa]\}.$$

As a result,

$$\mathcal{N}[z;\mathcal{C}_1^\kappa([T_0,T_0+\varepsilon])] \leq c_{\psi,x}\{1+\varepsilon^{\gamma-\alpha-\kappa}\mathcal{N}[y;\mathcal{C}_1^\kappa]\}.$$

By copying the arguments of the proof of Proposition 3.8, we then deduce the existence of a small ε, independent of $y^{(1)}$, and a radius N_1, such that the ball

$$\mathcal{B}_{y^{(1)},T_0,\varepsilon} := \{ y \in \mathcal{C}_1^\kappa([0,T_0+\varepsilon]) : \; y_{|[0,T_0]} = y^{(1)},\ \mathcal{N}[y;\mathcal{C}_1^\kappa] \leq N_1 \}$$

is invariant by Γ.

Step 2. *Contraction property.* Let $\eta \leq \varepsilon$, and consider $y, \tilde{y} \in \mathcal{C}_1^\kappa([0,T_0+\eta])$ such that $y_{|[0,T_0]} = \tilde{y}_{|[0,T_0]} = y^{(1)}$, $\mathcal{N}[y;\mathcal{C}_1^\kappa] \leq N_1$ and $\mathcal{N}[\tilde{y};\mathcal{C}_1^\kappa] \leq N_1$. Set $z=\Gamma(y)$, $\tilde{z} = \Gamma(\tilde{y})$.

Let $s,t \in [T_0,T_0+\eta]$ and consider the decomposition (38) of $\delta(z-\tilde{z})_{st}$. For III, use (39), together with (40)–(42), to obtain

$$\|III_{st}\| \leq c_{\psi,x} \eta^{\gamma-\alpha-\kappa} \left| t-s \right|^{\kappa} \{1+2N_1\} \mathcal{N}[y-\tilde{y};\mathcal{C}_1^\kappa].$$

As far as IV is concerned, the decomposition (43), together with (44), (45) and the fact that $\psi(y_0) = \psi(\tilde{y}_0)$, provides

$$\|IV_{st}\| \leq c_{\psi,x} \eta^{\lambda-\kappa} \left| t-s \right|^{\kappa} \{1+2N_1\} \mathcal{N}[y-\tilde{y};\mathcal{C}_1^\kappa].$$

Therefore,

$$\mathcal{N}[z-\tilde{z};\mathcal{C}_1^\kappa([T_0,T_0+\eta])] \leq c_{\psi,x} \eta^{\lambda-\kappa} \{1+2N_1\} \mathcal{N}[y-\tilde{y};\mathcal{C}_1^\kappa].$$

The end of the proof follows then exactly the same line as the proof of Proposition 3.8. □

Remark 4.9. Another natural approach to this singular Young case would have consisted in regularizing the kernel $K_{ts} = |t-s|^{-\alpha}$ into $K_{ts}^\varepsilon = |t-s+\varepsilon|^{-\alpha}$ and solving the associated Volterra system

$$y_t^\varepsilon = a + \int_0^t K_{tu}^\varepsilon \, \psi(y_u^\varepsilon)\, dx_u \tag{46}$$

in $\mathcal{C}_1^\gamma([0,T])$ by means of Theorem 3.4. The convergence of the solution y^ε in $\mathcal{C}_1^\kappa([0,T_0])$ can then be established thanks to the arguments of Proposition 4.5.

Indeed, following the proof of (25) (which involves the study of Riemann sums), it is not hard to check that

$$\mathcal{N}[y^{\varepsilon};\mathcal{C}_1^{\kappa_0}([0,T_0])] \le c_{\psi,x}|T_0|^{\gamma-\alpha-\kappa_0}\{1+\mathcal{N}[y^{\varepsilon};\mathcal{C}_1^{\kappa_0}([0,T_0])]\}$$

uniformly in $\varepsilon \in (0,1]$, provided $\gamma-\alpha-\kappa_0 > 0$. In particular, if T_0 is small enough, the sequence (y^{ε}) is bounded in $\mathcal{C}_1^{\kappa_0}([0,T_0])$, hence it converges (at least along a subsequence) to an element $y \in \mathcal{C}_1^{\kappa}([0,T_0])$, for any $\kappa < \kappa_0$.

To see that y actually satisfies our problem on $[0,T_0]$, it only remains to justify the passage to the limit in (46). This can be done using the arguments of Lemma 4.3, under the additional condition $(\gamma-\alpha)+\kappa > 1$, which ensures that the integral $\int_s^t K_{tu}\psi(y_u)\,dx_u$ is well-defined.

However, this regularization procedure only provides us with a local and (at this point) not necessarily unique solution y to (21). The uniqueness and extension of y then require a specific treatment: even with a compactness argument, the proof should follow the steps of Theorem 4.6, which means that we cannot avoid some lengthy estimations of Riemann sums.

Remark 4.10. As we have followed the same steps as in the proof of Theorem 3.4, it is quite obvious that the regularity result for the Itô map contained in Proposition 3.10 also holds true for this singular case. We do not repeat it though, for the sake of conciseness.

5. The Rough Case

In this section, we go back to Eq. (1), with a smooth and bounded coefficient σ. However, we will only assume that x belongs to $\mathcal{C}_1^{\gamma}([0,T];\mathbb{R}^n)$ for some $\gamma \in (1/3,1/2)$, which means in particular that we can no longer resort to Young's interpretation for $\int_0^t \sigma(t,u,y_u)\,dx_u$ and some rough path type considerations must come into the picture. We will thus briefly review the setting used in this context, and then prove a local existence and uniqueness result for our equation.

5.1. *Controlled processes*

For the sake of conciseness, we only recall here the key ingredients of the formalism introduced in [11] in order to handle integrals driven by an irregular signal x. First, as usual in the rough path theory, we will have to assume *a priori* the following hypothesis:

Hypothesis 1. The path x admits a Levy area, that is a process $x^2 \in \mathcal{C}_2^{2\gamma}([0,T];\mathbb{R}^{n,n})$ such that

$$\delta x^2 = \delta x \otimes \delta x, \quad \text{i.e. } (\delta x^2)_{sut}(i,j) = (\delta x^i)_{su} \otimes (\delta x^j)_{ut},$$

for all $s,u,t \in [0,T]$ and $i,j \in \{1,\ldots,n\}$.

As explained in [11], we are then incited to introduce a particular subspace of the space of Hölder continuous functions $\mathcal{C}_1^\gamma([0,T];\mathbb{R}^{1,k})$, which are the convenient processes to be integrated with respect to x:

Definition 5.1. Let $k \in \mathbb{N}^*$ and $\eta > \gamma$. A process $y \in \mathcal{C}_1^\gamma([0,T];\mathbb{R}^{1,k})$ is said to be (γ,η)-*controlled* by x if there exists $y' \in \mathcal{C}_1^{\eta-\gamma}([0,T];\mathcal{L}(\mathbb{R}^n,\mathbb{R}^{1,k}))$, $r^y \in \mathcal{C}_2^\eta([0,T];\mathbb{R}^{1,k})$ such that

$$(\delta y)_{st} = y'_s(\delta x)_{st} + r^y_{st}, \quad \text{for any } s,t \in [0,T]. \tag{47}$$

Remark 5.2. The decomposition (47) is not necessarily unique, but if we fix y, y', then, of course, the remainder r^y is uniquely determined. For this reason, define $\mathcal{Q}^{\gamma,\eta}([0,T];\mathbb{R}^{1,k})$ as the space of couples $(y,y') \in \mathcal{C}_1^\gamma([0,;\mathbb{R}^{1,k}) \times \mathcal{C}_1^{\eta-\gamma}([0,T];\mathcal{L}(\mathbb{R}^n,\mathbb{R}^{1,k}))$ such that the decomposition (47) holds. In the sequel, however, and for the sake of conciseness, we shall mostly write y instead of (y,y'). The space $\mathcal{Q}^{\gamma,\eta}([0,T];\mathbb{R}^{1,k})$ is endowed with the natural semi-norm

$$\begin{aligned}\mathcal{N}[y;\mathcal{Q}^{\gamma,\eta}([0,T];\mathbb{R}^{1,k})] &= \mathcal{N}[(y,y');\mathcal{Q}^{\gamma,\eta}([0,T];\mathbb{R}^{1,k})]\\ &:= \mathcal{N}[y;\mathcal{C}_1^\gamma([0,T];\mathbb{R}^{1,k})] + \mathcal{N}[y';\mathcal{C}_1^0([0,T];\mathcal{L}(\mathbb{R}^n,\mathbb{R}^{1,k})]\\ &\quad + \mathcal{N}[y';\mathcal{C}_1^{\gamma-\eta}([0,T];\mathcal{L}(\mathbb{R}^n,\mathbb{R}^{1,k})]\\ &\quad + \mathcal{N}[r^y;\mathcal{C}_2^\eta([0,T];\mathbb{R}^{1,k})].\end{aligned}$$

Observe that if $(y,y') \in \mathcal{Q}^{\gamma,\eta}([0,T];\mathbb{R}^{1,k})$, then

$$\mathcal{N}[y;\mathcal{C}_1^\gamma([0,T];\mathbb{R}^{1,d})] \le c_x\{\|y'_0\| + T^{\eta-\gamma}\mathcal{N}[y;\mathcal{Q}^{\gamma,\eta}([0,T];\mathbb{R}^{1,d})]\}. \tag{48}$$

Finally, let us denote $\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,k}) = \mathcal{Q}^{\gamma,2\gamma}([0,T];\mathbb{R}^{1,k})$.

With our main equation (13) in mind, it is important for us to get a stability property for controlled processes, when composed with the map σ. This is the object of the following proposition (for which we recall the notation on gradient of functions given at the end of the introduction).

Proposition 5.3. *Let $(y,y') \in \mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})$, with decomposition $\delta y = y'(\delta x) + r^y$, and consider $\sigma \in \mathcal{C}^{2,\mathbf{b}}([0,T]^2 \times \mathbb{R}^{1,d};\mathbb{R}^{d,n})$. For $i = 1,\dots,d$, denote by $\sigma_i(z)$ the ith line of $\sigma(z)$ when considered as a matrix. Then, for any $t \ge 0$, $(\sigma_i(t,.,y_.), D_3\sigma_i(t,.,y_.) \circ y') \in \mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,n})$ and*

$$\mathcal{N}[\sigma_i(t,.,y_.);\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,n})] \le c_\sigma\{1 + \mathcal{N}[y;\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})]^2\}, \tag{49}$$

where c_σ does not depend on t.

Proof. See the Appendix. □

Let us now turn to the integration of weakly controlled paths, which is summarized in the following proposition, borrowed from [11]. This result requires a little additional notation: if $\varphi \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{1,n})$ and $A \in \mathbb{R}^{n,n}$, we denote $\varphi \cdot A = \sum_{i,j=1}^n \langle \varphi e_i, e_j^* \rangle A_{ij}$.

Proposition 5.4. *Let x be a signal satisfying Hypothesis* 1, *and let also (z, z') be an element of $\mathcal{Q}^\gamma([0,T]; \mathbb{R}^{1,n})$ with decomposition $\delta z = z'(\delta x) + r^z$. One can define $A \in \mathcal{C}_1^\gamma([0,T]; \mathbb{R})$ by $A_0 = a \in \mathbb{R}$ and*

$$(\delta A)_{st} = z_s (\delta x)_{st} + z'_s \cdot x^2_{st} + \Lambda_{st}(r^z \delta x + \delta z' \cdot x^2),$$

and set $\mathcal{J}(z\, dx) = \mathcal{J}((z, z')\, dx) = \delta A$. Then $\mathcal{J}(z\, dx)$ coincides with the usual Riemann integral of z with respect to x in case of smooth functions. Moreover, it holds

$$\mathcal{J}(z\, dx) = \lim_{|\Pi_{st}| \to 0} \sum_i \{ z_{t_i} (\delta x)_{t_i t_{i+1}} + z'_{t_i} \cdot x^2_{t_i t_{i+1}} \},$$

for any $0 \le s < t \le T$, where the limit is taken over all the partitions $\Pi_{st} = \{s = t_0 < t_1 < \cdots < t_n = t\}$ of $[s,t]$, as the mesh of the partition goes to zero.

It only remains to enunciate the multidimensional version of the previous proposition:

Definition 5.5. Assume that $z \in \mathcal{C}_1^\gamma([0,T]; \mathbb{R}^{d,n})$ is such that for each z_i (ith line of z), there exists $z'_i \in \mathcal{C}_1^\gamma([0,T]; \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{1,n}))$ for which $(z_i, z'_i) \in \mathcal{Q}^\gamma([0,T]; \mathbb{R}^{1,n})$. Then we define $\mathcal{J}(z\, dx) = \mathcal{J}((z, z')\, dx) \in \mathcal{C}_1^\gamma([0,T]; \mathbb{R}^{1,d})$ by the natural relations

$$\mathcal{J}(z\, dx)^{(i)} = \mathcal{J}((z_i, z'_i)\, dx), \quad i = 1, \ldots, d.$$

5.2. *Rough Volterra equations*

Let us say a few words about the strategy to be used in order to solve Eq. (13) in case of a rough driving signal. First, this Volterra system will be interpreted according to Propositions 5.3 and 5.4 when (y, y') belongs to $\mathcal{Q}^\gamma([0,T]; \mathbb{R}^{1,d})$ and $\sigma \in \mathcal{C}^{2,\mathbf{b}}([0,T]^2 \times \mathbb{R}^{1,d}; \mathbb{R}^{d,n})$. Moreover, in order to settle a fixed point argument, we shall see that the process z defined by $z_0 = a$ and

$$(\delta z)_{st} = \mathcal{J}_{st}(\sigma(t, ., y_.)\, dx) + \mathcal{J}_{0s}([\sigma^t - \sigma^s](\mathcal{Y})\, dx)$$

is a controlled process (recall that $\mathcal{Y}$ stands for the multidimensional function $s \mapsto (s, y_s)$). Indeed, if we assume that the path $w_i = \sigma_i^t(\mathcal{Y})$ can be decomposed as

$$\delta w_i = \delta \sigma_i^t(\mathcal{Y}) = \sigma_i^t(\mathcal{Y})'(\delta x) + r^{\sigma_i^t(\mathcal{Y})},$$

which can be done owing to Proposition 5.3, and if we set $\delta z^{(i)} = \mathcal{J}(w_i\, dx)$, then one can write $(\delta z)^{(i)}_{st} = \sigma_i(s, s, y_s)(\delta x)_{st} + (r^z_{st})^{(i)}$ for $i = 1, \ldots, d$, with

$$\begin{aligned}(r^z_{st})^{(i)} &= [\sigma_i(t, s, y_s) - \sigma_i(s, s, y_s)](\delta x)_{st} + \sigma_i^t(\mathcal{Y})'_s \cdot x^2_{st} \\ &\quad + \Lambda_{st}(r^{\sigma_i^t(\mathcal{Y})} \delta x + \delta(\sigma_i^t(\mathcal{Y})') \cdot x^2) \\ &\quad + \mathcal{J}_{0s}([\sigma(t, ., y_.) - \sigma(s, ., y_.)]\, dx)^{(i)}.\end{aligned}$$

If we manage to show that $\sigma(., ., y_.)^* : x \mapsto (\sigma_1(., ., y_.)(x), \ldots, (\sigma_d(., ., y_.)(x))$ belongs to $\mathcal{C}_1^\gamma([0,T]; \mathcal{L}(\mathbb{R}^n, \mathbb{R}^{1,d}))$ and $r^z \in \mathcal{C}_2^{2\gamma}([0,T]; \mathbb{R}^{1,d})$ (which will be done in the course of the following proof), then $(z, \sigma(.,.,y_.)^*) \in \mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})$ and the application Γ introduced in the Young setting becomes here

$$\Gamma : \mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d}) \to \mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d}),\ (y,y') \mapsto (z, \sigma(.,.,y_.)^*). \tag{50}$$

With this notation, a solution of (13) corresponds to a fixed point of Γ.

We have now all the tools in hand to express the announced (local) result properly:

Theorem 5.6. *Let $\kappa \in (0,1)$ such that $\gamma(\kappa+2) > 1$, $\sigma \in \mathcal{C}^{3,\mathbf{b},\kappa}([0,T]^2 \times \mathbb{R}^d;\mathbb{R}^{d,n})$ and $a \in \mathbb{R}^{1,d}$. Then there exists $T_0 \in (0,T]$ such that the equation*

$$y_t = a + \mathcal{J}_{0t}(\sigma(t,.,y_.)\,dx),$$

interpreted in the sense of Definition 5.5, *admits a unique solution in $\mathcal{Q}^\gamma([0,T_0];\mathbb{R}^{1,d})$.*

As in the Young case, the result will stem from a contraction argument (Proposition 5.9) on some invariant ball (Proposition 5.8). Before we give details of these arguments, let us state an equivalent of Lemma 3.7:

Lemma 5.7. *Let $(y,y'),(\tilde{y},\tilde{y}') \in \mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})$ such that $y_0 = \tilde{y}_0$ and $y'_0 = \tilde{y}'_0$. Then, under the hypothesis of Theorem* 5.6, *for any $i \in \{1,\dots,d\}$ and any $s,t \in [0,T]$,*

$$\mathcal{N}[[\sigma_i^t - \sigma_i^s](\mathcal{Y});\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,n})] \le c_\sigma\,|t-s|\,\{1 + \mathcal{N}[y;\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})]^2\}, \tag{51}$$

the path $\sigma^t(\mathcal{Y}) - \sigma^t(\tilde{\mathcal{Y}})$ satisfies

$$\begin{aligned}\mathcal{N}[\sigma_i^t(\mathcal{Y}) - \sigma_i^t(\tilde{\mathcal{Y}});\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})]&\\ \le c_\sigma\{1 + \mathcal{N}[y;\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})]^2 + \mathcal{N}[\tilde{y};\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})]^2\}&\\ \times \mathcal{N}[y-\tilde{y};\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})]&\end{aligned} \tag{52}$$

and

$$\begin{aligned}\mathcal{N}[[\sigma_i^t - \sigma_i^s](\mathcal{Y}) - [\sigma_i^t - \sigma_i^s](\tilde{\mathcal{Y}});\mathcal{Q}^{\gamma,\gamma+\gamma\kappa}([0,T];\mathbb{R}^{1,d})]&\\ \le c_\sigma\,|t-s|\,\{1 + \mathcal{N}[y;\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})]^{1+\kappa} + \mathcal{N}[\tilde{y};\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})]^{1+\kappa}\}&\\ \times \mathcal{N}[y-\tilde{y};\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})].&\end{aligned} \tag{53}$$

Proof. See the Appendix. □

We can now state the result concerning the invariance of a ball for the map Γ:

Proposition 5.8. (Invariance of a ball) *Under the hypothesis of Theorem* 5.6, *there exists $T_0 \in (0,T]$ such that for each $T_1 \in (0,T_0]$, the ball*

$$\begin{aligned}B_{T_1}^{A_{T_1}} = \{(y,y') \in \mathcal{Q}^\gamma([0,T_1]) :\ y_0 = a,\ y'_0 = \sigma(0,0,a)^*,&\\ \mathcal{N}[(y,y');\mathcal{Q}^\gamma([0,T_1])] \le A_{T_1}\}&\end{aligned}$$

is invariant by Γ (defined by (50)) for some large enough radius A_{T_1}.

Proof. Fix a time $T_0 \le T$ and let $(y, y') \in B_{T_0}^{A_{T_0}}$ with decomposition $\delta y = y'\delta x + r^y$. Set $(z, z') = \Gamma(y, y')$. Then $\delta z = z'\delta x + r^z$, where r^z can be further decomposed into:

$$r^z = r^{z,0} + r^{z,1,1} + r^{z,1,2} + r^{z,2,1} + r^{z,2,2}, \tag{54}$$

with

$$r_{st}^{z,0,(i)} = [\sigma_i^t - \sigma_i^s](\mathcal{Y}_s)(\delta x)_{st}, \quad r_{st}^{z,1,1,(i)} = \sigma_i^t(\mathcal{Y})_s' \cdot x_{st}^2$$
$$r_{st}^{z,1,2,(i)} = \Lambda_{st}(r^{\sigma_i^t(\mathcal{Y})}\delta x + \delta(\sigma_i^t(\mathcal{Y})') \cdot x^2),$$

and

$$r_{st}^{z,2,1,(i)} = [\sigma_i^t - \sigma_i^s](\mathcal{Y}_0)(\delta x)_{0s} + [\sigma_i^t - \sigma_i^s](\mathcal{Y})_0' \cdot x_{0s}^2,$$
$$r_{st}^{z,2,2,(i)} = \Lambda_{0s}([r^{\sigma_i^t(\mathcal{Y})} - r^{\sigma_i^s(\mathcal{Y})}]\delta x + \delta([\sigma_i^t - \sigma_i^s](\mathcal{Y})') \cdot x^2).$$

Let us check that this decomposition actually identifies z as an element of $\mathcal{Q}^\gamma$, that is $z' \in \mathcal{C}_1^\gamma$ and $r^z \in \mathcal{C}_2^{2\gamma}$. For z', pick $0 \le s < t \le T_1$ and observe that

$$\begin{aligned}
\|(\delta z')_{st}\| &= \|\sigma(t,t,y_t)^* - \sigma(s,s,y_s)^*\| \\
&\le \|\sigma(t,t,y_t)^* - \sigma(s,t,y_t)^*\| + \|\sigma(s,t,y_t)^* - \sigma(s,s,y_s)^*\| \\
&\le \|D\sigma\|_\infty |t-s| + \sum_{i=1}^d \|\delta(\sigma_i^s(\mathcal{Y}))_{st}\|.
\end{aligned}$$

But, according to (48),

$$\begin{aligned}
\|\delta(\sigma_i^s(\mathcal{Y}))_{st}\| &\le c_x |t-s|^\gamma \{\|D_3\sigma_i(s,\mathcal{Y}_0) \circ y_0'\| + T_0^\gamma \mathcal{N}[\sigma_i^s(\mathcal{Y}); \mathcal{Q}^\gamma]\} \\
&\le c_{x,\sigma} |t-s|^\gamma \{1 + T_0^\gamma \mathcal{N}[\sigma_i^s(\mathcal{Y}); \mathcal{Q}^\gamma]\},
\end{aligned}$$

which, together with (49), leads to $\mathcal{N}[z'; \mathcal{C}_1^\gamma] \le c_{x,\sigma}\{1 + T_0^\gamma \mathcal{N}[y; \mathcal{Q}^\gamma]^2\}$.

Let us now estimate the 2γ-Hölder norm of the remaining terms.

Case $r^{z,0}$. Clearly, $\mathcal{N}[r^{z,0}; \mathcal{C}_2^{2\gamma}] \le \|D\sigma\|_\infty \mathcal{N}[x; \mathcal{C}_1^\gamma] T_0^{1-\gamma} \le c_{\sigma,x}$.

Case $r^{z,1,1}$. Since $\|\sigma_i^t(\mathcal{Y})_0'\| = \|D_3\sigma_i(t,\mathcal{Y}_0) \circ y_0'\| \le c_\sigma$, one has, owing to (49),

$$\begin{aligned}
\|r_{st}^{z,1,1,(i)}\| &\le c_\sigma |t-s|^{2\gamma} \mathcal{N}[x^2; \mathcal{C}_2^{2\gamma}]\{1 + T_0^\gamma \mathcal{N}[\sigma_i^t(\mathcal{Y})'; \mathcal{C}_1^\gamma]\} \\
&\le c_{\sigma,x} |t-s|^{2\gamma} \{1 + T_0^\gamma \mathcal{N}[\sigma_i^t(\mathcal{Y}); \mathcal{Q}^\gamma]\} \\
&\le c_{\sigma,x} |t-s|^{2\gamma} \{1 + T_0^\gamma \mathcal{N}[y; \mathcal{Q}^\gamma]^2\}.
\end{aligned}$$

Case $r^{z,1,2}$. It is readily checked, invoking (6) and (49), that

$$\begin{aligned}
\|r_{st}^{z,1,2,(i)}\| &\le c\,|t-s|^{3\gamma} \{\mathcal{N}[r^{\sigma_i^t(\mathcal{Y})}; \mathcal{C}_2^{2\gamma}]\mathcal{N}[x; \mathcal{C}_1^\gamma] + \mathcal{N}[(\sigma_i^t(\mathcal{Y}))'; \mathcal{C}_1^\gamma]\mathcal{N}[x^2; \mathcal{C}_2^{2\gamma}]\} \\
&\le c_x |t-s|^{3\gamma} \mathcal{N}[\sigma_i(t,\mathcal{Y}); \mathcal{Q}^\gamma] \;\le\; c_{x,\sigma} |t-s|^{2\gamma} T_0^\gamma \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^2\}.
\end{aligned}$$

Case $r^{z,2,1}$. The following elementary estimates hold true.

$$\|r_{st}^{z,2,1,(i)}\| \leq \|D\sigma_i\|_\infty |t-s| T_0^\gamma \mathcal{N}[x;\mathcal{C}_1^\gamma] + \|D_3\sigma_i(t,\mathcal{Y}_0) - D_3\sigma_i(s,\mathcal{Y}_0)\| \|y_0'\| \mathcal{N}[x^2;\mathcal{C}_2^{2\gamma}] T_0^{2\gamma} \leq c_{x,\sigma} |t-s|^{2\gamma}.$$

Case $r^{z,2,2}$. Owing to (6) and (51), we have

$$\begin{aligned}\|r_{st}^{z,2,2,(i)}\| &\leq c\, T_0^{3\gamma}\{\mathcal{N}[r^{\sigma_i^t(\mathcal{Y})} - r^{\sigma_i^s(\mathcal{Y})};\mathcal{C}_2^{2\gamma}]\mathcal{N}[x;\mathcal{C}_1^\gamma] \\ &\quad + \mathcal{N}[([\sigma_i^t - \sigma_i^s](\mathcal{Y}))';\mathcal{C}_1^\gamma]\mathcal{N}[x^2;\mathcal{C}_2^{2\gamma}]\} \\ &\leq c_x\, T_0^{3\gamma}\mathcal{N}[[\sigma_i^t - \sigma_i^s](\mathcal{Y});\mathcal{Q}^\gamma] \\ &\leq c_{x,\sigma}\, T_0^{3\gamma}\, |t-s|\, \{1+\mathcal{N}[y;\mathcal{Q}^\gamma]^2\}.\end{aligned}$$

Finally, gathering all our estimates for the terms in (54), it is easily seen that $\mathcal{N}[r^z;\mathcal{C}_2^{2\gamma}] \leq c_{\sigma,x}\{1+T_0^\gamma\mathcal{N}[y;\mathcal{Q}^\gamma]^2\}$. Hence we have obtained that $r^z \in \mathcal{C}_2^{2\gamma}$ and $(z,z') \in \mathcal{Q}^\gamma$.

Notice that the above estimations also easily lead to $\mathcal{N}[z;\mathcal{Q}^\gamma] \leq c_{x,\sigma}\{1 + T_0^\gamma\mathcal{N}[y;\mathcal{Q}^\gamma]^2\}$. Choose now for T_0 the greatest time $\tau \in (0,T]$ such that the equation $c_{\sigma,x}1+\tau^\gamma A^2 = A$ admits a unique solution A_τ. Then T_0 satisfies the property announced in our proposition. □

We can now prove the contraction property allowing to establish the existence and uniqueness of a local solution to Eq. (13).

Proposition 5.9. (Contraction property) *Under the hypothesis of Theorem* 5.6, *there exists* $T_1 \in (0,T_0]$ *such that for each* $T_2 < T_1$, *the application* Γ *is a strict contraction on the* (*stable*) *ball* $B_{T_2}^{A_{T_2}}$.

Proof. Let (y,y') and $(\tilde{y},\tilde{y}')$ of two elements of $B_{T_1}^{A_{T_1}}$, and set $(z,z') = \Gamma(y,y')$, $(\tilde{z},\tilde{z}') = \Gamma(\tilde{y},\tilde{y}')$. Thus, $\delta(z-\tilde{z}) = (z'-\tilde{z}')\delta x + (r^z - r^{\tilde{z}})$, where $z' = \sigma(.,.,y_.)^*$, $\tilde{z}' = \sigma(.,.,\tilde{y}_.)^*$, and r^z is given by (54), with a similar expression for $r^{\tilde{z}}$. Let us now estimate each term of

$$\mathcal{N}[z-\tilde{z};\mathcal{Q}^\gamma] = \mathcal{N}[z'-\tilde{z}';\mathcal{C}_1^0] + \mathcal{N}[z'-\tilde{z}';\mathcal{C}_1^\gamma] + \mathcal{N}[r^z - r^{\tilde{z}};\mathcal{C}_2^{2\gamma}] + \mathcal{N}[z-\tilde{z};\mathcal{C}_1^\gamma].$$

Case $\mathcal{N}[z'-\tilde{z}';\mathcal{C}_1^0]$. If $s \in [0,T_1]$, $\|z_s' - \tilde{z}_s'\| = \|\sigma(s,s,y_s)^* - \sigma(s,s,\tilde{y}_s)^*\| \leq \|D\sigma\|_\infty \|y_s - \tilde{y}_s\|$. But $y_0 = \tilde{y}_0$, so that $\|y_s - \tilde{y}_s\| \leq T_1^\gamma \mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]$ and $\mathcal{N}[z'-\tilde{z}';\mathcal{C}_1^0] \leq c_\sigma T_1^\gamma \mathcal{N}[y-\tilde{y};\mathcal{Q}^\gamma]$.

Case $\mathcal{N}[z' - \tilde{z}'; \mathcal{C}_1^\gamma]$. Pick $0 \le s < t \le T_1$ and observe that

$$\|(z'_t - \tilde{z}'_t) - (z'_s - \tilde{z}'_s)\| = \|(\sigma(t, \mathcal{Y}_t)^* - \sigma(t, \tilde{\mathcal{Y}}_t)^* - \sigma(s, \mathcal{Y}_s)^* + \sigma(s, \tilde{\mathcal{Y}}_s)^*\|$$
$$\le \|[\sigma^t - \sigma^s](\mathcal{Y}_t) - [\sigma^t - \sigma^s](\mathcal{Y}_t)\| + \|\delta(\sigma^s(\mathcal{Y}) - \sigma^s(\tilde{\mathcal{Y}}))_{st}\|.$$

Then

$$\|[\sigma^t - \sigma^s](\mathcal{Y}_t) - [\sigma^t - \sigma^s](\mathcal{Y}_t)\| \le \|D(\sigma^t - \sigma^s)\|_\infty \|y_t - \tilde{y}_t\|$$
$$\le \|D^2\sigma\|_\infty \, |t-s| \, \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\gamma] T_1^\gamma$$
$$\le c_\sigma \, |t-s|^\gamma \, \mathcal{N}[y - \tilde{y}; \mathcal{Q}^\gamma] T_1,$$

while, according to (48) and (52),

$$\|\delta(\sigma_i^s(\mathcal{Y}) - \sigma_i^s(\tilde{\mathcal{Y}}))_{st}\| \le |t-s|^\gamma \, \mathcal{N}[\sigma_i^s(\mathcal{Y}) - \sigma_i^s(\tilde{\mathcal{Y}}); \mathcal{C}_1^\gamma]$$
$$\le c_x \, |t-s|^\gamma \, \{\|(\sigma_i^s(\mathcal{Y}) - \sigma_i^s(\tilde{\mathcal{Y}}))'_0\| + T_1^\gamma \mathcal{N}[\sigma_i^s(\mathcal{Y}) - \sigma_i^s(\tilde{\mathcal{Y}}); \mathcal{Q}^\gamma]\}$$
$$\le c_{x,\sigma} \, |t-s|^\gamma \, T_1^\gamma \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^2 + \mathcal{N}[\tilde{y}; \mathcal{Q}^\gamma]^2\} \mathcal{N}[y - \tilde{y}; \mathcal{Q}^\gamma]$$

since $(\sigma_i^s(\mathcal{Y}) - \sigma_i^s(\tilde{\mathcal{Y}}))'_0 = 0$. Hence, thanks to the fact that we are working on the invariant ball $B_{T_1}^{A_{T_1}}$, we get $\mathcal{N}[z' - \tilde{z}'; \mathcal{C}_1^\gamma] \le c_{x,\sigma}\{1 + A_{T_1}^2\} \mathcal{N}[y - \tilde{y}; \mathcal{Q}^\gamma] T_1^\gamma$.

Case $\mathcal{N}[r^z - r^{\tilde{z}}; \mathcal{C}_2^{2\gamma}]$. Since $(y_0, y'_0) = (\tilde{y}_0, \tilde{y}'_0)$, $r^{z-\tilde{z}} = r^z - r^{\tilde{z}}$ reduces to the sum of

$$r_{st}^{z-\tilde{z},0,(i)} = \{[\sigma_i^t - \sigma_i^s](\mathcal{Y}_s) - [\sigma_i^t - \sigma_i^s](\tilde{\mathcal{Y}}_s)\}(\delta x)_{st},$$
$$r_{st}^{z-\tilde{z},1,1,(i)} = [\sigma_i^t(\mathcal{Y})'_s - \sigma_i^t(\tilde{\mathcal{Y}})'_s] \cdot x_{st}^2,$$
$$r_{st}^{z-\tilde{z},1,2,(i)} = \Lambda_{st}([r^{\sigma_i^t(\mathcal{Y})} - r^{\sigma_i^t(\tilde{\mathcal{Y}})}]\delta x + \delta(\sigma_i^t(\mathcal{Y})' - \sigma_i^t(\tilde{\mathcal{Y}})') \cdot x^2),$$
$$r_{st}^{z-\tilde{z},2,(i)} = \Lambda_{0s}([r^{\sigma_i^t(\mathcal{Y})} - r^{\sigma_i^s(\mathcal{Y})} - r^{\sigma_i^t(\tilde{\mathcal{Y}})} + r^{\sigma_i^s(\tilde{\mathcal{Y}})}]\delta x$$
$$+ \delta([\sigma_i^t - \sigma_i^s](\mathcal{Y})' - [\sigma_i^t - \sigma_i^s](\tilde{\mathcal{Y}})' \cdot x^2).$$

We will now bound each of these terms.

Study of $r_{st}^{z-\tilde{z},0}$. One has

$$\|r_{st}^{z-\tilde{z},0,(i)}\| \le c_x \, |t-s|^\gamma \, \|D(\sigma_i^t - \sigma_i^s)\|_\infty \|\mathcal{Y}_s - \tilde{\mathcal{Y}}_s\|$$
$$\le c_x \, |t-s|^{1+\gamma} \, \|D^2\sigma_i\|_\infty \|y_s - \tilde{y}_s\|$$
$$\le c_{x,\sigma} \, |t-s|^{2\gamma} \, \mathcal{N}[y - \tilde{y}; \mathcal{C}_1^\gamma] T_1^{1-\gamma}$$
$$\le c_{x,\sigma} \, |t-s|^{2\gamma} \, \mathcal{N}[y - \tilde{y}; \mathcal{Q}^\gamma] T_1^{1-\gamma}.$$

Study of $r_{st}^{z-\tilde{z},1,1}$. Since $(\sigma^t(\mathcal{Y}) - \sigma^t(\tilde{\mathcal{Y}}))'_0 = 0$, we get, owing to (52),

$$\|r_{st}^{z-\tilde{z},1,1,(i)}\| \le c_x \, |t-s|^{2\gamma} \, \|(\sigma_i^t(\mathcal{Y}) - \sigma_i^t(\tilde{\mathcal{Y}}))'_s\|$$
$$\le c_x \, |t-s|^{2\gamma} \, \mathcal{N}[\sigma_i^t(\mathcal{Y}) - \sigma_i^t(\tilde{\mathcal{Y}}); \mathcal{Q}^\gamma] T_1^\gamma$$
$$\le c_x \, |t-s|^{2\gamma} \, \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^2 + \mathcal{N}[\tilde{y}; \mathcal{Q}^\gamma]^2\} \mathcal{N}[y - \tilde{y}; \mathcal{Q}^\gamma] T_1^\gamma.$$

Study of $r^{z-\tilde{z},1,2}$. By (6) and (52),

$$\|r_{st}^{z-\tilde{z},1,2,(i)}\| \leq c_x\,|t-s|^{3\gamma}\,\mathcal{N}[\sigma_i^t(\mathcal{Y})-\sigma_i^t(\tilde{\mathcal{Y}});\mathcal{Q}^\gamma]$$
$$\leq c_{\sigma,x}\,|t-s|^{2\gamma}\,\{1+\mathcal{N}[y;\mathcal{Q}^\gamma]^2+\mathcal{N}[\tilde{y};\mathcal{Q}^\gamma]^2\}\mathcal{N}[y-\tilde{y};\mathcal{Q}^\gamma]T_1^\gamma.$$

Study of $r^{z-\tilde{z},2}$. By (6) and (53),

$$\|r_{st}^{z-\tilde{z},2,(i)}\| \leq c_x T_1^{\gamma(\kappa+2)}\mathcal{N}[[\sigma_i^t-\sigma_i^s](\mathcal{Y})-[\sigma_i^t-\sigma_i^s](\tilde{\mathcal{Y}});\mathcal{Q}^{\gamma,\gamma(1+\kappa)}]$$
$$\leq c_{x,\sigma}T_1^{\gamma(\kappa+2)}\,|t-s|\,\{1+\mathcal{N}[y;\mathcal{Q}^\gamma]^{1+\kappa}+\mathcal{N}[\tilde{y};\mathcal{Q}^\gamma]^{1+\kappa}\}\mathcal{N}[y-\tilde{y};\mathcal{Q}^\gamma].$$

Finally, putting together all our estimates of the remainder terms, we end up with the relation $\mathcal{N}[r^z-r^{\tilde{z}};\mathcal{C}_2^{2\gamma}] \leq c_{x,\sigma}\{1+A_{T_1}^2\}\mathcal{N}[y-\tilde{y};\mathcal{Q}^\gamma]T_1^\gamma$, which together with the above estimation of $\mathcal{N}[z'-\tilde{z}';\mathcal{C}_1^\gamma]$, gives

$$\mathcal{N}[z-\tilde{z};\mathcal{Q}^\gamma] \leq c_{x,\sigma}\{1+A_{T_1}^2\}\mathcal{N}[y-\tilde{y};\mathcal{Q}^\gamma]T_1^\gamma.$$

The greatest time $T_1 \in (0,T_0]$ such that $c_{x,\sigma}\{1+A_{T_1}^2\}T_1^\gamma \leq 1/2$ then clearly yields the contraction property for Γ on $[0,T_1]$. □

In the rough case, it is also easily seen that our existence and uniqueness result for Eq. (13) can be applied to the fractional Brownian motion:

Corollary 5.10. *Let B be an n-dimensional fractional Brownian motion with Hurst parameter $1/3 < H \leq 1/2$, defined on a complete probability space $(\Omega,\mathcal{F},P)$. Then almost surely, B fulfills the hypotheses of Theorem 5.6.*

Proof. We only have to show that B satisfies Hypothesis 1. But this kind of result is easily deduced from the convergence results contained in [6]. □

5.3. *Extending the solution*

To finish with, let us briefly evoke the technical difficulties we encounter when trying to extend the solution on $[0,T]$ along the same lines as in the Young case. Denote $(y^{(1)},(y^{(1)})')$ the solution on $[0,T_0]$.

The first step would consist in finding some small $\varepsilon > 0$, independent of $(y^{(1)},(y^{(1)})')$, and some radius N_1 such that the ball

$$\{(y,y') \in \mathcal{Q}^\gamma([0,T_0+\varepsilon]) : (y,y')_{|[0,T_0]}$$
$$= (y^{(1)},(y^{(1)})'),\ \mathcal{N}[(y,y');\mathcal{Q}^\gamma([0,T_0+\varepsilon])] \leq N_1\}$$

is invariant by Γ. In fact, if we set $(z,z') = \Gamma(y,y')$ for (y,y') in this ball, then some standard estimations, similar to those appearing in the proofs above, show that

$$\mathcal{N}[(z,z');\mathcal{Q}^\gamma([0,T_0+\varepsilon])]$$
$$\leq c_1\mathcal{N}[y^{(1)};\mathcal{Q}^\gamma([0,T_0])] + c_2\{1+\varepsilon^\lambda\mathcal{N}[(y,y');\mathcal{Q}^\gamma([0,T_0+\varepsilon])]^2\}, \quad (55)$$

for some $\lambda > 0$ and some constants c_1, c_2 with $c_1 > 2$. It is then rather clear that, owing to the exponent 2 in the latter expression, the constant ε ensuring the stability of the ball has to depend on $\mathcal{N}[y^{(1)};\mathcal{Q}^\gamma([0,T_0])]$.

More specifically, imagine the reasoning of the proof of Proposition 3.8 remains true when starting with (55), which means that we can find some constant $\varepsilon > 0$ and some sequence of radii (N_i) such that

$$c_1 N_i + c_2\{1 + \varepsilon^\lambda N_{i+1}^2\} \leq N_{i+1}. \tag{56}$$

Then $N_{i+1} \geq c_1 N_i \geq 2N_i$ and the sequence (N_i) diverges to infinity. On the other hand, if relation (56) is meant to admit solutions, then the relation $1 - 4\varepsilon^\lambda c_2(c_1 N_i + c_2) \geq 0$ must be fulfilled, so that (N_i) is bounded, hence a contradiction.

At this point, it is interesting to notice that even if ε is allowed to vary and becomes a sequence ε_i such that $\sum_i \varepsilon_i = \infty$ (in order to be sure that $[0,T]$ is covered), then we get $\frac{N_1}{2} 2^i \leq N_i \leq \frac{c}{\varepsilon_{i+1}^\lambda}$, so that $\varepsilon_i \leq \frac{c}{(2^{1/\lambda})^i}$, which of course contradicts $\sum_i \varepsilon_i = \infty$.

This failure in our apprehension of (1) motivated the study of a particular case of Volterra equations (see our companion paper [7]) for which some modifications of the δ-formalism enable one to get rid (in some way) of the past-dependent term in (2).

Remark 5.11. In case the driving process x is a usual Brownian motion, by means of an identification of our generalized integral with Itô's stochastic integral (see [11]), one could certainly obtain a global solution, filling thus a gap between this paper and the existing literature. It would in particular be interesting to compare our results with the ones contained in [21] , especially for the less stringent regularity conditions imposed on σ in the latter reference (roughly speaking, σ is only assumed to be $\mathcal{C}^{1,\mathbf{b}}$ in the first variable, and Lipschitz in the others).

Appendix

We gather in this section some regularity results for the functions and controlled processes we handle in throughout the paper.

Proof of Lemma 3.7. To obtain (15), pick $u < v$ and observe that

$$\begin{aligned}\|[\sigma^t - \sigma^s](\mathcal{Y}_v) - [\sigma^t - \sigma^s](\mathcal{Y}_u)\| &\leq \|D(\sigma^t - \sigma^s)\|_\infty \|\mathcal{Y}_v - \mathcal{Y}_u\| \\ &\leq \|D^2\sigma\|_\infty \,|t-s|\, (|v-u| + \mathcal{N}[y;\mathcal{C}_1^\gamma]\, |v-u|^\gamma)\,,\end{aligned}$$

which gives the result.

In order to establish (16), let us introduce the operator R defined for any $\varphi \in \mathcal{C}^{1,b}(\mathbb{R}^{d+1})$, $\xi, \xi' \in \mathbb{R}^{d+1}$, by

$$R\varphi(\xi,\xi') = \int_0^1 D\varphi(\alpha\xi + (1-\alpha)\xi')\, d\alpha.$$

Then of course $\|R\varphi\|_\infty \le \|D\varphi\|_\infty$ and $\|R\varphi(\xi_1,\xi_1') - R\varphi(\xi_2,\xi_2')\| \le \|D^2\varphi\|_\infty(\|\xi_1 - \xi_2\| + \|\xi_1' - \xi_2'\|)$. With this notation, if $0 < u < v < T$,

$$\begin{aligned}
&\|[\sigma^t(\mathcal{Y}_v) - \sigma^t(\tilde{\mathcal{Y}}_v)] - [\sigma^t(\mathcal{Y}_u) - \sigma^t(\tilde{\mathcal{Y}}_u)]\| \\
&\quad = \|R\sigma^t(\mathcal{Y}_v,\tilde{\mathcal{Y}}_v)(\mathcal{Y}_v - \tilde{\mathcal{Y}}_v) - R\sigma^t(\mathcal{Y}_u,\tilde{\mathcal{Y}}_u)(\mathcal{Y}_u - \tilde{\mathcal{Y}}_u)\| \\
&\quad \le \|R\sigma^t(\mathcal{Y}_v,\tilde{\mathcal{Y}}_v)([\mathcal{Y}_v - \tilde{\mathcal{Y}}_v] - [\mathcal{Y}_u - \tilde{\mathcal{Y}}_u])\| \\
&\qquad + \|[R\sigma^t(\mathcal{Y}_v,\tilde{\mathcal{Y}}_v) - R\sigma^t(\mathcal{Y}_u,\tilde{\mathcal{Y}}_u)](\mathcal{Y}_u - \tilde{\mathcal{Y}}_u)\| \\
&\quad \le \|D\sigma^t\|_\infty \|[y_v - \tilde{y}_v] - [y_u - \tilde{y}_u]\| \\
&\qquad + \|D^2\sigma^t\|_\infty (2\,|v-u| + \|y_v - y_u\| + \|\tilde{y}_v - \tilde{y}_u\|)\|y_u - \tilde{y}_u\| \\
&\quad \le \mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]\,|v-u|^\gamma \,\{\|D\sigma\|_\infty + \|D^2\sigma\|_\infty(2T^{1-\gamma} + \mathcal{N}[y;\mathcal{C}_1^\gamma] + \mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma])T^\gamma\},
\end{aligned}$$

where, in the last inequality, we have used the fact that $y_u - \tilde{y}_u = [y_u - \tilde{y}_u] - [y_0 - \tilde{y}_0]$. Inequality (16) follows easily. Notice that those are the same arguments as in the proof of [11, Lemma 5].

To prove (17), let us introduce the operator L defined for any $\varphi \in \mathcal{C}^{2,\mathbf{b},\kappa}(\mathbb{R}^{d+2})$ and any $s,t \in \mathbb{R}$, $\xi,\xi' \in \mathbb{R}^{d+1}$, as

$$L\varphi(s,t,\xi,\xi') = \int_0^1\int_0^1 D^2\varphi(s+\mu(t-s),\xi+\lambda(\xi'-\xi))\,d\mu\,d\lambda.$$

Thus, $L\varphi(s,t,\xi,\xi')$ is a bilinear mapping on $\mathbb{R}\times(\mathbb{R}\times\mathbb{R}^d)$ such that $\|L\varphi\|_\infty \le \|D^2\varphi\|_\infty$ and $\|L\varphi(s,t,\xi_1,\xi_1') - L\varphi(s,t,\xi_2,\xi_2')\| \le \|D^2\varphi\|_\kappa(\|\xi_1-\xi_2\|^\kappa + \|\xi_1'-\xi_2'\|^\kappa)$.

With this notation, it is readily checked that

$$\sigma(t,\xi) - \sigma(s,\xi) - \sigma(t,\xi') + \sigma(s,\xi') = L\sigma(s,t,\xi,\xi')((t-s,0),(0,\xi-\xi'))$$

for any $s,t \in [0,T]$, $\xi,\xi' \in [0,T]\times\mathbb{R}^d$, so that

$$\begin{aligned}
&\|[\sigma^t - \sigma^s](\mathcal{Y}_u) - [\sigma^t - \sigma^s](\tilde{\mathcal{Y}}_u) - [\sigma^t - \sigma^s](\mathcal{Y}_v) + [\sigma^t - \sigma^s](\tilde{\mathcal{Y}}_v)\| \\
&\quad = \|L\sigma(s,t,\mathcal{Y}_u,\tilde{\mathcal{Y}}_u)((t-s,0),(0,\mathcal{Y}_u - \tilde{\mathcal{Y}}_u)) \\
&\qquad - L\sigma(s,t,\mathcal{Y}_v,\tilde{\mathcal{Y}}_v)((t-s,0),(0,\mathcal{Y}_v - \tilde{\mathcal{Y}}_v))\| \\
&\quad \le \|L\sigma(s,t,\mathcal{Y}_u,\tilde{\mathcal{Y}}_u)((t-s,0),(0,[\mathcal{Y}_u - \tilde{\mathcal{Y}}_u] - [\mathcal{Y}_v - \tilde{\mathcal{Y}}_v]))\| \\
&\qquad + \|[L\sigma(s,t,\mathcal{Y}_u,\tilde{\mathcal{Y}}_u) - L\sigma(s,t,\mathcal{Y}_v,\tilde{\mathcal{Y}}_v)]((t-s,0),(0,\mathcal{Y}_v - \tilde{\mathcal{Y}}_v))\| \\
&\quad \le \|D^2\sigma\|_\infty\,|t-s|\,\|[y_u - \tilde{y}_u] - [y_v - \tilde{y}_v]\| \\
&\qquad + \|D^2\sigma\|_\kappa\,(2\,|u-v|^\kappa + \|y_u - y_v\|^\kappa + \|\tilde{y}_u - \tilde{y}_v\|^\kappa)\,|t-s|\,\|y_v - \tilde{y}_v\| \\
&\quad \le c_\sigma\,|t-s|\,\{\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\kappa]\,|u-v|^\gamma \\
&\qquad + (2\,|u-v|^\kappa + |u-v|^{\kappa\gamma}\,\{\mathcal{N}[y;\mathcal{C}_1^\gamma]^\kappa + \mathcal{N}[\tilde{y};\mathcal{C}_1^\gamma]^\kappa\})\mathcal{N}[y-\tilde{y};\mathcal{C}_1^\gamma]\,T^\gamma\}, \qquad (57)
\end{aligned}$$

which leads to the result. ☐

Proof of Proposition 5.3. This is a matter of elementary differential calculus. For the sake of conciseness, denote $\sigma = \sigma_i$ and $\varphi_{uv}(r) = \mathcal{Y}_u + r(\mathcal{Y}_v - \mathcal{Y}_u)$. Then

$$\begin{aligned}
(\delta(\sigma^t(\mathcal{Y}))_{uv} &= \sigma^t(\mathcal{Y}_v) - \sigma^t(\mathcal{Y}_u) \\
&= \int_0^1 dr\, D_2\sigma(t, \varphi_{uv}(r))(v-u) + \int_0^1 dr\, D_3\sigma(t, \varphi_{uv}(r))(\delta y)_{uv} \\
&= D_3\sigma(t, \mathcal{Y}_u)(\delta y)_{uv} + \int_0^1 dr\, [D_3\sigma(t, \varphi_{uv}(r)) - D_3\sigma(t, \mathcal{Y}_u)](\delta y)_{uv} \\
&\quad + \int_0^1 dr\, D_2\sigma(t, \varphi_{uv}(r))(v-u) \\
&:= (D_3\sigma(t, \mathcal{Y}_u) \circ y'_u)(\delta x)_{uv} + r_{uv}, \qquad (58)
\end{aligned}$$

where r has to be interpreted as a remainder, whose exact expression is given by:

$$\begin{aligned}
r_{uv} = D_3\sigma(t, \mathcal{Y}_u) r^y_{uv} &+ \int_0^1 dr\, [D_3\sigma(t, \varphi_{uv}(r)) - D_3\sigma(t, \mathcal{Y}_u)](\delta y)_{uv} \\
&+ \int_0^1 dr\, D_2\sigma(t, \varphi_{uv}(r))(v-u).
\end{aligned}$$

We will now bound the two terms in expression (58).

First, $\|D_3\sigma(t, \mathcal{Y}) \circ y'\|_\infty \le \|D_3\sigma\|_\infty \mathcal{N}[y'; \mathcal{C}_1^0] \le c_\sigma \mathcal{N}[y; \mathcal{Q}^\gamma]$, and if $0 \le u < v \le T$,

$$\begin{aligned}
&\|D_3\sigma(t, \mathcal{Y}_v) \circ y'_v - D_3\sigma(t, \mathcal{Y}_u) \circ y'_u\| \\
&\quad \le \|[D_3\sigma(t, \mathcal{Y}_v) - D_3\sigma(t, \mathcal{Y}_u)] \circ y'_v\| + \|D_3\sigma(t, \mathcal{Y}_u) \circ [y'_v - y'_u]\| \\
&\quad \le \|D^2\sigma\|_\infty \|\mathcal{Y}_v - \mathcal{Y}_u\| \mathcal{N}[y'; \mathcal{C}_1^0] + \|D_3\sigma\|_\infty \mathcal{N}[y'; \mathcal{C}_1^\gamma]\, |v-u|^\gamma \\
&\quad \le \|D^2\sigma\|_\infty (|v-u| + \mathcal{N}[y; \mathcal{C}_1^\gamma]\, |v-u|^\gamma) \mathcal{N}[y'; \mathcal{C}_1^0] + \|D_3\sigma\|_\infty \mathcal{N}[y'; \mathcal{C}_1^\gamma]\, |v-u|^\gamma \\
&\quad \le c_\sigma\, |v-u|^\gamma\, \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^2\},
\end{aligned}$$

hence $D_3\sigma(t, \mathcal{Y}) \circ y' \in \mathcal{C}_1^\gamma$ and $\mathcal{N}[D_3\sigma(t, \mathcal{Y}) \circ y'; \mathcal{C}_1^\gamma] \le c_\sigma \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^2\}$.

As for r, if $0 \le u < v \le T$,

$$\begin{aligned}
\|r_{uv}\| &\le \|D_3\sigma\|_\infty \mathcal{N}[r^y; \mathcal{C}_2^{2\gamma}]\, |v-u|^{2\gamma} + \|D^2\sigma\|_\infty \|\mathcal{Y}_v - \mathcal{Y}_u\| \mathcal{N}[y; \mathcal{C}_1^\gamma]\, |v-u|^\gamma \\
&\quad + \|D_2\sigma\|_\infty\, |v-u| \\
&\le c_\sigma\, |v-u|^{2\gamma}\, \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^2\},
\end{aligned}$$

so that $r \in \mathcal{C}_2^{2\gamma}$ and $\mathcal{N}[r; \mathcal{C}_2^{2\gamma}] \le c_\sigma \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^2\}$.

To get (49), it only remains to note that $\mathcal{N}[\sigma^t(\mathcal{Y}); \mathcal{C}_1^\gamma] \le c_\sigma \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]\}$. □

Proof of Lemma 5.7. According to the proof of Proposition 5.3, if $D_1\sigma^t := D_2\sigma(t, ., .)$ and $D_2\sigma^t := D_3\sigma(t, ., .)$, one has $[\sigma_i^t - \sigma_i^s](\mathcal{Y})'_u = D_2(\sigma_i^t - \sigma_i^s)(\mathcal{Y}_u) \circ y'_u$

and

$$r_{uv}^{[\sigma_i^t-\sigma_i^s](\mathcal{Y})} = D_2[\sigma_i^t-\sigma_i^s](\mathcal{Y}_u)(r_{uv}^y)$$
$$+\int_0^1 dr\,[D_2(\sigma^t-\sigma^s)(\mathcal{Y}_u+r(\mathcal{Y}_v-\mathcal{Y}_u))-D_2(\sigma^t-\sigma^s)(\mathcal{Y}_u)](\delta y)_{uv}$$
$$+\int_0^1 dr\,D_1(\sigma^t-\sigma^s)(\mathcal{Y}_u)(v-u).$$

Recall that in order to bound $(\sigma_i^t-\sigma_i^s)(\mathcal{Y}_u)$ in $\mathcal{Q}^\gamma$, the main steps consist in estimating $\mathcal{N}[(\sigma_i^t-\sigma_i^s)(\mathcal{Y}_u)';\mathcal{C}_1^\gamma]$ and $\mathcal{N}[r;\mathcal{C}_2^{2\gamma}]$. However,

$$\begin{aligned}
&\|[\sigma_i^t-\sigma_i^s](\mathcal{Y})'_v-[\sigma_i^t-\sigma_i^s](\mathcal{Y})'_u]\| \\
&\quad\le \|[D_2(\sigma_i^t-\sigma_i^s)(\mathcal{Y}_v)-D_2(\sigma_i^t-\sigma_i^s)(\mathcal{Y}_u)]\circ y'_v\| \\
&\qquad+\|D_2(\sigma_i^t-\sigma_i^s)(\mathcal{Y}_u)\circ[y'_v-y'_u]\| \\
&\quad\le \|D^2(\sigma_i^t-\sigma_i^s)\|_\infty(|v-u|+\mathcal{N}[y;\mathcal{C}_1^\gamma]\,|v-u|^\gamma)\mathcal{N}[y';\mathcal{C}_1^0] \\
&\qquad+\|D_2(\sigma_i^t-\sigma_i^s)\|_\infty\mathcal{N}[y';\mathcal{C}_1^\gamma]\,|v-u|^\gamma \\
&\quad\le \|D^3\sigma_i\|\,|t-s|\,(|v-u|+|v-u|^\gamma\,\mathcal{N}[y;\mathcal{C}_1^\gamma])\mathcal{N}[y';\mathcal{C}_1^0] \\
&\qquad+\|D^2\sigma_i\|_\infty\,|t-s|\,\mathcal{N}[y';\mathcal{C}_1^\gamma]\,|v-u|^\gamma \\
&\quad\le c_\sigma\,|t-s|\,|v-u|^\gamma\,\{1+\mathcal{N}[y;\mathcal{Q}^\gamma]^2\},
\end{aligned}$$

and

$$\begin{aligned}
\|r_{uv}^{[\sigma_i^t-\sigma_i^s](\mathcal{Y})}\| &\le \|D_1(\sigma_i^t-\sigma_i^s)\|_\infty\,|v-u|+\|D_2(\sigma_i^t-\sigma_i^s)\|_\infty\,|v-u|^{2\gamma}\,\mathcal{N}[r^y;\mathcal{C}_2^{2\gamma}] \\
&\quad+\|D^2(\sigma_i^t-\sigma_i^s)\|_\infty(|v-u|+\mathcal{N}[y;\mathcal{C}_1^\gamma]\,|v-u|^\gamma)\mathcal{N}[y;\mathcal{C}_1^\gamma]\,|v-u|^\gamma \\
&\le c_\sigma\,|t-s|\,|v-u|^{2\gamma}\,\{1+\mathcal{N}[y;\mathcal{Q}^\gamma]^2\}.
\end{aligned}$$

The upper bound (51) is now easily obtained.

Inequality (52) is in fact a direct consequence of [11, Proposition 4]. Indeed, if $y\in\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d})$, then of course $\mathcal{Y}\in\mathcal{Q}^\gamma([0,T];\mathbb{R}^{1,d+1})$ with decomposition

$$(\delta\mathcal{Y})_{st}=(0,y'_s)(\delta x)_{st}+(t-s,r_{st}^y).$$

Then, according to the aforementioned proposition,

$$\mathcal{N}[\sigma^t(\mathcal{Y})-\sigma^t(\tilde{\mathcal{Y}});\mathcal{Q}^\gamma]\le c_{\sigma,x}\{1+\mathcal{N}[\mathcal{Y};\mathcal{Q}^\gamma]^2+\mathcal{N}[\tilde{\mathcal{Y}};\mathcal{Q}^\gamma]^2\}\mathcal{N}[\mathcal{Y}-\tilde{\mathcal{Y}};\mathcal{Q}^\gamma].$$

It is then readily checked that $\mathcal{N}[\mathcal{Y};\mathcal{Q}^\gamma]\le c\{1+\mathcal{N}[y;\mathcal{Q}^\gamma]\}$ and $\mathcal{N}[\mathcal{Y}-\tilde{\mathcal{Y}};\mathcal{Q}^\gamma]=\mathcal{N}[y-\tilde{y};\mathcal{Q}^\gamma]$.

Let us now prove inequality (53). To this end, denote $\zeta^{st}:=D_2(\sigma_i^t-\sigma_i^s)$ and use the fact that $[(\sigma_i^t-\sigma_i^s)(\mathcal{Y})]'=\zeta^{st}(\mathcal{Y})\circ\mathcal{Y}'$. This yields the decomposition

$$[(\sigma_i^t-\sigma_i^s)(\mathcal{Y})]'-[(\sigma_i^t-\sigma_i^s)(\tilde{\mathcal{Y}})]')_{uv}=A_{uv}^{st}+B_{uv}^{st}+C_{uv}^{st}+D_{uv}^{st},\text{ with}$$

$$A_{uv}^{st}=\delta(\zeta^{st}(\mathcal{Y}))_{uv}\circ[y_v'-\tilde{y}_v'],\quad B_{uv}^{st}=\zeta^{st}(\mathcal{Y}_u)\circ\delta([y'-\tilde{y}'])_{uv},$$

$$C_{uv}^{st}=[\zeta^{st}(\mathcal{Y}_v)-\zeta^{st}(\tilde{\mathcal{Y}}_v)]\circ(\delta\tilde{y}')_{uv},\quad D_{uv}^{st}=\delta([\zeta^{st}(\mathcal{Y})-\zeta^{st}(\tilde{\mathcal{Y}})])_{uv}\circ\tilde{y}_u'.$$

Owing to the regularity of σ, we are in position to apply Lemma 3.7 with $D_3\sigma_i$, which gives

$$\begin{aligned}\mathcal{N}[A^{st},\mathcal{C}_2^{\kappa\gamma}]&\le\mathcal{N}[D_2(\sigma_i^t-\sigma_i^s)(\mathcal{Y});\mathcal{C}_1^{\gamma}]T^{\gamma(1-\kappa)}\mathcal{N}[y-\tilde{y};\mathcal{Q}^{\gamma}]\\&\le c_\sigma\,|t-s|\,\{1+\mathcal{N}[y;\mathcal{Q}^{\gamma}]\}\mathcal{N}[y-\tilde{y};\mathcal{Q}^{\gamma}]\end{aligned}$$

and

$$\begin{aligned}\mathcal{N}[D^{st};\mathcal{C}_2^{\kappa\gamma}]&\le\mathcal{N}[D_2(\sigma_i^t-\sigma_i^s)(\mathcal{Y})-D_2(\sigma_i^t-\sigma_i^s)(\tilde{\mathcal{Y}});\mathcal{C}_1^{\kappa\gamma}]\mathcal{N}[\tilde{y};\mathcal{Q}^{\gamma}]\\&\le c_\sigma\,|t-s|\,\{1+\mathcal{N}[y;\mathcal{Q}^{\gamma}]^{\kappa}+\mathcal{N}[\tilde{y};\mathcal{Q}^{\gamma}]^{\kappa}\}\mathcal{N}[y-\tilde{y};\mathcal{Q}^{\gamma}]\mathcal{N}[\tilde{y};\mathcal{Q}^{\gamma}].\end{aligned}$$

Besides, it is easy to see that $\mathcal{N}[B^{st};\mathcal{C}_1^{\kappa\gamma}]\le c_\sigma\,|t-s|\,\mathcal{N}[y-\tilde{y};\mathcal{Q}^{\gamma}]$, while $\mathcal{N}[C^{st};\mathcal{C}_1^{\kappa\gamma}]\le c_\sigma\,|t-s|\,\mathcal{N}[\tilde{y};\mathcal{Q}^{\gamma}]\mathcal{N}[y-\tilde{y};\mathcal{Q}^{\gamma}]$, hence

$$\begin{aligned}&\mathcal{N}[([\sigma_i^t-\sigma_i^s](\mathcal{Y})-[\sigma_i^t-\sigma_i^s](\tilde{\mathcal{Y}}))';\mathcal{C}_1^{\kappa\gamma}]\\&\quad\le c_\sigma\,|t-s|\,\{1+\mathcal{N}[y;\mathcal{Q}^{\gamma}]^{1+\kappa}+\mathcal{N}[\tilde{y};\mathcal{Q}^{\gamma}]^{1+\kappa}\}\mathcal{N}[y-\tilde{y};\mathcal{Q}^{\gamma}].\end{aligned}\tag{59}$$

As for $r_{uv}^{st}:=r^{[\sigma_i^t-\sigma_i^s](\mathcal{Y})}-r^{[\sigma_i^t-\sigma_i^s](\tilde{\mathcal{Y}})}$, we know from (58) that, if $\varphi_{uv}(r)=\mathcal{Y}_u+r(\mathcal{Y}_v-\mathcal{Y}_u)$, $\tilde{\varphi}_{uv}(r):=\tilde{\mathcal{Y}}_u+r(\tilde{\mathcal{Y}}_v-\tilde{\mathcal{Y}}_u)$ and $\sigma_i^{st}:=\sigma_i^t-\sigma_i^s$, then $r_{uv}^{st}=r_{uv}^{st,1}+r_{uv}^{st,2}+r_{uv}^{st,3}$, with

$$\begin{aligned}r_{uv}^{st,1}&=\int_0^1 dr\,[D_1\sigma_i^{st}(\varphi_{uv}(r))-D_1\sigma_i^{st}(\tilde{\varphi}_{uv}(r))](v-u),\\r_{uv}^{st,2}&=D_2\sigma_i^{st}(\mathcal{Y}_u)(r_{uv}^{y})-D_2\sigma_i^{st}(\tilde{\mathcal{Y}}_u)(r_{uv}^{\tilde{y}}),\\r_{uv}^{st,3}&=\int_0^1 dr\,\{[D_2\sigma_i^{st}(\varphi_{uv}(r))-D_2\sigma_i^{st}(\mathcal{Y}_u)](\delta y)_{uv}\\&\qquad-[D_2\sigma_i^{st}(\tilde{\varphi}_{uv}(r))-D_2\sigma_i^{st}(\tilde{\mathcal{Y}}_u)](\delta\tilde{y})_{uv}\}.\end{aligned}$$

Obvious arguments allow to assert that $\mathcal{N}[r^{st,1};\mathcal{C}_2^{\gamma+\gamma\kappa}]\le c_\sigma\,|t-s|\,\mathcal{N}[y-\tilde{y};\mathcal{Q}^{\gamma}]$. To deal with $r^{st,2}$, write of course

$$r_{uv}^{st,2}=[D_2\sigma_i^{st}(\mathcal{Y}_u)-D_2\sigma_i^{st}(\tilde{\mathcal{Y}}_u)](r_{uv}^{y})+D_2\sigma_i^{st}(\tilde{\mathcal{Y}}_u)([r_{uv}^{y}-r_{uv}^{\tilde{y}}]),$$

which leads to $\mathcal{N}[r^{st,2};\mathcal{C}_2^{\gamma+\gamma\kappa}]\le c_\sigma\,|t-s|\,\{1+\mathcal{N}[y;\mathcal{Q}^{\gamma}]\}\mathcal{N}[y-\tilde{y};\mathcal{Q}^{\gamma}]$. Finally, decompose $r^{st,3}$ into $r^{st,3}=r^{st,3,1}+r^{st,3,2}$, with

$$r_{uv}^{st,3,1}=\int_0^1 dr\,[D_2\sigma_i^{st}(\varphi_{uv}(r))-D_2\sigma_i^{st})(\mathcal{Y}_u)]\delta(y-\tilde{y})_{uv},$$

$$r_{uv}^{st,3,2}=\int_0^1 dr\,[D_2\sigma_i^{st}(\varphi_{uv}(r))-D_2\sigma_i^{st}(\mathcal{Y}_u)-D_2\sigma_i^{st}(\tilde{\varphi}_{uv}(r))+D_2\sigma_i^{st}(\tilde{\mathcal{Y}}_u)](\delta\tilde{y})_{uv}.$$

Clearly, $\mathcal{N}[r^{st,3,1}; \mathcal{C}_2^{\gamma+\gamma\kappa}] \le c_\sigma |t-s| \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]\} \mathcal{N}[y - \tilde{y}; \mathcal{Q}^\gamma]$. To conclude, observe that the double increment appearing into brackets in $r_{uv}^{st,3,2}$ can be dealt with just as (57) (replace $[\sigma^t - \sigma^s]$ with $D_2[\sigma_i^t - \sigma_i^s]$ and $\mathcal{Y}_v$ with $\varphi_{uv}(r)$). This gives

$$\mathcal{N}[r^{st,3,2}; \mathcal{C}_2^{\gamma+\gamma\kappa}] \le c_\sigma |t-s| \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^\kappa + \mathcal{N}[\tilde{y}; \mathcal{Q}^\gamma]^\kappa\} \mathcal{N}[y - \tilde{y}; \mathcal{Q}^\gamma] \mathcal{N}[\tilde{y}; \mathcal{Q}^\gamma].$$

We have thus shown that

$$\mathcal{N}[r^{st}; \mathcal{C}_2^{\gamma+\gamma\kappa}] \le c_\sigma |t-s| \{1 + \mathcal{N}[y; \mathcal{Q}^\gamma]^{1+\kappa} + \mathcal{N}[\tilde{y}; \mathcal{Q}^\gamma]^{1+\kappa}\} \mathcal{N}[y - \tilde{y}; \mathcal{Q}^\gamma],$$

which, together with (59), entails (53). □

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