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Skorohod and Stratonovich integrals for controlled processes

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Abstract

Given a continuous Gaussian process x which gives rise to a p -geometric rough path for $p \in (2, 3)$, and a general continuous process y controlled by x , under proper conditions we establish the relationship between the Skorohod integral $\int_0^t y_s d^\diamond x_s$ and the Stratonovich integral $\int_0^t y_s d\mathbf{x}_s$. Our strategy is to employ the tools from rough paths theory and Malliavin calculus to analyze discrete sums of the integrals. © 2022 Elsevier B.V. All rights reserved.

1. Introduction

For sake of clarity, we will divide this introduction in 3 parts. In Section 1.1 we motivate our problem and recall some previous contributions giving Stratonovich–Skorohod corrections. Section 1.2 is devoted to a description of our main result, the strategy employed in the article, and some perspectives for future works. At the end, some notations used in this article are introduced in Section 1.3.

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1.1. Background

In recent decades, two approaches for the analysis of dynamical systems driven by Gaussian processes have been greatly developed: (i) the “probabilistic” approach, which invokes stochastic analysis tools and leads to Itô–Skorohod integration, and (ii) the “pathwise” approach which employs the theory of rough paths and gives rise to Stratonovich integration. In general, one gets a more transparent understanding of the system by using the pathwise approach, while it is more convenient to explore probabilistic properties (e.g. compute the moments for the solution of a noisy dynamical system driven by a Gaussian noise) via the probabilistic approach. One key ingredient to understand the connection between these two approaches is the relationship between Skorohod and Stratonovich integrals.

For a standard Brownian motion, the relationship between Itô and Stratonovich integrals is well-known. It is classically obtained by Itô calculus, although rough paths theory can also be invoked by observing that both Itô and Stratonovich integrals can be regarded as integrals against rough paths lifted from a Brownian motion with different second order terms. For general Gaussian processes (consider fractional Brownian motion as a typical example), however, it is non-trivial to obtain the relationship. Indeed, for a general Gaussian process Itô calculus (or martingale calculus) is not available, and moreover Skorohod integrals cannot be regarded as integrals against rough paths lifted from the corresponding Gaussian processes. We briefly recall some results giving Skorohod–Stratonovich corrections below.

Let $x = (x^1, \dots, x^d)$ be a d -dimensional centered Gaussian process with i.i.d components giving rise to a p -geometric rough path, where $2 < p < 3$ (see Section 2.1 for more details about geometric rough paths). Denote by R the covariance function of x , namely $R(s, t) = \mathbf{E}[x_t^1 x_s^1]$. We also set $R_t = R(t, t)$. The correction terms between Skorohod and Stratonovich integrals with respect to x have been considered in the following cases:

(i) In [15], the Skorohod–Stratonovich corrections were computed for integrals of the form $\int_s^t \nabla f(x_u) d^\circ x_u$ for a smooth function f defined on \mathbb{R}^d . More specifically,

$$\int_0^t \nabla f(x_r) dx_r = \int_0^t \nabla f(x_r) d^\circ x_r + \frac{1}{2} \int_0^t \Delta f(x_r) dR_r, \tag{1}$$

where the integral with respect to x on the left-hand side is a Stratonovich integral while the one on the right-hand side is a Skorohod integral. The strategy in [15] relied on the fact that $\int_0^t \nabla f(x_u) d^\circ x_u$ is obtained by taking limits of Riemann–Wick sums of the form:

$$S^{It, \diamond} = \sum_{i=0}^{n-1} \sum_{k=1}^N \frac{1}{k!} f^{(k)}(x_{t_i}) \diamond (x_{t_{i+1}}^1)^{\diamond k}, \tag{2}$$

where \diamond stands for the Wick product. Then the Skorohod–Stratonovich corrections in [15] were analyzed thanks to a computation of the Wick corrections in (2). Notice that an extension of this result to the case of a Gaussian process indexed by $[0, 1]^2$ with Hölder exponent greater than $1/3$ was handled in [22]. There some change of variable formulas were derived for the Stratonovich and Skorohod integrals respectively. As a consequence, the correction terms were computed explicitly. Also note that some preliminary cases for a 1-d fractional Brownian motion had been considered in [21].

(ii) Ref. [4] is concerned with solutions of rough differential equations driven by x , where x is again a d -dimensional centered Gaussian process with i.i.d components giving rise to a

p -geometric rough path (recall that $2 < p < 3$). The equation can be written as

$$dy_r = \sigma(y_r)dx_r, \tag{3}$$

with a smooth enough coefficient $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$, and we refer to [9] for more details about this object. Below we denote by y_0 the initial condition of (3), and J_u^x is designated as the Jacobian of the flow map $y_0 \rightarrow y_u$. Then the formula for the correction terms in [4] can be read as

$$\int_0^t y_r dx_r = \int_0^t y_r d^\circ x_r + \frac{1}{2} \int_0^t \text{tr}[\sigma(y_r)] dR_r + \int_{\{0 < r_1 < r_2 < t\}} \text{tr} \left[J_{r_2}^x (J_{r_1}^x)^{-1} \sigma(y_{r_1}) - \sigma(y_{r_2}) \right] dR(r_1, r_2). \tag{4}$$

Consider the i th column $\sigma_i(x)$ of the coefficient matrix $\sigma(x)$ as a vector field on \mathbb{R}^d for $1 \leq i \leq d$. If the Lie bracket $[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = 0$ for $1 \leq i \leq j \leq d$, then the solution y_t to (3) is of the form $y_t = \varphi(x_t, y_0)$ with $(\nabla_x \cdot \varphi)(x, y_0) = \text{tr}[\sigma(x, y_0)]$ and $J_t^x = \sigma(y_t)$ (see [1, Proposition 24]). Clearly in this case, (4) coincides with (1), noting that the last term on the right-hand side of (4) now vanishes. Therefore, relation (4) is indeed compatible with (1).

1.2. Main result and strategy

In this paper, we consider a d -dimensional centered Gaussian process x with i.i.d components. Let y be a *controlled* process relative to x . That is, the increments $\delta y_{st} := y_t - y_s$ can be decomposed along the increments of x as follows:

$$\delta y_{st}^i = \sum_{j=1}^d y_s^{x;ij} x_{st}^{1;j} + r_{st}^i, \text{ for } i = 1, \dots, d, \tag{5}$$

where y^x has finite p -variation and the remainder r has finite $\frac{p}{2}$ -variation (one can alternatively use Hölder spaces in this definition). Notice that controlled processes are the natural class of functions for which a proper rough integration with respect to x can be constructed (see e.g [13]). The following is the main result of this paper (see Theorem 3.1 for a more precise statement). Under proper conditions on x and y ,

$$\int_0^t y_r dx_r = \int_0^t y_r d^\circ x_r + \frac{1}{2} \sum_{i=1}^d \int_0^t y_r^{x;ii} dR_r + \sum_{i=1}^d \int_{\{0 < r_1 < r_2 < t\}} \left(\mathbf{D}_{r_1}^i y_{r_2}^i - y_{r_2}^{x;ii} \right) dR(r_1, r_2). \tag{6}$$

On the left-hand side of (6), the integral $\int_0^t y_r dx_r$ is understood in the rough path sense (see Proposition 2.20 for further details). On the right-hand side of the same equation, $\int_0^t y_r d^\circ x_r$ stands for the Skorohod integral, y^x is defined by (5), R is the covariance function alluded to above and \mathbf{D} represents the Malliavin derivative (notions of Malliavin calculus will be recalled in Section 2.4).

Note that our formula (6) unifies the previous cases (1) and (4). Indeed, we have argued that (4) can be seen as an extension of (1). Furthermore, note that the solution y to the rough differential equation (3) with a sufficiently regular coefficient function $\sigma(y)$ is a controlled process with $y_s^{x;ij} = (\sigma(y_s))_{ij}$ and $\mathbf{D}_s y_t = J_t^x (J_s^x)^{-1} \sigma(y_s)$ for $s \leq t$. Therefore it is easy to see that (6) is an extension of (4), which is the main result of [4]. This point will be detailed in Section 3.2.2

Inspired by [4,15], our proof of the main result is based on the discrete sums method combined with tools from rough paths theory and Malliavin calculus, in which some discrete techniques developed in [18] are also invoked. We outline the idea as follows.

Consider a controlled process y with a decomposition given by (5), satisfying some path regularity and Malliavin differentiability conditions. Let $\pi = \pi^n$ denote the uniform partition of $[0, T]$ and \mathcal{H} be the Hilbert space associated to x . Denote

$$y^\pi(t) = \sum_{k=0}^{n-1} y_{t_k} \mathbf{1}_{[t_k, t_{k+1})}(t).$$

We first prove that (see Lemma 3.2)

$$\lim_{n \rightarrow \infty} y^\pi = y \quad \text{in } \mathbb{D}^{1,2}(\mathcal{H}).$$

This enables us to show the convergence of the discrete Skorohod integral $\delta^\diamond(y^\pi)$ to the Skorohod integral $\delta^\diamond(y) = \int_0^T y_r \, d^\diamond x_r$ in $L^2(\Omega)$, i.e.,

$$\int_s^t y_r \, d^\diamond x_r = \lim_{n \rightarrow \infty} \sum_{m=0}^{n-1} \left[\sum_{i=1}^d y_{t_m}^i \diamond \mathbf{x}_{t_m t_{m+1}}^{1:i} \right] \text{ in } L^2(\Omega), \tag{7}$$

where we have written $\pi = \{t_0, \dots, t_n\}$ with $t_m = s + m(t - s)/n$ for $m = 0, \dots, n$. It is also known from the rough paths theory that the following holds true almost surely:

$$\int_s^t y_r \, dx_r = \lim_{|\pi| \rightarrow 0} \sum_{i=1}^d \sum_{k=0}^{n-1} \left(y_{t_k}^i \mathbf{x}_{t_k t_{k+1}}^{1:i} + \frac{1}{2} \sum_{j=1}^d y_{t_k}^{x;ij} \mathbf{x}_{t_k t_{k+1}}^{2;ji} \right), \tag{8}$$

where the left-hand side above stands for the rough paths integral of y with respect to x .

The Stratonovich–Skorohod correction terms in (6) now can be obtained by computing the difference between the right-hand sides in (7) and (8). When computing the difference, one key ingredient will be the forthcoming Proposition 2.28. This proposition is inspired by the analogous results in [18] and establishes a general estimate for weighted sums in the second chaos of the Gaussian process x .

To end this subsection, we provide some perspectives for future works. On the one hand, as an application, some central limit theorems for Skorohod integrals could be obtained with the help of our main result, generalizing the results in [18,17]. On the other hand, noting that in this article the Gaussian rough paths with finite p -variation for $p \in (1, 3)$ are handled, we believe that our methodology can be carried out for rougher Gaussian paths with $p \geq 3$. It is also interesting to consider the correction terms for the processes arising from delay equations [19], Volterra equations [6,7,14], etc. We will go back to those extensions in Section 3.2.3.

1.3. Notation

Let $\pi : 0 = t_0 < t_1 < \dots < t_n = T$ be a partition on $[0, T]$. Take $s, t \in [0, T]$. We write $\llbracket s, t \rrbracket$ for the discrete interval that consists of t_k 's such that $t_k \in [s, t]$. We denote by $\mathcal{S}_k(\llbracket s, t \rrbracket)$ the simplex $\{(t_1, \dots, t_k) \in [s, t]^k; t_1 \leq \dots \leq t_k\}$. In contrast, whenever we deal with a discrete interval, we set $\mathcal{S}_k(\llbracket s, t \rrbracket) = \{(t_1, \dots, t_k) \in \llbracket s, t \rrbracket^k; t_1 < \dots < t_k\}$. For $t = t_k$ we denote $t- := t_{k-1}$, $t+ := t_{k+1}$. We also denote by $\mathcal{D}(\llbracket s, t \rrbracket)$ the set of all dissections of $[s, t]$.

For $x = (x^1, \dots, x^n)$ and $y = (y^1, \dots, y^n)$ in \mathbb{R}^n , we write xy for their dot product $x \cdot y = \sum_{i=1}^n x^i y^i$ and write $|x|$ for the Euclid norm $(\sum_{i=1}^n x_i^2)^{1/2}$. The L^p -norm $(\mathbb{E}[|\xi|^p])^{1/p}$ of a random variable ξ is denoted by $\|\xi\|_p$, for $p \geq 1$.

Generally speaking, we will write C for a generic constant whose exact value can change from line to line.

2. Preliminary material

This section contains some basic tools from rough paths theory and Malliavin calculus, as well as some analytical results, which are crucial for the definition and integration of controlled processes.

2.1. Rough path above \mathbf{x}

In this subsection we shall recall the notion of a rough path above a signal x , and how this applies to Gaussian signals. The interested reader is referred to [8,9,13] for further details.

As mentioned in Section 1.3, for $s < t$ and $m \geq 1$, we consider the simplex $\mathcal{S}_m([s, t]) = \{(u_1, \dots, u_m) \in [s, t]^m; u_1 < \dots < u_m\}$. For notational sake, we just write \mathcal{S}_m for $\mathcal{S}_m([0, T])$. The definition of a rough path above a signal x relies on the following notion of increments.

Definition 2.1. Let $k \geq 1$. Then the space of $(k - 1)$ -increments, denoted by $\mathcal{C}_k([0, T], \mathbb{R}^d)$ or simply $\mathcal{C}_k(\mathbb{R}^d)$, is defined as

$$\mathcal{C}_k(\mathbb{R}^d) \equiv \left\{ g \in C(\mathcal{S}_k; \mathbb{R}^d); \lim_{t_i \rightarrow t_{i+1}} g_{t_1 \dots t_k} = 0, i \leq k - 1 \right\}.$$

We now introduce a finite difference operator called δ , which acts on increments and is useful to split iterated integrals into simpler pieces.

Definition 2.2. Let $g \in \mathcal{C}_1(\mathbb{R}^d)$, $h \in \mathcal{C}_2(\mathbb{R}^d)$. Then for $(s, u, t) \in \mathcal{S}_3$, we set

$$\delta g_{st} = g_t - g_s, \quad \text{and} \quad \delta h_{sut} = h_{st} - h_{su} - h_{ut}.$$

The regularity of increments in $\mathcal{C}_2(\mathbb{R}^d)$ will be measured in terms of p -variation as follows.

Definition 2.3. For $f \in \mathcal{C}_2(\mathbb{R}^d)$ and $p > 0$, we define

$$\|f\|_{p\text{-var}} = \|f\|_{p\text{-var};[0,T]} = \sup_{(t_i) \in \mathcal{D}([0,T])} \left(\sum_i |f_{t_i t_{i+1}}|^p \right)^{1/p}.$$

The set of increments in $\mathcal{C}_2(\mathbb{R}^d)$ with finite p -variation is denoted by $\mathcal{C}_2^{p\text{-var}}(\mathbb{R}^d)$.

Note that for a continuous function $g : [0, T] \rightarrow \mathbb{R}^d$ with finite p -variation, if we set $\|g\|_{p\text{-var};[0,T]} = \|\delta g\|_{p\text{-var};[0,T]}$, then we recover its usual p -variation.

With these preliminary definitions in hand, we can now introduce the notion of a rough path.

Definition 2.4. Let x be a continuous \mathbb{R}^d -valued path with finite p -variation for some $p \geq 1$. We say that x gives rise to a geometric p -rough path if there exists a family

$$\left\{ \mathbf{x}_{st}^{n;i_1, \dots, i_n}, (s, t) \in \mathcal{S}_2, n \leq \lfloor p \rfloor, i_1, \dots, i_n \in \{1, \dots, d\} \right\},$$

such that $\mathbf{x}_{st}^1 = \delta x_{st}$ and

(1) *Regularity:* For all $n \leq \lfloor p \rfloor$, each component of \mathbf{x}^n has finite $\frac{p}{n}$ -variation in the sense of Definition 2.3.

(2) *Multiplicativity*: With $\delta \mathbf{x}^n$ as in Definition 2.2, we have

$$\delta \mathbf{x}_{sut}^{n;i_1, \dots, i_n} = \sum_{n_1=1}^{n-1} \mathbf{x}_{su}^{n_1;i_1, \dots, i_{n_1}} \mathbf{x}_{ut}^{n-n_1;i_{n_1+1}, \dots, i_n}. \tag{9}$$

(3) *Geometricity*: Let x^ε be a sequence of piecewise smooth approximations of x . For any $n \leq [p]$ and any set of indices $i_1, \dots, i_n \in \{1, \dots, d\}$, we assume that $\mathbf{x}^{\varepsilon, n; i_1, \dots, i_n}$ converges in $\frac{p}{n}$ -variation to $\mathbf{x}^{n; i_1, \dots, i_n}$, where $\mathbf{x}_{st}^{\varepsilon, n; i_1, \dots, i_n}$ is defined for $(s, t) \in \mathcal{S}_2$ by

$$\mathbf{x}_{st}^{\varepsilon, n; i_1, \dots, i_n} = \int_{(u_1, \dots, u_n) \in \mathcal{S}_n([s, t])} dx_{u_1}^{\varepsilon, i_1} \dots dx_{u_n}^{\varepsilon, i_n}.$$

We are now ready to state one of the main assumptions on our standing process x .

Hypothesis 2.5. Throughout the paper, x will designate a continuous \mathbb{R}^d -valued path with finite p -variation for $p \geq 1$. We assume that x gives rise to a geometric rough path in the sense of Definition 2.4.

On top of Hypothesis 2.5, we assume that $x_t = (x_t^1, \dots, x_t^d)$ is a continuous centered Gaussian process with i.i.d. components, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The covariance function of x is given by

$$R(s, t) := \mathbf{E} \left[x_s^j x_t^j \right], \tag{10}$$

for any $j \in \{1, \dots, d\}$. Throughout the paper, we will also set $R_t := R(t, t)$.

The information on the path regularity of x is mostly contained in the rectangular increments R_{uv}^{st} of its covariance function R , which are defined as

$$R_{uv}^{st} := \mathbf{E} \left[(x_t^j - x_s^j)(x_v^j - x_u^j) \right]. \tag{11}$$

The regularity of R is expressed thanks to some 2d-variation type quantities. For sake of clarity we first recall the definition of the 2d ρ -variation.

Definition 2.6. Let $\rho \in [1, \infty)$. For a general continuous function $R : [0, T]^2 \rightarrow \mathbb{R}$, its 2d ρ -variation is defined as

$$\|R\|_{\rho\text{-var}; [s, t] \times [u, v]} := \sup_{\substack{(t'_j) \in \mathcal{D}([s, t]) \\ (t''_j) \in \mathcal{D}([u, v])}} \left(\sum_{t'_j} \sum_{t''_j} \left| R_{t'_j t''_j}^{t'_j t''_j+1} \right|^\rho \right)^{\frac{1}{\rho}}. \tag{12}$$

where

$$R_{t'_j t''_j}^{t'_j t''_j+1} = R(t_{i+1}, t'_{j+1}) - R(t_{i+1}, t'_j) - R(t_i, t'_{j+1}) + R(t_i, t'_j). \tag{13}$$

Observe that, whenever the function R in Definition 2.6 is a covariance function as in (10), the rectangular increment $R_{t'_j t''_j}^{t'_j t''_j+1}$ can also be written as in (11).

In the following definition, we consider each element $((s, t), (u, v))$ in $\mathcal{S}_2 \times \mathcal{S}_2$ as a rectangle and denote it by $[s, t] \times [u, v]$.

Definition 2.7. A continuous function $\omega : \mathcal{S}_2 \times \mathcal{S}_2 \rightarrow \mathbb{R}^+$ is called a 2d control, if it is zero on degenerate rectangles, and super-additive in the sense that for all rectangles A, B and C

contained in \mathcal{S}_2 satisfying $A \cup B \subset C$ and $A \cap B = \emptyset$,

$$\omega(A) + \omega(B) \leq \omega(C).$$

With these elementary notions at hand, we next introduce a hypothesis which allows the use of both rough paths techniques and tools from stochastic analysis for the underlying process x .

Hypothesis 2.8. Let x be a d -dimensional continuous and centered Gaussian process with i.i.d. components, whose initial value is 0 and covariance R is given by (10). We assume that for some $\rho \in [1, 2)$, the function R admits a finite 2d ρ -variation.

It is well known that for a continuous function $g : [0, T] \rightarrow \mathbb{R}$ with finite p -variation, the function $[a, b] \mapsto \|g\|_{p\text{-var};[a,b]}^p$ is a control. However, for a continuous function $R : [0, T]^2 \rightarrow \mathbb{R}$ with finite 2d ρ -variation, the function $[a, b] \times [c, d] \mapsto \|R\|_{\rho\text{-var};[a,b] \times [c,d]}^\rho$ may fail to be super-additive for $\rho > 1$ (see [11, Theorem 1]). To regain this property, here we introduce the so-called *controlled 2d ρ -variation* for $1 \leq \rho < \infty$ (this notion is also introduced in [11]).

Definition 2.9. Let $\rho \in [1, \infty)$. For a continuous function $R : [0, T]^2 \rightarrow \mathbb{R}$, its controlled 2d ρ -variation is defined as

$$\|R\|_{\rho\text{-var};[s,t] \times [u,v]} := \sup_{\Pi \in \mathcal{P}([s,t] \times [u,v])} \left(\sum_{[t_i, t_{i+1}] \times [t'_j, t'_{j+1}] \in \Pi} \left| R_{t_i t_{i+1}}^{t'_j t'_{j+1}} \right|^\rho \right)^{\frac{1}{\rho}},$$

where $R_{t_i t_{i+1}}^{t'_j t'_{j+1}}$ is given in (13), Π is a partition of $[s, t] \times [u, v]$ which is a finite set of essentially disjoint rectangles whose union is $[s, t] \times [u, v]$, and $\mathcal{P}([s, t] \times [u, v])$ is the collection of all such partitions.

The norms $\|\cdot\|$ and $\|\|\cdot\|\|$ are comparable thanks to the following property borrowed from [11, Theorem 1]: for all $\rho' > \rho$ there exists a constant $C_{\rho, \rho'}$ such that

$$C_{\rho, \rho'} \|f\|_{\rho'\text{-var};[s_1, s_2] \times [t_1, t_2]} \leq \|f\|_{\rho\text{-var};[s_1, s_2] \times [t_1, t_2]} \leq \|\|f\|\|_{\rho\text{-var};[s_1, s_2] \times [t_1, t_2]}. \tag{14}$$

Moreover, the function $[a, b] \times [c, d] \mapsto \|\|f\|\|_{\rho\text{-var};[a,b] \times [c,d]}$ is a 2d control ([11, Theorem 1]).

Remark 2.10. Owing to (14), any continuous function $R : [0, T]^2 \rightarrow \mathbb{R}$ with finite 2d ρ -variation also has a finite *controlled 2d ρ' -variation* for all $\rho' > \rho$. Furthermore, for all $((s, t), (u, v)) \in \mathcal{S}_2 \times \mathcal{S}_2$,

$$\|R\|_{\rho'\text{-var};[s,t] \times [u,v]}^{\rho'} \leq \omega([s, t] \times [u, v]),$$

where ω is the 2d control (as introduced in Definition 2.7) given by

$$\omega([s, t] \times [u, v]) = \|\|R\|\|_{\rho'\text{-var};[s,t] \times [u,v]}. \tag{15}$$

Remark 2.11. As an example, if the Gaussian process x is a fractional Brownian motion with Hurst parameter $H \in (0, \frac{1}{2}]$, the covariance R of x has finite 2d ρ -variation with $\rho = \frac{1}{2H}$ and Hypothesis 2.8 is satisfied (see [11, Example 1]). If we choose $\rho' > \rho = \frac{1}{2H}$, then the quantity $\|R\|_{\rho'\text{-var};[s,t] \times [u,v]}^{\rho'}$ is controlled by the 2d control $\|\|R\|\|_{\rho'\text{-var};[s,t] \times [u,v]}$. Note that $\|R\|_{\rho\text{-var};[0,T]^2} = \infty$ if we choose $\rho \leq \frac{1}{2H}$ (as shown in [11, Example 2]).

In the sequel we will also request the function $t \mapsto R(t, t)$ to be Hölder continuous. We now state an additional assumption which guarantees this Hölder continuity (see e.g [4,12] for a similar hypothesis).

Hypothesis 2.12. Let $\rho \in [1, 2)$ be given in Hypothesis 2.8. We assume that there exists $C < \infty$ such that for all $s, t \in [0, T]$ the covariance function R satisfies

$$\|R(t, \cdot) - R(s, \cdot)\|_{\rho\text{-var};[0,T]}^\rho \leq C|t - s|. \tag{16}$$

Remark 2.13. A direct consequence of Hypothesis 2.12 is that $R_t := R(t, t)$ has finite ρ -variation, by [4, Lemma 2.14]. Moreover, recall that by Hypothesis 2.8, we have $x_0 = 0$ and hence $R(0, \cdot) = R(\cdot, 0) \equiv 0$. This together with (16) implies that $R(t, \cdot)$ and $R(\cdot, t)$ have finite ρ -variation for each fixed $t \in [0, T]$.

Remark 2.14. Given $\rho \in [1, 2)$, clearly we have, for $0 \leq s_1 \leq s_2 \leq T$ and $0 \leq t_1 \leq t_2 \leq T$,

$$\|R\|_{\rho\text{-var};[s_1,s_2] \times [t_1,t_2]}^2 \leq \|R\|_{\rho\text{-var};[s_1,s_2] \times [0,T]} \|R\|_{\rho\text{-var};[0,T] \times [t_1,t_2]}.$$

Furthermore, it is a direct consequence of (16) that for $0 \leq s \leq t \leq T$,

$$\|R\|_{\rho\text{-var};[s,t] \times [0,T]}^\rho \leq C(t - s).$$

Combining the two inequalities above, we have the following control on the 2d ρ -variation of R : for some positive constant C ,

$$\|R\|_{\rho\text{-var};[s_1,s_2] \times [t_1,t_2]}^{2\rho} \leq C(s_2 - s_1)(t_2 - t_1). \tag{17}$$

Remark 2.15. Note that for $1 \leq \gamma \leq \gamma' < \infty$, $\|R\|_{\gamma'\text{-var};[s,t] \times [u,v]} \leq \|R\|_{\gamma\text{-var};[s,t] \times [u,v]}$. Therefore, under Hypothesis 2.12, inequality (16) and hence (17) hold with ρ replaced by $\rho' \in (\rho, 2)$ and C depending on (ρ, ρ', T) .

Remark 2.16. Clearly (17) yields the following relations on squares of the form $[s, t]^2$,

$$\|R\|_{\rho\text{-var};[s,t]^2}^\rho \leq C(t - s). \tag{18}$$

We say that R has finite Hölder-controlled 2d ρ -variation if R satisfies both Hypothesis 2.8 and (18). An important consequence of R having finite Hölder controlled 2d ρ -variation is that x has $1/p$ -Hölder continuous sample paths for every $p > 2\rho$. It is also readily checked that, whenever x satisfies (18), we have

$$\mathbf{E} \left[\left(\mathbf{x}_{st}^{1:i} \right)^2 \right] \leq c(t - s)^{\frac{1}{\rho}}. \tag{19}$$

Remark 2.17. Similarly to the argument in [3, Remark 2.4], for any process x whose covariance function R admits a finite ρ -variation one can introduce a deterministic time-change $\tau : [0, T] \rightarrow [0, T]$ such that $\tilde{X} = X \circ \tau$ has finite Hölder-controlled 2d ρ -variation. That is the time changed process \tilde{X} satisfies Hypothesis 2.8 and Eq. (18).

The following result (stated e.g. in [9, Theorem 15.33]) relates the 2d ρ -variation of R with the pathwise assumptions allowing to apply the abstract rough paths theory.

Proposition 2.18. Let $x = (x^1, \dots, x^d)$ be a continuous centered Gaussian process with i.i.d. components and covariance function R defined by (10). If R satisfies Hypothesis 2.8, then x also satisfies Hypothesis 2.5 provided $p > 2\rho$.

Proposition 2.18 asserts that under **Hypothesis 2.8**, the Gaussian process x is amenable to rough path analysis. In particular, a rough path integral with respect to x can be constructed. In this context, the natural class of integrand one might want to consider is the family of controlled processes. Its definition is recalled below.

Definition 2.19. Consider a continuous \mathbb{R}^d -valued path x with finite p -variation for some $p \geq 1$. We say that a continuous \mathbb{R}^d -valued path y of finite p -variation is controlled by x , if there exist a continuous \mathbb{R}^{d^2} -valued path y^x of finite p -variation and a 1-increment process $r \in \mathcal{C}_2^{\frac{p}{2}-\text{var}}(\mathbb{R}^d)$ as defined in **Definition 2.3**, such that

$$\delta y_{st}^i = \sum_{j=1}^d y_s^{x;ij} \mathbf{x}_{st}^{1;j} + r_{st}^i, \quad \text{for } i = 1, \dots, d. \tag{20}$$

We are now ready to state the basic integration result for controlled processes, which can be found e.g. in [8,9,13].

Proposition 2.20. Let $T > 0$ be fixed. Let \mathbf{x} be a geometric p -rough path lifted from a continuous \mathbb{R}^d -valued path with finite p -variation for some $p \in [1, 3)$, and let y be a continuous \mathbb{R}^d -valued path of finite p -variation that is controlled by x in the sense of **Definition 2.19**. Then for $0 \leq s < t \leq T$, one can define the integral $\int_s^t y_r d\mathbf{x}_r$ as the limit of the following Riemann sums,

$$\int_s^t y_r d\mathbf{x}_r = \lim_{|\pi_n| \rightarrow 0} \sum_{k=0}^{n-1} \left(\sum_{i=1}^d y_{t_k}^i \mathbf{x}_{t_k t_{k+1}}^{1;i} + \sum_{i=1}^d \sum_{j=1}^d y_{t_k}^{x;ij} \mathbf{x}_{t_k t_{k+1}}^{2;ij} \right), \tag{21}$$

where $\pi_n = [s = t_0 < t_1 < \dots < t_n = t]$ is a partition of $[s, t]$ and $|\pi_n| = \max_{k \in \{0, \dots, n-1\}} |t_{k+1} - t_k|$. In (21), observe that we have also used the convention on inner products put forward in Section 1.3. Moreover, there exists a constant $C = C(T, p)$ depending only on (T, p) such that for all $0 \leq s < t \leq T$ we have

$$\left| \int_s^t y_r d\mathbf{x}_r - y_s \mathbf{x}_{st}^1 - \sum_{i=1}^d \sum_{j=1}^d y_s^{x;ij} \mathbf{x}_{st}^{2;ij} \right| \leq C \left(\|\mathbf{x}^1\|_{p\text{-var}} \|r\|_{\frac{p}{2}\text{-var}} + \|\mathbf{x}^2\|_{\frac{p}{2}\text{-var}} \|y^x\|_{p\text{-var}} \right) |t - s|^{3/p},$$

where we recall that r is the increment introduced in (20).

Recall that our main objective is to compute some Skorohod–Stratonovich corrections as in [4]. To this aim we will need a more detailed description of the increments of y than the ones given in (20). Namely we will assume that y is a second order controlled process as defined below (for the definition of controlled processes of general order, we refer to [8, Definition 4.17] or [3, Definition 5.1]).

Definition 2.21. Consider a continuous \mathbb{R}^d -valued path x with finite p -variation for some $p \geq 1$. We say that a continuous \mathbb{R}^d -valued path y of finite p -variation is a second-order controlled process with respect to x , if there exist a continuous \mathbb{R}^{d^2} -valued path y^x , a continuous \mathbb{R}^{d^3} -valued path y^{xx} , both of which are of finite p -variation, and 1-increment processes $r \in \mathcal{C}_2^{\frac{p}{3}-\text{var}}(\mathbb{R}^d), r^x \in \mathcal{C}_2^{\frac{p}{2}-\text{var}}(\mathbb{R}^{d^2})$ as defined in **Definition 2.3**, such that for

$i = 1, \dots, d$ and $(s, t) \in \mathcal{S}_2([0, T])$ we have

$$\delta y_{st}^i = \sum_{j=1}^d y_s^{x;ij} \mathbf{x}_{st}^{1;j} + \sum_{j,k=1}^d y_s^{xx;ijk} \mathbf{x}_{st}^{2;jk} + r_{st}^i. \tag{22}$$

In addition, the increment y^x in (22) is a controlled process of order 1, that is for $i, j = 1, \dots, d$ and $(s, t) \in \mathcal{S}_2([0, T])$ we have

$$\delta y_{st}^{x;ij} = \sum_{k=1}^d y_s^{xx;ijk} \mathbf{x}_{st}^{1;k} + r_{st}^{x;ij}. \tag{23}$$

2.2. Higher dimensional Young integrals

In this subsection, we gather some inequalities for Young integrals in \mathbb{R}^n which will feature in our computations throughout the paper. We start by a relation for integrals in the plane borrowed from [10,23].

Theorem 2.22. *Let $f, R : [0, T]^2 \rightarrow \mathbb{R}$ be continuous functions with finite p -variation and finite q -variation respectively for $\frac{1}{p} + \frac{1}{q} > 1$. Specifically recalling our Definition 2.6, we assume $\|f\|_{p\text{-var};[0,T]^2} < \infty$ and $\|R\|_{q\text{-var};[0,T]^2} < \infty$. Moreover, assume that for all $s_1, s_2 \in [0, T]$, both $f(s_1, \cdot)$ and $f(\cdot, s_2)$ have finite 1-dimensional p -variation as given in Definition 2.3. Then the 2d Young–Stieltjes integral of f with respect to R exists and the following Young’s inequality holds, for $[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2] \subset [0, T]^2$,*

$$\begin{aligned} \left| \int_{[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} f(s_1, s_2) dR(s_1, s_2) \right| &\leq C_{p,q} \left(|f(\underline{s}_1, \underline{s}_2)| + \|f(\underline{s}_1, \cdot)\|_{p\text{-var};[\underline{s}_2, \bar{s}_2]} \right. \\ &\quad \left. + \|f(\cdot, \underline{s}_2)\|_{p\text{-var};[\underline{s}_1, \bar{s}_1]} + \|f\|_{p\text{-var};[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} \right) \\ &\quad \times \|R\|_{q\text{-var};[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]}. \end{aligned} \tag{24}$$

We now state a lemma about integration in \mathbb{R}^4 which will be invoked in order to analyze discretization properties for the Malliavin derivative of a controlled process y . Although its proof might be traced back to [23], we include it here for the sake of clarity since Lemma 2.23 is tailored for our specific needs.

Lemma 2.23. *Let f, g, R be continuous functions defined on $[0, T]^2$. Similarly to Theorem 2.22, we assume that f, g have finite p -variation, as well as $f(s_1, \cdot), f(\cdot, s_3), g(s_2, \cdot)$ and $g(\cdot, s_4)$ for fixed arbitrary $s_1, s_2, s_3, s_4 \in [0, T]$. We also suppose that R has finite q -variation on $[0, T]^2$, with p, q satisfying $\frac{1}{p} + \frac{1}{q} > 1$. Then for $\underline{s}_1, \bar{s}_1, \dots, \underline{s}_4, \bar{s}_4 \in [0, T]$ such that $\underline{s}_j < \bar{s}_j$ for $j = 1, \dots, 4$, the following Young integral in \mathbb{R}^4 is well defined:*

$$\begin{aligned} &I^{f,g,R}(\underline{s}_1, \bar{s}_1, \dots, \underline{s}_4, \bar{s}_4) \\ &:= \int_{[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2] \times [\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]} f(s_1, s_3) g(s_2, s_4) dR(s_1, s_2) dR(s_3, s_4). \end{aligned}$$

Moreover, $I^{f,g,R}(\underline{s}_1, \bar{s}_1, \dots, \underline{s}_4, \bar{s}_4)$ can be bounded as

$$\begin{aligned} |I^{f,g,R}(\underline{s}_1, \bar{s}_1, \dots, \underline{s}_4, \bar{s}_4)| &\leq C_{p,q} \|R\|_{q\text{-var};[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} \|R\|_{q\text{-var};[\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]} \\ &\times \left(|f(\underline{s}_1, \underline{s}_3)| + \|f(\cdot, \underline{s}_3)\|_{p\text{-var};[\underline{s}_1, \bar{s}_1]} + \|f(\underline{s}_1, \cdot)\|_{p\text{-var};[\underline{s}_3, \bar{s}_3]} + \|f\|_{p\text{-var};[\underline{s}_1, \bar{s}_1] \times [\underline{s}_3, \bar{s}_3]} \right) \\ &\times \left(|g(\underline{s}_2, \underline{s}_4)| + \|g(\cdot, \underline{s}_4)\|_{p\text{-var};[\underline{s}_2, \bar{s}_2]} + \|g(\underline{s}_2, \cdot)\|_{p\text{-var};[\underline{s}_4, \bar{s}_4]} + \|g\|_{p\text{-var};[\underline{s}_2, \bar{s}_2] \times [\underline{s}_4, \bar{s}_4]} \right). \end{aligned} \tag{25}$$

Proof. We will divide this proof in several steps.

Step 1: Decomposition of the integral. We can write

$$I^{f,g,R}(\underline{s}_1, \bar{s}_1, \dots, \underline{s}_4, \bar{s}_4) = \int_{[\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]} F(s_3, s_4) dR(s_3, s_4) \tag{26}$$

where the function F is defined on $[0, T]^2$ by

$$F(s_3, s_4) = \int_{[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} f(s_1, s_3) g(s_2, s_4) dR(s_1, s_2), \tag{27}$$

and where we observe that the right-hand side of (27) is well defined thanks to [Theorem 2.22](#). Our strategy in order to estimate $I^{f,g,R}$ will rely on some successive applications of (24). Specifically, with (26) in mind, relation (24) yields

$$\begin{aligned} |I^{f,g,R}(\underline{s}_1, \bar{s}_1, \underline{s}_2, \bar{s}_2, \underline{s}_3, \bar{s}_3, \underline{s}_4, \bar{s}_4)| &\leq C_{p,q} \left(|F(\underline{s}_3, \underline{s}_4)| + \|F(\underline{s}_3, \cdot)\|_{p\text{-var};[\underline{s}_4, \bar{s}_4]} \right. \\ &\quad \left. + \|F(\cdot, \underline{s}_4)\|_{p\text{-var};[\underline{s}_3, \bar{s}_3]} + \|F\|_{p\text{-var};[\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]} \right) \\ &\quad \times \|R\|_{q\text{-var};[\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]}. \end{aligned} \tag{28}$$

We will now estimate the terms in right-hand side of (28) separately.

Step 2: Upper bound for $F(\underline{s}_3, \underline{s}_4)$. Given $(\underline{s}_3, \underline{s}_4) \in [0, T]^2$ and recalling the definition (27) of F , another application of (24) enables to write

$$\begin{aligned} |F(\underline{s}_3, \underline{s}_4)| &\leq C_{p,q} \left(|f(\underline{s}_1, \underline{s}_3)g(\underline{s}_2, \underline{s}_4)| + |f(\underline{s}_1, \underline{s}_3)| \|g(\cdot, \underline{s}_4)\|_{p\text{-var};[\underline{s}_2, \bar{s}_2]} \right. \\ &\quad \left. + |g(\underline{s}_2, \underline{s}_4)| \|f(\cdot, \underline{s}_3)\|_{p\text{-var};[\underline{s}_1, \bar{s}_1]} \right. \\ &\quad \left. + \|f(\cdot, \underline{s}_3)\|_{p\text{-var};[\underline{s}_1, \bar{s}_1]} \|g(\cdot, \underline{s}_4)\|_{p\text{-var};[\underline{s}_2, \bar{s}_2]} \right) \|R\|_{q\text{-var};[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} \end{aligned}$$

and we notice that the above expression can be simplified as

$$\begin{aligned} |F(\underline{s}_3, \underline{s}_4)| &\leq C_{p,q} \left(|f(\underline{s}_1, \underline{s}_3)| + \|f(\cdot, \underline{s}_3)\|_{p\text{-var};[\underline{s}_1, \bar{s}_1]} \right) \\ &\quad \times \left(|g(\underline{s}_2, \underline{s}_4)| + \|g(\cdot, \underline{s}_4)\|_{p\text{-var};[\underline{s}_2, \bar{s}_2]} \right) \|R\|_{q\text{-var};[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]}. \end{aligned} \tag{29}$$

Step 3: Upper bound for $\|F(\underline{s}_3, \cdot)\|_{p\text{-var};[\underline{s}_4, \bar{s}_4]}$. Recall [Definition 2.3](#) of p -variation. We thus have

$$\|F(\underline{s}_3, \cdot)\|_{p\text{-var};[\underline{s}_4, \bar{s}_4]} = \sup_{\pi} \left(\sum_i |F(\underline{s}_3, v_{i+1}) - F(\underline{s}_3, v_i)|^p \right)^{1/p}.$$

Plugging expression (27) into the above relation, we get

$$\begin{aligned} & \|F(\underline{s}_3, \cdot)\|_{p\text{-var}, [\underline{s}_4, \bar{s}_4]}^p \\ &= \sup_{\pi} \sum_i \left| \int_{[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} f(s_1, \underline{s}_3) \left(g(s_2, v_{i+1}) - g(s_2, v_i) \right) dR(s_1, s_2) \right|^p. \end{aligned}$$

We now apply (24) again and we end up with

$$\|F(\underline{s}_3, \cdot)\|_{p\text{-var}, [\underline{s}_4, \bar{s}_4]} \leq C_{p,q} \|R\|_{q\text{-var}, [\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} \sum_{k=1}^4 V_k, \tag{30}$$

where the terms V_1, V_2 are respectively defined by

$$\begin{aligned} V_1 &= |f(\underline{s}_1, \underline{s}_3)| \sup_{\pi} \left(\sum_i |g(\underline{s}_2, v_{i+1}) - g(\underline{s}_2, v_i)|^p \right)^{1/p}; \\ V_2 &= |f(\underline{s}_1, \underline{s}_3)| \sup_{\pi} \left(\sum_i \|g(\cdot, v_{i+1}) - g(\cdot, v_i)\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2]}^p \right)^{1/p}, \end{aligned}$$

and similarly the terms V_3, V_4 are expressed as

$$\begin{aligned} V_3 &= \sup_{\pi} \left(\sum_i |g(\underline{s}_2, v_{i+1}) - g(\underline{s}_2, v_i)|^p \|f(\cdot, \underline{s}_3)\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1]}^p \right)^{1/p}; \\ V_4 &= \sup_{\pi} \left(\sum_i \|f(\cdot, \underline{s}_3)(g(*, v_{i+1}) - g(*, v_i))\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]}^p \right)^{1/p}. \end{aligned}$$

In addition, the terms V_1, V_2, V_3 are easily bounded. Indeed, resorting again to Definition 2.3, we get

$$V_1 = |f(\underline{s}_1, \underline{s}_3)| \|g(\underline{s}_2, \cdot)\|_{p\text{-var}, [\underline{s}_4, \bar{s}_4]}, \quad V_2 \leq |f(\underline{s}_1, \underline{s}_3)| \|g\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2] \times [\underline{s}_4, \bar{s}_4]}, \tag{31}$$

and

$$V_3 = \|f(\cdot, \underline{s}_3)\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1]} \|g(\underline{s}_2, \cdot)\|_{p\text{-var}, [\underline{s}_4, \bar{s}_4]}. \tag{32}$$

For the term V_4 , by Definition 2.6, it is readily checked that

$$V_4 \leq \|f(\cdot, \underline{s}_3)\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1]} \|g\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2] \times [\underline{s}_4, \bar{s}_4]}. \tag{33}$$

Hence, plugging (31)–(33) into (30), we end up with

$$\begin{aligned} \|F(\underline{s}_3, \cdot)\|_{p\text{-var}, [\underline{s}_4, \bar{s}_4]} &\leq C_{p,q} \left(|f(\underline{s}_1, \underline{s}_3)| + \|f(\cdot, \underline{s}_3)\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1]} \right) \\ &\quad \left(\|g(\underline{s}_2, \cdot)\|_{p\text{-var}, [\underline{s}_4, \bar{s}_4]} + \|g\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2] \times [\underline{s}_4, \bar{s}_4]} \right) \\ &\quad \times \|R\|_{q\text{-var}, [\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]}. \end{aligned} \tag{34}$$

Furthermore, notice that in a similar way we get

$$\begin{aligned} \|F(\cdot, \underline{s}_4)\|_{p\text{-var}, [\underline{s}_4, \bar{s}_4]} &\leq C_{p,q} \left(|g(\underline{s}_2, \underline{s}_4)| + \|g(\cdot, \underline{s}_4)\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2]} \right) \\ &\quad \left(\|f(\underline{s}_1, \cdot)\|_{p\text{-var}, [\underline{s}_3, \bar{s}_3]} + \|f\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1] \times [\underline{s}_3, \bar{s}_3]} \right) \\ &\quad \times \|R\|_{q\text{-var}, [\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]}. \end{aligned} \tag{35}$$

Step 4: Upper bound for $\|F\|_{p\text{-var}, [\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]}$. According to Definition 2.6, one can write

$$\|F\|_{p\text{-var}, [\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]}^p = \sup_{\pi} \sum_{t_i, t'_j} |F(t_i, t'_j) + F(t_{i+1}, t'_{j+1}) - F(t_i, t'_{j+1}) - F(t_{i+1}, t'_j)|^p,$$

where we recall that π takes the form $\pi \in \mathcal{D}([\underline{s}_3, \bar{s}_3]) \times \mathcal{D}([\underline{s}_4, \bar{s}_4])$ and the notation $\mathcal{D}([s, t])$ is introduced in Section 1.3. Hence with the expression (27) of F in mind we get

$$\begin{aligned} & \|F\|_{p\text{-var}, [\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]}^p \\ &= \sup_{\pi} \sum_{t_i, t'_j} \left| \int_{[\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} (f(s_1, t_{i+1}) - f(s_1, t_i))(g(s_2, t'_{j+1}) - g(s_2, t'_j)) dR(s_1, s_2) \right|^p. \end{aligned}$$

In this context, relation (24) can thus be read as

$$\|F\|_{p\text{-var}, [\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]} \leq C \|R\|_{q\text{-var}, [\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} \sup_{\pi} \left(\sum_{t_i, t'_j} |Q_{ij'}|^p \right)^{1/p},$$

where the term $Q_{ij'}$ is defined by

$$\begin{aligned} Q_{ij'} &= |(f(\underline{s}_1, t_{i+1}) - f(\underline{s}_1, t_i))(g(\underline{s}_2, t'_{j+1}) - g(\underline{s}_2, t'_j))| \\ &\quad + |f(\underline{s}_1, t_{i+1}) - f(\underline{s}_1, t_i)| \|g(\cdot, t'_{j+1}) - g(\cdot, t'_j)\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2]} \\ &\quad + \|f(\cdot, t_{i+1}) - f(\cdot, t_i)\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1]} |g(\underline{s}_2, t'_{j+1}) - g(\underline{s}_2, t'_j)| \\ &\quad + \|f(\cdot, t_{i+1}) - f(\cdot, t_i)\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1]} \|g(\cdot, t'_{j+1}) - g(\cdot, t'_j)\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2]}, \end{aligned}$$

and we notice that $Q_{ij'}$ can easily be simplified as

$$\begin{aligned} Q_{ij'} &= \left(|f(\underline{s}_1, t_{i+1}) - f(\underline{s}_1, t_i)| + \|f(\cdot, t_{i+1}) - f(\cdot, t_i)\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1]} \right) \\ &\quad \times \left(|g(\underline{s}_2, t'_{j+1}) - g(\underline{s}_2, t'_j)| + \|g(\cdot, t'_{j+1}) - g(\cdot, t'_j)\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2]} \right). \end{aligned}$$

Summarizing our computations in this step, we have found that

$$\begin{aligned} \|F\|_{p\text{-var}, [\underline{s}_3, \bar{s}_3] \times [\underline{s}_4, \bar{s}_4]} &\leq C \|R\|_{q\text{-var}, [\underline{s}_1, \bar{s}_1] \times [\underline{s}_2, \bar{s}_2]} \\ &\quad \times \left(\|f(\underline{s}_1, \cdot)\|_{p\text{-var}, [\underline{s}_3, \bar{s}_3]} + \|f\|_{p\text{-var}, [\underline{s}_1, \bar{s}_1] \times [\underline{s}_3, \bar{s}_3]} \right) \\ &\quad \times \left(\|g(\underline{s}_2, \cdot)\|_{p\text{-var}, [\underline{s}_4, \bar{s}_4]} + \|g\|_{p\text{-var}, [\underline{s}_2, \bar{s}_2] \times [\underline{s}_4, \bar{s}_4]} \right). \end{aligned} \tag{36}$$

Step 5: Conclusion. Let us gather our estimates (29), (34), (35) and (36) into (28). Then we let the patient reader check that (25) is achieved. This finishes the proof. \square

2.3. The Hilbert space associated to x

Consider a continuous d -dimensional centered Gaussian process x on $[0, T]$ with covariance function R given by (10). Every component of x (say x^1) is a 1-dimensional centered Gaussian process with covariance R . In this section we review some basic facts about the related Hilbert space \mathcal{H} of functions for which Wiener integrals with respect to x (see e.g. [20]) are well defined.

The Hilbert space \mathcal{H} is the completion of the set of step functions

$$\mathcal{E} = \left\{ \sum_{i=1}^n a_i \mathbf{1}_{[0, t_i]} : a_i \in \mathbb{R}, t_i \in [0, T], i = 1, \dots, n \text{ for } n \in \mathbb{N} \right\},$$

with respect to the inner product

$$\left\langle \sum_{i=1}^n a_i \mathbf{1}_{[0,t_i]}, \sum_{j=1}^m b_j \mathbf{1}_{[0,s_j]} \right\rangle_{\mathcal{H}} = \sum_{i=1}^n \sum_{j=1}^m a_i b_j R(t_i, s_j).$$

Observe that this inner product can also be written as

$$\left\langle \sum_{i=1}^n a_i \mathbf{1}_{[0,t_i]}, \sum_{j=1}^m b_j \mathbf{1}_{[0,s_j]} \right\rangle_{\mathcal{H}} = \int_0^T \int_0^T \left(\sum_{i=1}^n a_i \mathbf{1}_{[0,t_i]}(t) \right) \left(\sum_{j=1}^m b_j \mathbf{1}_{[0,s_j]}(s) \right) dR(t, s). \tag{37}$$

One can further relate \mathcal{H} to our driving process x in the following way: let \mathbf{H} be the closure of the set

$$\mathbf{E} = \left\{ \sum_{i=1}^n a_i x_{t_i}^1 : a_i \in \mathbb{R}, t_i \in [0, T], i = 1, \dots, n \text{ for } n \in \mathbb{N} \right\},$$

in $L^2(\Omega, \mathcal{F}, \mathbf{P})$. Then the linear map $x^1 : \mathcal{E} \rightarrow \mathbf{E}$ defined by $x^1(\mathbf{1}_{[0,t]}) = x_t^1$ extends to a linear isometry between \mathcal{H} and \mathbf{H} . Hence, $\mathbf{H} = \{x^1(h), h \in \mathcal{H}\}$ and this family is known as the isonormal Gaussian process related to x^1 (see [20, Definition 1.1.1]). Note that $x^1(h)$ for $h \in \mathcal{H}$ is called the Wiener integral of h with respect to x^1 and is usually denoted by $\int_0^T h(s) dx_s^1$.

Remark 2.24. Recall that we have assumed $x_0 = 0$ and thus $R(0, 0) = 0$. Thus relation (37) suggests

$$\langle h_1, h_2 \rangle_{\mathcal{H}} = \int_0^T \int_0^T h_1(s) h_2(t) dR(s, t) \quad \text{for } h_1, h_2 \in \mathcal{H}, \tag{38}$$

whenever the 2D Young’s integral on the right-hand side is well-defined (see, e.g., [2, Proposition 4] for details).

Remark 2.25. Denoting by $\mathcal{E}([a, b])$ the set of step functions in \mathcal{E} restricted on $[a, b] \subset [0, T]$, the closure $\mathcal{H}([a, b])$ of $\mathcal{E}([a, b])$ with respect to the inner product (37) then coincides with \mathcal{H} restricted on $[a, b]$, and for $f, g \in \mathcal{H}$,

$$\langle f \mathbf{1}_{[a,b]}, g \mathbf{1}_{[a,b]} \rangle_{\mathcal{H}} = \langle f, g \rangle_{\mathcal{H}([a,b])}. \tag{39}$$

2.4. Malliavin calculus for Gaussian processes

In this subsection, we collect some basic concepts of Malliavin calculus, and we refer to [20] for more details.

Recall that x_t is a continuous centered d -dimensional Gaussian process with i.i.d. components, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For the sake of simplicity, we assume that \mathcal{F} coincides with the σ -algebra generated by $\{x_t; t \in [0, T]\}$. For the d -dimensional process x , we define an extension of the Wiener integral defined as follows: let $\varphi = (\varphi^1, \dots, \varphi^d)$ be an element of \mathcal{H}^d where we recall that \mathcal{H} has been introduced in Section 2.3. Then we set

$$x(\varphi) = \sum_{j=1}^d x^j(\varphi^j), \tag{40}$$

where each term $x^j(\varphi^j)$ is a 1-d Wiener integral as in Section 2.3.

A smooth functional of x is a random variable of the form $F = f(x(\varphi_1), \dots, x(\varphi_n))$, where $n \geq 1$, $\{\varphi_1, \dots, \varphi_n\}$ is a family of elements of \mathcal{H}^d and each $x(\varphi_i)$ is understood as in (40). Moreover, we assume that the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is smooth and its partial derivatives grow at most polynomially fast. Then, the Malliavin derivative $\mathbf{D}F$ of F is the \mathcal{H}^d -valued random variable defined by

$$\mathbf{D}F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(x(\varphi_1), \dots, x(\varphi_n))\varphi_k. \tag{41}$$

One can show that \mathbf{D} is closable from $L^2(\Omega)$ to $L^2(\Omega; \mathcal{H})$, and thus one may span the space of the smooth and cylindrical random variables under the norm

$$\|F\|_{1,2} = (\mathbb{E}[F^2] + \mathbb{E}[\|\mathbf{D}F\|_{\mathcal{H}}^2])^{\frac{1}{2}}.$$

The resulting closure is called Sobolev space $\mathbb{D}^{1,2}$.

Remark 2.26. As seen in (41), the Malliavin derivative $\mathbf{D}F$ of a functional F is a \mathbb{R}^d -valued process. The i th coordinate of $\mathbf{D}F$ corresponds to the Malliavin derivative of F with respect to the randomness in x^i only. It will be denoted by $\mathbf{D}^i F$ in the sequel.

The divergence operator δ° (also known as the Skorohod integral) is the adjoint operator of the Malliavin derivative operator \mathbf{D} defined by the duality relation

$$\mathbb{E}[F\delta^\circ(u)] = \mathbb{E}[\langle \mathbf{D}F, u \rangle_{\mathcal{H}^d}], \quad \text{for all } F \in \mathbb{D}^{1,2} \text{ and for all } u \in \text{Dom } \delta^\circ.$$

Here $\text{Dom } \delta^\circ$ is the domain of the divergence operator δ° , which is the space of \mathcal{H} -valued random variables $u \in L^2(\Omega; \mathcal{H}^d)$ such that $|\mathbb{E}[\langle \mathbf{D}F, u \rangle_{\mathcal{H}^d}]| \leq c_F \|F\|_2$ with some constant c_F depending on F , for all $F \in \mathbb{D}^{1,2}$. In particular, $\mathbb{D}^{1,2}(\mathcal{H}^d) \subset \text{Dom } \delta^\circ$. Note that for $u \in \text{Dom } \delta^\circ$, we have $\delta^\circ(u) \in L^2(\Omega)$ and $\mathbb{E}[\delta^\circ(u)] = 0$. By convention, we also take the following notation, for $u \in \text{Dom } \delta^\circ$,

$$\int_0^T u_t d^\circ x_t := \delta^\circ(u). \tag{42}$$

For our main computations below we shall invoke the following relation taken from [21]: for any $G \in \mathbb{D}^{1,2}(\mathbb{R}^d)$ and $0 \leq a < b \leq T$ we have

$$\delta^\circ(G \mathbf{1}_{[a,b]}) = \sum_{i=1}^d \int_a^b G^i d^\circ x_t^i = \sum_i G^i \diamond \delta x_{ab}^i, \tag{43}$$

where \diamond stands for the Wick product (see [15] for a brief account on Wick products). Moreover, according to [16, Proposition 4.7], relation (43) can be simplified as

$$\delta^\circ(G \mathbf{1}_{[a,b]}) = \sum_{i=1}^d G^i \delta x_{ab}^i - \langle \mathbf{D}^i G^i, \mathbf{1}_{[a,b]} \rangle_{\mathcal{H}}. \tag{44}$$

2.5. Discrete rough paths techniques

In this subsection, we develop some inequalities about discrete sums in a rough paths context. This kind of sum will feature prominently in the analysis of our Skorohod–Stratonovich corrections.

We state a crucial lemma about convergence of discrete sums in the second chaos of x . It generalizes [18, Lemma 3.4] to a generic Gaussian process (as opposed to the fractional Brownian motion case handled in [18]).

Proposition 2.27. *Let x be a \mathbb{R}^d -valued Gaussian process satisfying [Hypotheses 2.8](#) and [2.12](#). For $n \geq 1$ we consider the uniform partition on $[0, T]$, namely $t_k = \frac{k}{n}T$. We define a process $F = \{F_t^{ij}; t \in \llbracket 0, T \rrbracket, i, j = 1, \dots, d\}$ by $F_0^{ij} = 0$, and for all $t > 0$,*

$$F_t^{ij} = \sum_{t_k=0}^{t_-} \left(\mathbf{x}_{t_k t_{k+1}}^{2;ij} - \mathbf{E}[\mathbf{x}_{t_k t_{k+1}}^{2;ij}] \right) = \begin{cases} \sum_{t_k=0}^{t_-} \mathbf{x}_{t_k t_{k+1}}^{2;ij}, & i \neq j, \\ \sum_{t_k=0}^{t_-} \left(\mathbf{x}_{t_k t_{k+1}}^{2;ii} - \mathbf{E}[\mathbf{x}_{t_k t_{k+1}}^{2;ii}] \right), & i = j, \end{cases} \tag{45}$$

where we recall the notation t_- from [Section 1.3](#) and where \mathbf{x}^2 is introduced in [Definition 2.4](#). Then for all $q \geq 1$, $\rho' \in (\rho, 2)$, $(s, t) \in S_2(\llbracket 0, T \rrbracket)$ and $n \geq 1$ the following inequality holds true

$$\left(\mathbf{E} \left[\left| \delta F_{st}^{ij} \right|^q \right] \right)^{1/q} \leq C \frac{(t-s)^{\frac{1}{2}}}{n^{\beta-\frac{1}{2}}}, \tag{46}$$

where $C = C(q, \rho, T)$, $\beta = \frac{1}{\rho} \in (1/2, 1]$ for $i \neq j$, and $C = C(q, \rho, \rho', T)$, $\beta = \frac{1}{\rho'} \in (1/2, 1)$ for $i = j$.

Proof. Due to the hyper-contractivity property of the second Wiener chaos, it suffices to show the case $q = 2$. In addition, we assume (without loss of generality) that $s = t_{m_1} < t = t_{m_2}$ for $0 \leq m_1 < m_2 \leq n$.

Case 1: $i = j$. In this case, due to the definition (45) of F and the geometric nature of \mathbf{x} assumed in [Definition 2.4](#), we have

$$\mathbf{E} \left[(\delta F_{st}^{ii})^2 \right] = \mathbf{E} \left(\sum_{k=m_1}^{m_2-1} \left[(\mathbf{x}_{t_k t_{k+1}}^{1;i})^2 - \mathbf{E}[(\mathbf{x}_{t_k t_{k+1}}^{1;i})^2] \right] \right)^2,$$

and expanding the square on the right hand side above we get

$$\mathbf{E} \left[(\delta F_{st}^{ii})^2 \right] = \sum_{k,l=m_1}^{m_2-1} \left\{ \mathbf{E}[(\mathbf{x}_{t_k t_{k+1}}^{1;i})^2 (\mathbf{x}_{t_l t_{l+1}}^{1;i})^2] - \mathbf{E}[(\mathbf{x}_{t_k t_{k+1}}^{1;i})^2] \mathbf{E}[(\mathbf{x}_{t_l t_{l+1}}^{1;i})^2] \right\}. \tag{47}$$

In order to evaluate the right-hand side of (47) we apply a particular case of Wick’s formula for centered Gaussian random variables X and Y , which can be stated as:

$$\mathbf{E}[X^2 Y^2] - \mathbf{E}[X^2] \mathbf{E}[Y^2] = 2 (\mathbf{E}[X Y])^2.$$

Plugging this result into (47) and recalling the definition (11) of R_{st}^{uv} we obtain

$$\mathbf{E} \left[(\delta F_{st}^{ii})^2 \right] = 2 \sum_{k,l=m_1}^{m_2-1} \left(\mathbf{E} \left[\mathbf{x}_{t_k t_{k+1}}^{1;i} \mathbf{x}_{t_l t_{l+1}}^{1;i} \right] \right)^2 = 2 \sum_{k,l=m_1}^{m_2-1} \left(R_{t_l t_{l+1}}^{t_k t_{k+1}} \right)^2. \tag{48}$$

Therefore invoking elementary properties of p -variations we end up with

$$\mathbf{E} \left[(\delta F_{st}^{ii})^2 \right] \leq 2 \sup_{k,l} \left(R_{t_l t_{k+1}}^{t_k t_{k+1}} \right)^{2-\rho} \|R\|_{\rho\text{-var};[s,t]^2}^\rho. \tag{49}$$

On the right-hand side of (49), notice that under [Hypothesis 2.12](#), $\|R\|_{\rho\text{-var};[s,t]^2}^\rho$ can be upper bounded by $C(t-s)$ thanks to (18). Moreover, a simple use of Cauchy–Schwarz inequality, together with (19), shows that

$$|R_{t_l t_{k+1}}^{t_k t_{k+1}}| = \left| \mathbf{E}[\mathbf{x}_{t_l t_{k+1}}^{1,i} \mathbf{x}_{t_k t_{k+1}}^{1,i}] \right| \leq \left(\mathbf{E} \left[\left| \mathbf{x}_{t_l t_{k+1}}^{1,i} \right|^2 \right] \mathbf{E} \left[\left| \mathbf{x}_{t_k t_{k+1}}^{1,i} \right|^2 \right] \right)^{\frac{1}{2}} \leq \frac{C_T}{n^{\frac{1}{\rho}}}.$$

Reporting this information into (49) and recalling that $\beta = \frac{1}{\rho}$, it is seen that

$$\mathbf{E} \left[(\delta F_{st}^{ii})^2 \right] \leq C T^{2\beta-1} \frac{(t-s)}{n^{2\beta-1}}. \tag{50}$$

This ends our proof for the case $i = j$.

Case 2: $i \neq j$. According to our definition (45), if $i \neq j$ we have

$$\mathbf{E} \left[(\delta F_{st}^{ij})^2 \right] = \mathbf{E} \left[\left(\sum_{k=m_1}^{m_2-1} \mathbf{x}_{t_k t_{k+1}}^{2:ij} \right)^2 \right] = \sum_{k,l=m_1}^{m_2-1} \mathbf{E} \left[\mathbf{x}_{t_k t_{k+1}}^{2:ij} \mathbf{x}_{t_l t_{l+1}}^{2:ij} \right].$$

Therefore, invoking the proofs of [9, Theorem 15.33 and Proposition 15.28] for the computation of $\mathbf{E}[\mathbf{x}_{t_k t_{k+1}}^{2:ij} \mathbf{x}_{t_l t_{l+1}}^{2:ij}]$, we end up with

$$\mathbf{E} \left[(\delta F_{st}^{ij})^2 \right] = \sum_{k,l=m_1}^{m_2-1} \int_{t_k}^{t_{k+1}} \int_{t_l}^{t_{l+1}} R_{t_k v_1}^{t_l v_2} dR(v_1, v_2). \tag{51}$$

We now fix (k, l) and denote $G(v_1, v_2) = R_{t_k v_1}^{t_l v_2}$. Then $G(t_k, \cdot) = G(\cdot, t_l) = 0$. For any $\rho' \in (\rho, 2)$, [Hypothesis 2.8](#) implies R has finite 2d ρ' -variation, and [Hypothesis 2.12](#) implies both $R(t, \cdot)$ and $R(\cdot, t)$ have finite ρ' -variation for all $t \in [0, T]$. Hence resorting to [Theorem 2.22](#), we have for some fixed $\rho' \in (\rho, 2)$,

$$\left| \int_{t_k}^{t_{k+1}} \int_{t_l}^{t_{l+1}} R_{t_k v_1}^{t_l v_2} dR(v_1, v_2) \right| \leq C \|R\|_{\rho'\text{-var};[t_k, t_{k+1}] \times [t_l, t_{l+1}]}^2,$$

for some constant $C = C(\rho', T)$ depending on (ρ', T) only. Plugging this inequality into (51) we obtain

$$\begin{aligned} \mathbf{E} \left[(\delta F_{st}^{ij})^2 \right] &\leq C(\rho', T) \sum_{k,l=m_1}^{m_2-1} \|R\|_{\rho'\text{-var};[t_k, t_{k+1}] \times [t_l, t_{l+1}]}^2 \\ &\leq C(\rho', T) \sup_{k,l} \|R\|_{\rho'\text{-var};[t_k, t_{k+1}] \times [t_l, t_{l+1}]}^{2-\rho'} \\ &\quad \times \sum_{k,l=m_1}^{m_2-1} \|R\|_{\rho'\text{-var};[t_k, t_{k+1}] \times [t_l, t_{l+1}]}^{\rho'}. \end{aligned}$$

Therefore, thanks to [Remark 2.10](#), we have

$$\mathbf{E} \left[(\delta F_{st}^{ij})^2 \right] \leq C(\rho', T) \sup_{k,l} \|R\|_{\rho'\text{-var};[t_k, t_{k+1}] \times [t_l, t_{l+1}]}^{2-\rho'} \sum_{k,l=m_1}^{m_2-1} \omega([t_k, t_{k+1}] \times [t_l, t_{l+1}]),$$

where ω is a control given in (15). Furthermore, the super-additivity of ω yields

$$\begin{aligned} \mathbf{E} \left[\left(\delta F_{st}^{ij} \right)^2 \right] &\leq C(\rho', T) \sup_{k,l} \|R\|_{\rho' \text{-var}; [t_k, t_{k+1}] \times [t_l, t_{l+1}]}^{2-\rho'} \omega([s, t]^2) \\ &\leq C(\rho, \rho', T) \sup_{k,l} \|R\|_{\rho' \text{-var}; [t_k, t_{k+1}] \times [t_l, t_{l+1}]}^{2-\rho'} (t - s), \end{aligned} \tag{52}$$

where the last inequality is due to (15), (14), and (17). Finally, by Hypothesis 2.12 (and Remark 2.15), we have

$$\|R\|_{\rho' \text{-var}; [s, t] \times [u, v]}^{2\rho'} \leq C(\rho, \rho', T)(t - s)(u - v),$$

and therefore setting $\beta = 1/\rho'$, inequality (52) becomes

$$\mathbf{E} \left[\left(\delta F_{st}^{ij} \right)^2 \right] \leq C(\rho, \rho', T) \left(\frac{T}{n} \right)^{2\beta-1} (t - s). \tag{53}$$

With (50) and (53) in hand our claim (46) is now easily achieved, which concludes the proof. \square

Note that (46) is still valid for both cases of $i = j$ and $i \neq j$, if we choose $\beta = \frac{1}{\rho'}$ for any $\rho' \in (\rho, 2)$. We now give a weighted version of Proposition 2.27, which plays an important role in our correction computations.

Proposition 2.28. *Let x be a \mathbb{R}^d -valued Gaussian process satisfying Hypotheses 2.8 and 2.12. Let $\rho' \in (\rho, 2)$ be fixed. For $n \geq 1$ we consider the uniform partition on $[0, T]$, namely $t_k = \frac{k}{n}T$, as well as the process F defined by (45). Let now f be a controlled process in the $L^q(\Omega)$ sense, namely such that there exists a process g fulfilling (in the matrix sense), for some $\gamma \in (\frac{1}{4}, \frac{1}{2\rho})$ and for all $q \geq 1$,*

$$\|f_t\|_q + \|g_t\|_q \leq C, \quad \|\delta f_{st} - g_s \mathbf{x}_{st}^1\|_q \leq C(t - s)^{2\gamma}, \quad \|\delta g_{st}\|_q \leq C(t - s)^\gamma. \tag{54}$$

Then the following estimate holds true for $(s, t) \in \mathcal{S}_2(\llbracket 0, T \rrbracket)$:

$$\left\| \sum_{t_k=s}^{t-} f_{t_k} \otimes \delta F_{t_k t_{k+1}} \right\|_q \leq C \frac{(t - s)^{\frac{1}{2}}}{n^{\beta - \frac{1}{2}}},$$

where $C = C(q, \rho, \rho', T)$ and $\beta = \frac{1}{\rho'} \in (1/2, 1)$.

Proof. This proposition was proved in [18, Corollary 4.9] when x is a fractional Brownian motion. Although we generalize this result to a wider class of Gaussian processes, our proof goes along the same lines. Therefore we shall omit the details for sake of conciseness. \square

3. Correction terms in the case $2 \leq p < 3$

In this section, we derive a correction formula for controlled processes which are also in the domain of the Skorohod integral and then provide some examples.

3.1. Correction formula

As mentioned in the introduction, we have restricted our analysis to the case $p < 3$. Although we believe that our methodology could be extended to $p < 4$, this generalization would require a cumbersome study of third order integrals and related weighted sums.

Theorem 3.1. *Let x be a Gaussian rough path with covariance given by (10) satisfying Hypotheses 2.8 and 2.12 with $\rho \in [1, \frac{3}{2})$. This implies that x has finite p -variation for $p > 2\rho$. We can assume $\frac{1}{p} + \frac{1}{\rho} > 1$, noting that $\rho < \frac{3}{2}$.*

Let y be a second-order controlled process in the sense of Definition 2.21, and we assume $\mathbb{E}[\|y\|_{p\text{-var};[0,T]}^2] < \infty$. In particular, the rough integral $\int_0^t y_r d\mathbf{x}_r$ is defined as in Proposition 2.20, resorting to the convention on inner products of Section 1.3. We also assume that $y \in \mathbb{D}^{1,2}(\mathcal{H}^d)$, so that the Skorohod integral of y given in (42) is well defined. Furthermore, we suppose that $\mathbf{D}_0 y$ has finite p -variation with $\mathbb{E}[\|\mathbf{D}_0 y\|_{p\text{-var};[0,T]}^2] < \infty$, and $\mathbf{D}y$ has finite $2d$ p -variation with $\mathbb{E}[\|\mathbf{D}y\|_{p\text{-var};[0,T]^2}^2] < \infty$. Then for all $t \in [0, T]$ we have almost surely

$$\int_0^t y_r d\mathbf{x}_r = \int_0^t y_r d^\circ x_r + \frac{1}{2} \sum_{i=1}^d \int_0^t y_r^{x;ii} dR_r + \sum_{i=1}^d \int_{\mathcal{S}_2([0,t])} (\mathbf{D}_{r_1}^i y_{r_2}^i - y_{r_2}^{x;ii}) dR(r_1, r_2), \tag{55}$$

where we recall from Section 2.1 that $R_r := R(r, r)$ and where the Malliavin derivative \mathbf{D}^i is introduced in Remark 2.26.

Proof. Let $\pi = \pi_n$ be the uniform partition of order n of $[0, t]$, whose generic element is still denoted by $t_k = \frac{k}{n}t$. A natural discretization of y along π is given by

$$y^\pi(r) = \sum_{k=0}^{n-1} y_{t_k} \mathbf{1}_{[t_k, t_{k+1})}(r), \quad r \in [0, t]. \tag{56}$$

Notice that we have assumed that $y \in \mathbb{D}^{1,2}(\mathcal{H}^d)$. Hence both divergence integrals $\delta^\circ(y^\pi)$ and $\delta^\circ(y)$, as given in (42), are well defined. Moreover, according to (43), we have

$$\int_0^t y_r^\pi d^\circ x_r = \sum_{i=1}^d \sum_{k=0}^{n-1} y_{t_k}^i \diamond \mathbf{x}_{t_k t_{k+1}}^{1;i},$$

and owing to (44) this can be recast as

$$\int_0^t y_r^\pi d^\circ x_r = \sum_{i=1}^d \sum_{k=0}^{n-1} y_{t_k}^i \mathbf{x}_{t_k t_{k+1}}^{1;i} - \langle \mathbf{D}^i y_{t_k}^i, \mathbf{1}_{[t_k, t_{k+1})} \rangle_{\mathcal{H}}. \tag{57}$$

In addition, we will prove in the forthcoming Lemma 3.2 that $\delta^\circ(y^\pi)$ converges in $L^2(\Omega)$ to $\delta^\circ(y)$. Otherwise stated, for $t \in [0, T]$ we have

$$\int_0^t y_r d^\circ x_r = \lim_{n \rightarrow \infty} \int_0^t y_r^\pi d^\circ x_r. \tag{58}$$

Therefore combining (57) and (58), we get the following limit in $L^2(\Omega)$:

$$\int_0^t y_r d^\circ x_r = \lim_{n \rightarrow \infty} \sum_{i=1}^d \sum_{k=0}^{n-1} \left(y_{t_k}^i \mathbf{x}_{t_k t_{k+1}}^{1;i} - \langle \mathbf{D}^i y_{t_k}^i, \mathbf{1}_{[t_k, t_{k+1})} \rangle_{\mathcal{H}} \right). \tag{59}$$

On the other hand, owing to the fact that y is a controlled process in the sense of Definition 2.19, Proposition 2.20 asserts that $\int_0^t y_r d\mathbf{x}_r$ is defined as a rough paths integral and hence almost surely we have

$$\int_0^t y_r d\mathbf{x}_r = \lim_{n \rightarrow \infty} \left(\sum_{i=1}^d \sum_{k=0}^{n-1} y_{t_k}^i \mathbf{x}_{t_k t_{k+1}}^{1;i} + \sum_{i,j=1}^d \sum_{k=0}^{n-1} y_{t_k}^{x;ij} \mathbf{x}_{t_k t_{k+1}}^{2;ij} \right). \tag{60}$$

Gathering relations (59) and (60), we get the following expression for the Stratonovich–Skorohod correction term:

$$\int_0^t y_r d\mathbf{x}_r - \int_0^t y_r d^\circ x_r = \lim_{n \rightarrow \infty} \sum_{i=1}^d \sum_{k=0}^{n-1} \left(\sum_{j=1}^d y_{t_k}^{x;ij} \mathbf{x}_{t_k t_{k+1}}^{2;ij} + \langle \mathbf{D}^i y_{t_k}^i, \mathbf{1}_{[t_k, t_{k+1})} \rangle_{\mathcal{H}} \right), \tag{61}$$

where the limit on the right-hand side above is understood in probability. In (61), notice that the left-hand side is well defined thanks to the standing assumptions of our theorem. Hence the right-hand side of (61) also makes sense, and we will now identify the limits therein.

In order to compute the limit for the terms $y_{t_k}^{x;ij} \mathbf{x}_{t_k t_{k+1}}^{2;ij}$ in (61), observe that y is a second order controlled process according to Definition 2.21. Hence y^x is a controlled process satisfying relation (23). Since we have assumed that Hypotheses 2.8 and 2.12 are fulfilled, Proposition 2.28 for the increment F can be applied with $f = y^x$. Recalling (see (45)) that

$$\delta F_{t_k t_{k+1}}^{ij} = \mathbf{x}_{t_k t_{k+1}}^{2;ij} - \mathbb{E}[\mathbf{x}_{t_k t_{k+1}}^{2;ij}],$$

we end up with the following relation, valid for $i, j = 1, \dots, d$, where the limit has to be considered in the $L^1(\Omega)$ sense:

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} y_{t_k}^{x;ij} \left(\mathbf{x}_{t_k t_{k+1}}^{2;ij} - \mathbb{E} \left[\mathbf{x}_{t_k t_{k+1}}^{2;ij} \right] \right) = 0. \tag{62}$$

In particular, going back to (61), we get that for $i \neq j$ we have

$$\lim_{n \rightarrow \infty} \sum_{i \neq j} \sum_{k=0}^{n-1} y_{t_k}^{x;ij} \mathbf{x}_{t_k t_{k+1}}^{2;ij} = 0. \tag{63}$$

Let us deal with the left-hand side of (62) when $i = j$. Specifically, we will express the limit of the sums $\sum_{k=0}^{n-1} y_{t_k}^{x;ii} \mathbb{E} \left[\mathbf{x}_{t_k t_{k+1}}^{2;ii} \right]$ as a Young integral. To this aim, notice that $2\mathbf{x}_{t_k t_{k+1}}^{2,ii} = \left(\mathbf{x}_{t_k t_{k+1}}^{1,i} \right)^2$ due to the geometric assumption in Definition 2.4. Hence invoking the fact that $R_{t_k} = R(t_k, t_k)$ we have

$$\begin{aligned} 2\mathbb{E} \left[\mathbf{x}_{t_k t_{k+1}}^{2;ii} \right] &= \mathbb{E} \left[(x_{t_{k+1}}^i - x_{t_k}^i)^2 \right] = R_{t_{k+1}} - 2R(t_{k+1}, t_k) + R_{t_k} \\ &= (R_{t_{k+1}} - R_{t_k}) - 2(R(t_{k+1}, t_k) - R(t_k, t_k)). \end{aligned}$$

Therefore for all $i = 1, \dots, d$, we obtain a decomposition of the form

$$\sum_{k=0}^{n-1} y_{t_k}^{x;ii} \mathbb{E} \left[\mathbf{x}_{t_k t_{k+1}}^{2;ii} \right] = \frac{1}{2} I_n^i - J_n^i \tag{64}$$

where I_n^i, J_n^i are respectively defined by

$$I_n^i = \sum_{k=0}^{n-1} y_{t_k}^{x;ii} \delta R_{t_k t_{k+1}}, \quad \text{and} \quad J_n^i = \sum_{k=0}^{n-1} y_{t_k}^{x;ii} (R(t_{k+1}, t_k) - R(t_k, t_k)). \tag{65}$$

The limit of for the term I_n^i in (64) can be computed easily. Indeed, thanks to Remark 2.13 we know that $t \rightarrow R_t$ has finite ρ -variation. Furthermore, since y is a second order controlled process, Definition 2.21 entails that y^x has finite p -variation. We have also mentioned in Theorem 3.1 that $p^{-1} + \rho^{-1} > 1$. Hence classical Young integration arguments reveal that

for $i = 1, \dots, d$ we have almost surely,

$$\lim_{n \rightarrow \infty} J_n^i = \int_0^t y_r^{x;ii} dR_r. \tag{66}$$

As far as the term J_n^i in (65) is concerned, let us recast this expression in terms of a 2-d Riemann sum. Namely we define another uniform partition $\{v_l; 0 \leq l \leq n - 1\}$ of $[0, t]$, with $v_l = \frac{l}{n}t$. Then we start by writing

$$J_n^i = \sum_{k=0}^{n-1} y_{t_k}^{x;ii} (R(t_{k+1}, v_k) - R(t_k, v_k)). \tag{67}$$

In addition, notice that thanks to Remark 2.13 we have $R(\cdot, 0) = 0$. Thus an immediate telescoping sum argument yields the following relation, valid for $k = 0, \dots, n - 1$:

$$R(t_{k+1}, v_k) - R(t_k, v_k) = \sum_{l=0}^{k-1} R_{v_l v_{l+1}}^{t_k t_{k+1}}.$$

Reporting this identity into (67), we get

$$J_n^i = \sum_{k=0}^{n-1} y_{t_k}^{x;ii} \sum_{l=0}^{k-1} R_{v_l v_{l+1}}^{t_k t_{k+1}} = \sum_{0 \leq l < k \leq n-1} y_{t_k}^{x;ii} R_{v_l v_{l+1}}^{t_k t_{k+1}}. \tag{68}$$

This decomposition prompts us to define a degenerate function f in the plane as $f^i(u, v) = y_u^{x;ii} \mathbf{1}_{[0 < v < u < t]}$. With this notation in hand, relation (68) reads

$$J_n^i = \sum_{k,l=0}^{n-1} f^i(t_k, v_l) R_{v_l v_{l+1}}^{t_k t_{k+1}}.$$

In order to analyze the convergence of J_n^i , we now argue as follows: first R has a finite 2-dimensional ρ -variation. The function $f^i(u, v) = y_u^{x;ii} \mathbf{1}_{[0 < v < u < t]}$ is also easily seen to have a finite 2-dimensional p -variation (owing to the fact that $y^{x;ii}$ has finite p -variation), and recall that $p^{-1} + \rho^{-1} > 1$. Hence standard convergence procedures for 2d-Young integrals show that almost surely

$$\lim_{n \rightarrow \infty} J_n^i = \int_0^t \int_0^t f^i(u, v) dR(u, v) = \int_{S_2([0,t])} y_{r_2}^{x;ii} dR(r_1, r_2). \tag{69}$$

Summarizing our considerations for the case $i = j$, we gather (66) and (69) into the decomposition (64). We conclude that almost surely,

$$\lim_{n \rightarrow \infty} \sum_{i=1}^d \sum_{k=0}^{n-1} y_{t_k}^{x;ii} \mathbb{E}[x_{t_k t_{k+1}}^{2;ii}] = \frac{1}{2} \sum_{i=1}^d \int_0^t y_r^{x;ii} dR_r - \sum_{i=1}^d \int_{S_2([0,t])} y_{r_2}^{x;ii} dR(r_1, r_2). \tag{70}$$

We now go back to (61), and handle the terms $\langle \mathbf{D}^i y_{t_k}^i, \mathbf{1}_{[t_k, t_{k+1}]} \rangle_{\mathcal{H}}$ therein. We write the inner product in \mathcal{H} in an explicit way thanks to (38), which yields

$$\langle \mathbf{D}^i y_{t_k}^i, \mathbf{1}_{[t_k, t_{k+1}]} \rangle_{\mathcal{H}} = \int_0^t \int_0^t \mathbf{D}_{r_1}^i y_{t_k}^i \mathbf{1}_{[0, t_k]}(r_1) \mathbf{1}_{[t_k, t_{k+1}]}(r_2) dR(r_1, r_2).$$

We thus have

$$\begin{aligned} & \lim_{n \rightarrow \infty} \sum_{i=1}^d \sum_{k=0}^{n-1} \langle \mathbf{D}^i y_{t_k}^i, \mathbf{1}_{[t_k, t_{k+1}]} \rangle_{\mathcal{H}} \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^d \sum_{k=0}^{n-1} \int_0^t \int_0^t \mathbf{D}_{r_1}^i y_{t_k}^i \mathbf{1}_{[0, t_k]}(r_1) \mathbf{1}_{[t_k, t_{k+1}]}(r_2) dR(r_1, r_2). \end{aligned}$$

We now argue similarly to what we did for (69). Namely one of our standing assumptions is that $(r_1, r_2) \rightarrow \mathbf{D}_{r_1}^i y_{r_2}^i \mathbf{1}_{\mathcal{S}_2}(r_1, r_2)$ has a finite 2-dimensional p -variation. Since R admits a finite ρ -variation and $p^{-1} + \rho^{-1} > 1$, standard results concerning convergence of Riemann sums to Young integrals show that almost surely we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^d \sum_{k=0}^{n-1} \langle \mathbf{D}^i y_{t_k}^i, \mathbf{1}_{[t_k, t_{k+1}]} \rangle_{\mathcal{H}} = \sum_{i=1}^d \int_{\mathcal{S}_2([0, t])} \mathbf{D}_{r_1}^i y_{r_2}^i dR(r_1, r_2). \tag{71}$$

We can now conclude our proof easily. That is plugging (62), (63), (70) and (71) into (61), we end up with, almost surely,

$$\begin{aligned} & \int_0^t y_r dx_r - \int_0^t y_r d^\diamond x_r \\ &= \frac{1}{2} \sum_{i=1}^d \int_0^t y_r^{x; ii} dR_r - \sum_{i=1}^d \int_{\mathcal{S}_2([0, t])} y_{r_2}^{x; ii} dR(r_1, r_2) + \sum_{i=1}^d \int_{\mathcal{S}_2([0, t])} \mathbf{D}_{r_1}^i y_{r_2}^i dR(r_1, r_2), \end{aligned}$$

from which the claim (55) is immediately deduced. This concludes the proof. \square

We close this section by proving a technical result which has been used in order to derive relation (58).

Lemma 3.2. *Assume the same conditions as in Theorem 3.1. Then y^π defined in (56) converges to y in $\mathbb{D}^{1,2}(\mathcal{H}^d)$, i.e. $\lim_{|\pi| \rightarrow 0} \mathbb{E}[\|y^\pi - y\|_{\mathcal{H}^d}^2 + \|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d) \otimes \mathbb{R}^d}^2] = 0$.*

Proof. According to (38), we have

$$\|y^\pi - y\|_{\mathcal{H}^d}^2 = \sum_{i,j=0}^{n-1} \int_{[t_i, t_{i+1}] \times [t_j, t_{j+1}]} \langle y_{t_i} - y_s, y_{t_j} - y_t \rangle dR(s, t),$$

where we recall that $\pi = \{0 = t_0 < t_1 < \dots < t_n = t\}$. On each rectangle $[t_i, t_{i+1}] \times [t_j, t_{j+1}]$ we apply Theorem 2.22 to the function

$$f_{ij}(s, t) = \langle y_s - y_{t_i}, y_t - y_{t_j} \rangle,$$

which is allowed since f_{ij} is easily seen to be a function in $C_2^{p-\text{var}}$.

Recall that we have assumed $p^{-1} + \rho^{-1} > 1$. Throughout the proof, we choose $p' > p$ and $\rho'' > \rho > \rho$ satisfying

$$(p')^{-1} + (\rho')^{-1} > 1 \quad \text{and} \quad (p')^{-1} + (\rho'')^{-1} > 1.$$

Since we also have $f_{ij}(t_i, \cdot) = 0$ and $f_{ij}(\cdot, t_j) = 0$, we get

$$\|y^\pi - y\|_{\mathcal{H}^d}^2 \leq C \sum_{i,j=0}^{n-1} \left(\|y\|_{p'-\text{var}; [t_i, t_{i+1}]} \|y\|_{p'-\text{var}; [t_j, t_{j+1}]} \right) \|R\|_{\rho''-\text{var}; [t_i, t_{i+1}] \times [t_j, t_{j+1}]} \tag{72}$$

In order to bound the right-hand side of (72), we introduce a new function ω_1 , defined by

$$\omega_1([a, b] \times [c, d]) = \|y\|_{p'-\text{var};[a,b]}^{p'} \|y\|_{p'-\text{var};[c,d]}^{p'} \tag{73}$$

Then it is readily checked that ω_1 is also a 2d-control in the sense of Definition 2.7. The following is easily deduced from (72):

$$\begin{aligned} \|y^\pi - y\|_{\mathcal{H}^d}^2 &\leq C \sup_{i,j} \left(\omega([t_i, t_{i+1}] \times [t_j, t_{j+1}]) \right)^{\frac{1}{p'} - \frac{1}{p''}} \\ &\quad \times \sum_{i,j=0}^{n-1} \left(\omega_1([t_i, t_{i+1}] \times [t_j, t_{j+1}]) \right)^{\frac{1}{p'}} \left(\omega([t_i, t_{i+1}] \times [t_j, t_{j+1}]) \right)^{\frac{1}{p''}}, \end{aligned} \tag{74}$$

where ω is the control defined in (15). Now both ω_1 and ω above are 2d-controls. Hence an easy extension of [9, Exercise 1.9] to a 2d setting shows that $\omega_1^{1/p'} \omega^{1/p''}$ is also a 2d-control. Hence one can resort to the super-additivity property of $\omega_1^{1/p'} \omega^{1/p''}$ in order to deduce the following from (74):

$$\|y^\pi - y\|_{\mathcal{H}^d}^2 \leq C \sup_{i,j} \left(\omega([t_i, t_{i+1}] \times [t_j, t_{j+1}]) \right)^{\frac{1}{p'} - \frac{1}{p''}} \left(\omega_1([0, t]^2) \right)^{\frac{1}{p'}} \left(\omega([0, t]^2) \right)^{\frac{1}{p''}}. \tag{75}$$

We now turn to an upper bound on $\|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d)^{\otimes 2}}$. To this aim we first express this quantity using the norm in $(\mathcal{H}^d)^{\otimes 2}$ induced by (38). This yields

$$\begin{aligned} &\|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d)^{\otimes 2}}^2 \\ &= \sum_{i,j=0}^{n-1} \int_{[0,t_{i+1}] \times [0,t_{j+1}] \times [t_i,t_{i+1}] \times [t_j,t_{j+1}]} \langle \mathbf{D}_u y_{t_i} - \mathbf{D}_u y_s, \mathbf{D}_v y_{t_j} - \mathbf{D}_v y_t \rangle \, dR(u, v) dR(s, t). \end{aligned} \tag{76}$$

We apply Lemma 2.23 to the right-hand side of (76) and get

$$\begin{aligned} \|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d)^{\otimes 2}}^2 &\leq C \sum_{i,j=0}^{n-1} \|R\|_{\rho'-\text{var};[0,t_{i+1}] \times [0,t_{j+1}]} \|R\|_{\rho'-\text{var};[t_i,t_{i+1}] \times [t_j,t_{j+1}]} \\ &\quad \times \left(\|\mathbf{D}_0 y_{t_i} - \mathbf{D}_0 y\|_{\rho'-\text{var};[t_i,t_{i+1}]} + \|\mathbf{D}\cdot y_{t_i} - \mathbf{D}\cdot y\|_{\rho'-\text{var};[0,t_{i+1}] \times [t_i,t_{i+1}]} \right) \\ &\quad \times \left(\|\mathbf{D}_0 y_{t_j} - \mathbf{D}_0 y\|_{\rho'-\text{var};[t_j,t_{j+1}]} + \|\mathbf{D}\cdot y_{t_j} - \mathbf{D}\cdot y\|_{\rho'-\text{var};[0,t_{j+1}] \times [t_j,t_{j+1}]} \right). \end{aligned}$$

As a preliminary step, we also bound the variations on intervals of the form $[0, t_j]$ by variations on $[0, T]$. Thus one can bound $\|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d)^{\otimes 2}}^2$ by

$$\begin{aligned} &C \|R\|_{\rho'-\text{var};[0,T]}^2 \sum_{i,j=0}^{n-1} \|R\|_{\rho'-\text{var};[t_i,t_{i+1}] \times [t_j,t_{j+1}]} \\ &\quad \times \left(\|\mathbf{D}_0 y\|_{\rho'-\text{var};[t_i,t_{i+1}]} + \|\mathbf{D}y\|_{\rho'-\text{var};[0,T] \times [t_i,t_{i+1}]} \right) \\ &\quad \times \left(\|\mathbf{D}_0 y\|_{\rho'-\text{var};[t_j,t_{j+1}]} + \|\mathbf{D}y\|_{\rho'-\text{var};[0,T] \times [t_j,t_{j+1}]} \right). \end{aligned} \tag{77}$$

We now wish to apply super-additivity properties of the p -variations, as we did for (75). However, note that the function $[a, b] \times [c, d] \mapsto \|\mathbf{D}y\|_{\rho'-\text{var};[a,b] \times [c,d]}^{p'}$ may fail to be super-additive (see [11, Theorem 1]). Hence we need to resort to the controlled 2d variation as

introduced in Definition 2.9. Specifically, it follows from (77) that $\|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d)^{\otimes 2}}^2$ can be upper bounded by

$$\begin{aligned}
 C \|R\|_{\rho' \text{-var}; [0, T]^2} & \sum_{i, j=0}^{n-1} \|R\|_{\rho' \text{-var}; [t_i, t_{i+1}] \times [t_j, t_{j+1}]} \\
 & \times \left(\|\mathbf{D}_0 y\|_{\rho' \text{-var}; [t_i, t_{i+1}]} + \|\mathbf{D}y\|_{\rho' \text{-var}; [0, T] \times [t_i, t_{i+1}]} \right) \\
 & \times \left(\|\mathbf{D}_0 y\|_{\rho' \text{-var}; [t_j, t_{j+1}]} + \|\mathbf{D}y\|_{\rho' \text{-var}; [0, T] \times [t_j, t_{j+1}]} \right). \tag{78}
 \end{aligned}$$

Notice that the right-hand side of (78) is finite, noting that $p < p'$ and owing to (14). Furthermore, noting that the function $[c, d] \mapsto \|\mathbf{D}y\|_{\rho' \text{-var}; [0, T] \times [c, d]}^{p'}$ is a control, we can define the following 2d controls (where we use [9, Exercise 1.9] again):

$$\omega_2([a, b] \times [c, d]) = \|\mathbf{D}_0 y\|_{\rho' \text{-var}; [a, b]}^{p'} \|\mathbf{D}_0 y\|_{\rho' \text{-var}; [c, d]}^{p'} \tag{79}$$

$$\omega_3([a, b] \times [c, d]) = \|\mathbf{D}y\|_{\rho' \text{-var}; [0, T] \times [a, b]}^{p'} \|\mathbf{D}y\|_{\rho' \text{-var}; [0, T] \times [c, d]}^{p'} \tag{80}$$

$$\omega_4([a, b] \times [c, d]) = \|\mathbf{D}_0 y\|_{\rho' \text{-var}; [a, b]}^{p'} \|\mathbf{D}y\|_{\rho' \text{-var}; [0, T] \times [c, d]}^{p'}. \tag{81}$$

Now, similarly to (75), relation (78) entails

$$\begin{aligned}
 \|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d)^{\otimes 2}}^2 & \tag{82} \\
 \leq C \|R\|_{\rho' \text{-var}; [0, T]^2} & \sup_{i, j} \left(\omega([t_i, t_{i+1}] \times [t_j, t_{j+1}]) \right)^{\frac{1}{p'} - \frac{1}{\rho'}} \left(\omega([0, t]^2) \right)^{\frac{1}{\rho'}} \\
 & \times \sum_{k=2}^4 \left(\omega_k([0, t]^2) \right)^{\frac{1}{p'}},
 \end{aligned}$$

where ω is the control given in (15). This is our desired bound for the difference $\mathbf{D}y^\pi - \mathbf{D}y$.

Let us summarize our considerations so far. Gathering inequalities (75) and (82), we have proved that

$$\begin{aligned}
 \|y^\pi - y\|_{\mathcal{H}^d}^2 + \|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d)^{\otimes 2}}^2 & \tag{83} \\
 \leq C \left(1 + \|R\|_{\rho' \text{-var}; [0, T]^2} \right) & \left(\omega([0, t]^2) \right)^{\frac{1}{\rho'}} \sum_{k=1}^4 \left(\omega_k([0, t]^2) \right)^{\frac{1}{p'}} \\
 & \times \sup_{i, j} \left(\omega([t_i, t_{i+1}] \times [t_j, t_{j+1}]) \right)^{\frac{1}{\rho'} - \frac{1}{p'}},
 \end{aligned}$$

where the controls $\omega, \omega_1, \omega_2, \omega_3, \omega_4$ are respectively defined by (15), (73), (79), (80) and (81). We can now argue as follows: first, according to (17), (14) and (15) we have

$$\lim_{n \rightarrow \infty} \sup_{i, j} \omega([t_i, t_{i+1}] \times [t_j, t_{j+1}]) = 0.$$

Next we have assumed in Theorem 3.1 that

$$\mathbb{E} \left[\|y\|_{p \text{-var}; [0, T]}^2 + \|\mathbf{D}_0 y\|_{p \text{-var}; [0, T]}^2 + \|\mathbf{D}y\|_{p \text{-var}; [0, T]^2}^2 \right] < \infty.$$

Therefore one can take expected valued in (83) in order to get

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\|y^\pi - y\|_{\mathcal{H}^d}^2 + \|\mathbf{D}y^\pi - \mathbf{D}y\|_{(\mathcal{H}^d)^{\otimes 2}}^2 \right] = 0,$$

which is our claim. This concludes our proof. \square

3.2. Comments and examples

We close this section by comparing our main result [Theorem 3.1](#) with existing Skorohod–Stratonovich integral correction formulae, and we also provide examples that satisfy the conditions proposed in [Theorem 3.1](#).

Before starting our comparison, note that Skorohod–Stratonovich correction formulae have been studied for integrals of processes of specific forms (see [\[15,4\]](#)). In contrast, in [Theorem 3.1](#) we consider integrals of generic processes satisfying some proper conditions. This is obviously a more general setting. Below we will discuss the connections between our result and those of [\[15,4\]](#).

3.2.1. Generality of the Gaussian process x

In [\[15\]](#), the relationship [\(1\)](#) is obtained for a γ -Hölder Gaussian process x with $\gamma \in (0, 1)$. The process y considered in [\[15\]](#) is of the special form $y = f(x)$ for $f \in C^{2N}$ with $N = \lfloor \frac{1}{\gamma} \rfloor$.

Our main result [Theorem 3.1](#) holds for Gaussian processes possessing a finite p -variation with $p \in (2, 3)$ (or alternatively the covariance function R satisfies $\rho \in [1, \frac{3}{2})$). This set of assumptions is very close to the one used in [\[4\]](#). The p -variation setting is more general than the Hölder framework put forward in [\[15\]](#), but we do have the restriction $p < 3$ while [\[15\]](#) could handle cases with an arbitrary $p \geq 1$. Below we give some hints about the necessary steps in order to cover more irregular cases for our [Theorem 3.1](#).

Noting that [Propositions 2.27](#) and [2.28](#) hold under [Hypotheses 2.8](#) and [2.12](#) for $p \in (2, 4)$, we believe that our approach could be extended to $p \in (2, 4)$. A key difference between $p \in (2, 3)$ and $p \in [3, 4)$ is that a weighted sum in the third chaos of x will be involved in the rough integral [\(60\)](#) for $p \in [3, 4)$. Thus for $p \in (2, 4)$, to calculate the Skorohod–Stratonovich correction term, we also need to develop an estimation procedure for the weighted sum in the third chaos of x , which is parallel to [Proposition 2.28](#).

Observe that the condition $p \in (2, 3)$ (or $\rho \in [1, \frac{3}{2})$) is also used to define the Young integrals appearing in [\(66\)](#), [\(69\)](#) and [\(71\)](#). However, if we further assume that R and $R(\cdot, t)$ for each $t \in [0, T]$ are absolutely continuous as in [\[15, Hypothesis 3.1\]](#), which is satisfied by fractional Brownian motion B^H with Hurst parameter $H \in (0, 1)$, then it is easily checked that the integrals in [\(66\)](#), [\(69\)](#) and [\(71\)](#) are well-defined as Riemann integrals.

3.2.2. Generality of the integrand y

Apart from the aforementioned case $y = f(x)$ handled in [\[15\]](#), the only rough-Skorohod integral correction formula we are aware of is contained in [\[4\]](#). More specifically in [\[4\]](#) the relationship between $\int_0^t y_r dx_r$ and $\int_0^t y_r d^\circ x_r$ is studied, where x satisfies [Hypotheses 2.8](#) and [2.12](#) and y is the solution to the rough differential equation [\(3\)](#) with σ being sufficiently regular. This is a typical example of application for our [Theorem 3.1](#), if one can verify that the process y considered in [\[4\]](#) satisfy all conditions therein. This task is far from being trivial and is indeed one of the main achievements in [\[4\]](#). Otherwise stated, the main ingredients that are needed in the verification of the assumptions for [Theorem 3.1](#) have already been obtained in [\[4\]](#). To be more precise:

- (i) The condition $\mathbb{E}[\|y\|_{p\text{-var};[0,T]}^2] < \infty$ is a direct consequence of [\[4, Theorem 2.25\]](#).
- (ii) The assumption $\mathbb{E}[\|\mathbf{D}_0 y\|_{p\text{-var};[0,T]}^2] + \mathbb{E}[\|\mathbf{D}y\|_{p\text{-var};[0,T]^2}^2] < \infty$ follows from the relation

$$\mathbf{D}_s y_t = \mathbf{1}_{[0,t)}(s) J_t^x (J_s^x)^{-1} \sigma(y_s), \tag{84}$$

plus an application of [\[4, Theorem 2.27\]](#). See also the end of the proof of [\[4, Proposition 4.10\]](#).

With those relations in mind, our main [Theorem 3.1](#) covers the analysis of y given as the solution to a rough differential equation.

3.2.3. Rough differential equations with drift and other possible extensions

The reader might wonder if [Theorem 3.1](#) encompasses new cases of application, beyond the aforementioned Refs. [\[4,15\]](#). An easy step in this direction is given by rough differential equations with drift, of the form

$$dy_r = b(y_r)dr + \sigma(y_r)d\mathbf{x}_r, \tag{85}$$

where we assume that b and σ are C^∞ -bounded vector fields. In this case the main conditions in [Theorem 3.1](#) are verified as follows:

(i) The bound $\mathbb{E}[\|y\|_{p\text{-var};[0,T]}^2] < \infty$ stems directly from [\[3, Theorem 7.2\]](#).

(ii) The relation

$$\mathbb{E}[\|\mathbf{D}_0 y\|_{p\text{-var};[0,T]}^2] + \mathbb{E}[\|\mathbf{D}y\|_{p\text{-var};[0,T]^2}^2] < \infty \tag{86}$$

also holds for Eq. [\(85\)](#). Indeed, identity [\(84\)](#) still holds true for the rough differential equation with drift. In addition, the p -variation $\|J^x\|_{p\text{-var};[0,T]}$ has finite n -moment for all $n \geq 0$ (see [\[12, Proposition 2.26\]](#) and [\[5, Theorem 6.5\]](#)). This easily yields [\(86\)](#).

In conclusion, [Theorem 3.1](#) also covers the case of rough differential equations with drift.

Let us briefly comment on other possible extensions. As mentioned in the introduction, we firmly believe that our general methodology based on controlled paths with Malliavin calculus conditions (plus convergence of weighted sums of processes with finite chaos expansions) can be applied to a wider variety of contexts. To name a few, our framework can be extended to delay equations [\[19\]](#), Volterra equations [\[6,7,14\]](#) and even to rough PDEs. However, this kind of extension would require some significant additional effort. Indeed, the controlled path structure as well as the building blocks of the rough path are quite different in the cases mentioned above. A detailed analysis would thus be in order. We leave this kind of development to a subsequent publication for the sake of conciseness.

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