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Volterra equations driven by rough signals[☆]

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Abstract

This article is devoted to the extension of the theory of rough paths in the context of Volterra equations with possibly singular kernels. We begin to describe a class of two parameter functions defined on the simplex called Volterra paths. These paths are used to construct a so-called Volterra-signature, analogously to the signature used in Lyon's theory of rough paths. We provide a detailed algebraic and analytic description of this object. Interestingly, the Volterra signature does not have a multiplicative property similar to the classical signature, and we introduce an integral product behaving like a convolution extending the classical tensor product. We show that this convolution product is well defined for a large class of Volterra paths, and we provide an analogue of the extension theorem from the theory of rough paths (which guarantees in particular the existence of a Volterra signature). Moreover the concept of convolution product is essential in the construction of Volterra controlled paths, which is the natural class of processes to be integrated with respect to the driving noise in our situation. This leads to a rough integral given as a functional of the Volterra signature and the Volterra controlled paths, combined through the convolution product. The rough integral is then used in the construction of unique solutions to Volterra equations driven by Hölder noises with singular kernels. An example concerning Brownian noises and a singular kernel is treated.

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1. Introduction and main results

Consider a Volterra equation of the second kind written as

$$u_t = f_t + \int_0^t k_1(t,r)\, b(u_r)\, dr + \int_0^t k_2(t,r)\, \sigma(u_r)\, dx_r, \tag{1.1}$$

where f is some initial condition, b and σ are sufficiently smooth functions, k_1 and k_2 are possibly singular kernels, and x is an irregular signal (typically a (fractional) Brownian motion). Integral equations on this form have several applications to physics, biology or even finance. For example in physics, such equations are used to model viscoelastic materials [12], or in biology these equations may be used to model the spread of epidemics [6]. Volterra equations also play a crucial role in renewal theory [11], and is frequently used in stochastic volatility modelling where singular Volterra kernels in combination with Brownian noise, might be used to generate rough behaviour of the volatility process [1,10].

From a mathematical point of view, Volterra equations have been studied for a long time. At a heuristic level, in order to obtain existence and uniqueness of (1.1) one is typically confronted with the regularity assumption of b and σ, and the regularity of the initial data f as well as the driving noise x. Additionally one needs some type of regularity on the kernels k_1 and k_2. Although the conditions on b, σ and f ensuring existence and uniqueness in (1.1) are generally similar to the case of classical ODEs, the assumption on the noise x and the kernels k_1 and k_2 are more challenging objects to analyse in this context. Typically, one searches for the most general conditions on k_1, k_2 and x in order to still obtain existence and uniqueness for Eq. (1.1).

The introduction of irregular controls in terms of a random or irregular path x, as illustrated in the second integral term in (1.1), has been investigated in stochastic analysis for decades. Most of the early analysis in this field has been done under the assumption that x is a semi-martingale (see e.g. [2,24]), and high regularity of k_2 (i.e. non-singular cases). During the 1990's these equations received much attention from the perspective of white noise theory, see for example [22] for the case of linear equations with non-singular kernels k_2, and [5] for the case of linear equations with singular k_2.

A new direction in stochastic differential equations, called rough path theory, has been initiated in the late 1990's by Terry Lyons (see in particular [20]). In contrast to white noise analysis, the theory of rough paths gives a completely path-wise perspective on differential equations driven by irregular signals. In fact Terry Lyons showed (see [3]) that given an irregular Hölder continuous path x, the construction of a differential calculus with respect to x relied mostly on the ability to define iterated integrals of x. In particular the solution to an ordinary differential equation of the form

$$\dot{y}_t = \sigma(y_t)\dot{x}_t, \qquad y_0 = \xi, \tag{1.2}$$

is obtained as a continuous functional of the noise x, together with its iterated integrals and the initial data ξ. That is, if we let $\mathbf{x} = (\mathbf{x}^1, \mathbf{x}^2, \ldots, \mathbf{x}^n)$ for some $n \geq 1$ with $\mathbf{x}^1 = x$ be the collection of the path x together with its iterated integrals, then the solution y can be viewed as $y_t = \mathcal{I}(\mathbf{x}, \xi)_t$ where $\mathcal{I}$ is a Lipschitz continuous functional in both arguments. Therefore the theory of rough paths not only opens up the analysis of stochastic differential equations to a vast new class of driving stochastic processes, but it also provides simple stability results with respect to that noise. The cost of the improved analytical tractability of the solutions is that non-linear functions in the diffusion term (corresponding to σ in (1.2)) need to be better behaved than in Itô's theory. Typically one requires coefficients which are at least C^2 and bounded in order to get existence and uniqueness of solutions for rough differential equations driven by Brownian motion. This is in contrast to the Lipschitz and linear growth assumptions well known from classical Itô stochastic analysis.

In order to test the robustness of rough path theory, a natural endeavour has been to explore more general differential systems than the ordinary differential equation (1.2). One can think for example of delay equations [21] and cases of stochastic PDEs [7,16], culminating in the theory of regularity structures [17]. During the years 2009–2011, A. Deya and the second author of this paper provided in [8] and [9] a rough path perspective on Volterra equations driven by irregular signals x. In particular they proved that existence and uniqueness hold whenever k_2 is sufficiently regular (i.e. non-singular) and the driving rough path is Hölder continuous with Hölder exponent greater than $1/3$. Notice that in these papers, the authors also discussed the challenges of extending the theory of rough paths in order to include singular kernels in Eq. (1.1). This remained an open question until late 2018, when Prömel and Trabs [23] gave a para-controlled perspective on Volterra equations driven by irregular signals. Highly influenced by the theory of rough paths, the theory of para-controlled distributions developed by Gubinelli, Imkeller and Perkowski [15] gives a pathwise perspective on SDEs and SPDEs through Paley–Littlewood para-controlled calculus, and Bony's para-product. Although the result of Prömel and Trabs is very interesting in itself, it seems to be currently limited in the same way as for the theory of para-controlled calculus. Namely one has to assume that the regularity of the noise x, minus the order of the singularity of the kernel k_2 must be greater than $\frac{1}{3}$. Thus a full rough path "picture" in terms of Lyons' theory is not available at this time through the paracontrolled methodology. It should also be mentioned that paracontrolled distributions are mostly expressed through Fourier modes, which is usually not the natural way to handle nonlinear Volterra type equations.

With the above preliminary considerations in mind, this article is devoted to a complete and comprehensive picture of the theory of rough paths in a Volterra setting with singular kernels. The main idea in order to achieve this goal is to extend the concept of a path $t \mapsto z_t$ to a two variable object $(t,\tau) \mapsto z_t^\tau$ for $(t,\tau) \in \Delta_2$, where Δ_2 is a simplex of two variables. This extension of the notion of path is motivated from the generic form of a Volterra integral

$$z_t^\tau = \int_0^t k(\tau, r)dx_r, \tag{1.3}$$

for some (possibly singular) kernel k and a Hölder continuous function x. Note that by considering the mapping $t \mapsto z_t^t$ we recover the classical well known Volterra integral. However, the main advantage with the splitting of the variables into one variable coming from the kernel and the other coming from the integration limit is the following: the regularity of the mapping $\tau \mapsto z_t^\tau$ is then completely determined by the regularity/singularity of the kernel k, while on the other hand the mapping $t \mapsto z_t^\tau$ is completely determined by the regularity/singularity of the driving noise. While it is the composition of these regularities which yields the regularity of $t \mapsto z_t^t$, the separation of the two arguments allows us to give a framework for Volterra rough paths, similar to the classical rough path framework. More specifically, let E be a Banach space, and consider a two parameter E-valued path z as defined in (1.3). Let Δ_3 denote the three dimensional simplex in $[0,T]^3$. Our main assumption will be the existence of a n-tuple of the form

$$\mathbf{z} = (\mathbf{z}^1, \mathbf{z}^2, \dots, \mathbf{z}^n) : \Delta_3 \to \bigotimes_{i=1}^n E^{\otimes i} \tag{1.4}$$

with $\mathbf{z}^1 = z$, and satisfying a modified Chen type relation

$$\mathbf{z}^{\tau}_{ts} = \mathbf{z}^{\tau}_{tu} * \mathbf{z}^{\cdot}_{us}. \tag{1.5}$$

Notice that in (1.4) the classical tensor product $\otimes$ used in rough path theory is replaced by a bilinear convolution operation $*$. We will go back to this convolution product (which is one of our main ingredients) below. For the time being, let us just notice that it can be defined as a component-wise operation similarly to the classical tensor algebra, i.e.

$$\mathbf{z}^{m,\tau}_{ts} = \sum_{i=0}^{m} \mathbf{z}^{m-i,\tau}_{tu} * \mathbf{z}^{i,\cdot}_{us}. \tag{1.6}$$

With the Volterra structure for $(t,\tau) \mapsto z^{\tau}_{t}$ and the proper definition of the convolution product $*$, we will argue that the solution to a Banach valued Volterra equation

$$y_t = \xi + \int_0^t k(t,r)\sigma(y_r)dx_r, \qquad \xi \in V, \tag{1.7}$$

where V is a Banach space, can be viewed as a continuous functional of the noise $\mathbf{z}$ and the initial data $\xi \in V$. That is, the solution y is given by $y = \mathcal{I}(\mathbf{z}, \xi)$ where $\mathcal{I}$ is Lipschitz continuous in both arguments. It is worth noting that for $k \neq 1$, the element $\mathbf{z} \in \bigotimes_{i=1}^{n} E^{\otimes i}$ given as in (1.4) is fundamentally different from the classical iterated integrals in the theory of rough paths, both algebraically and analytically.

Let us go back to our first goal, namely the path-wise construction of the Volterra paths in (1.3) as well as the algebraic and analytical properties of the associated Volterra-signature (as generalized from the concept of signatures in the theory of rough paths). We begin to show that given an α-Hölder continuous path x and a singular kernel k such that $|k(t,s)| \lesssim |t-s|^{-\gamma}$ and $\alpha - \gamma > 0$, then the path $(t,\tau) \mapsto z^{\tau}_{t}$ is well defined and is contained in a space of two-variable Volterra–Hölder paths which will be specified later. Starting from this object, we will prove that the convolution product given in (1.5) is well defined for any two Volterra paths z and $\tilde{z}$ built from Volterra kernels k and $\tilde{k}$ and driving noise x and $\tilde{x}$ respectively. In fact, intuitively one can think of this operation between z and $\tilde{z}$ as

$$z^{\tau}_{tu} * \tilde{z}^{\cdot}_{us} = \int_u^t dz^{\tau}_{r} \otimes \tilde{z}^{r}_{us}, \tag{1.8}$$

where the increment z^{τ}_{tu} is defined by $z^{\tau}_{tu} = z^{\tau}_{t} - z^{\tau}_{u}$. In (1.8), note that the integration is done with respect to the upper parameter in $\tilde{z}$ (corresponding to a regularity coming from the kernel $\tilde{k}$) and the lower variable in z (representing the regularity coming from the driving noise x). This operation will be extended to any two Volterra type objects in the n-tuple $\mathbf{z}$, and leads naturally to the algebraic relation in (1.5). Let us also mention at this stage that the Hölder type norm under consideration in this paper, taking into account both the regularity coming from the kernel k and the noise x, will be given in the following way for the component $\mathbf{z}^i$ of $\mathbf{z}$ (below we have $\alpha, \gamma \in (0,1)$),

$$|\mathbf{z}^{i,\tau}_{ts}| \lesssim |\tau - t|^{-\gamma}|t-s|^{i(\alpha-\gamma)+\alpha}, \tag{1.9}$$

where we omit some of the other regularities to be considered for sake of clarity. As mentioned above, expression (1.9) is thus separating a singularity of order γ on the diagonal $t = \tau$ from the α-Hölder regularity in $t - s$. The object $\mathbf{z}$ satisfying (1.8) and (1.9) is called a Volterra rough path. In order to provide a full picture of the construction of these objects, we include in this article a generalization of the Sewing lemma [14, Proposition 1], as well as of the rough path extension theorem (see e.g. [3, Theorem 3.7]) in the Volterra context.

Once the construction of a Volterra rough path is secured, our second goal is concerned with the construction of solutions to (1.7). To this end, we will extend the theory of controlled rough paths, as described by Gubinelli in [14], to the Volterra rough path setting. Observe that this extension also relies upon the convolution product $*$ introduced in (1.5). In particular, a Volterra path $(t, \tau) \mapsto y_t^\tau$ controlled by the Volterra noise $(t, \tau) \mapsto z_t^\tau$ given as in (1.3) satisfies

$$y_{ts}^\tau = z_{ts}^\tau * y_s^{\prime,\tau,\cdot} + R_{ts}^\tau, \tag{1.10}$$

where we recall the notation $y_{ts}^\tau = y_t^\tau - y_s^\tau$, and where R_{ts}^τ is a sufficiently regular remainder term. Processes of the form (1.10) are the ones which can be naturally integrated with respect to x in the rough Volterra sense. Furthermore, once a rough integral is defined for a large enough class of processes and one can prove the stability of the structure (1.10) under composition with a nonlinear mapping, equations like (1.7) are solved thanks to a standard fixed point argument.

Let us now say a few words about the regularity of x and the singularity of k on the diagonal in Eq. (1.7). We believe that, provided they can be pushed to arbitrary orders, expansions like (1.10) yield a proper notion of Volterra rough type integral as long as k is a singular kernel of order $-\gamma$ and x is a α-Hölder continuous noise with $\alpha - \gamma > 0$. However, for sake of conciseness, this article is restricted to the case $\alpha - \gamma > \frac{1}{3}$. In this situation one only needs to assume the existence of the second step Volterra iterated integral $\mathbf{z}^2$, and the first order controlled path structure (1.10) is enough for our purposes. In the forthcoming article [19] we investigate this problem in more details, and extend the theory presented in the current article to the case of $\alpha - \gamma > \frac{1}{4}$. Rougher situations and more singular kernels are deferred to a further publication. For the construction of a Volterra rough path, one should also be aware of the fact that the concept of geometric rough paths is not directly transferable to the Volterra setting. Simply put, if k is a singular kernel one cannot expect to have a satisfying integration by parts formula (at least not in a classical sense) when integrating against k. Therefore the Volterra rough path $\mathbf{z}$ defined by (1.3) and (1.4) is in general not a continuous function of the classical rough path above x. We thus expect some of the algebraic considerations related to the Volterra case to be different from the classical rough path theory, possibly requiring the regularity structures techniques of [17]. The construction of Volterra rough paths above standard stochastic processes is deferred to a subsequent publication.

Below we give a brief outline of the sections in the paper.

(i) Section 2 provides the elementary tools of rough path theory and fractional calculus needed in order to develop our framework in the sequel.

(ii) Section 3 gives an introduction to the concept of Volterra iterated integrals and Volterra signatures in the case of smooth driving noise x, possibly involving a singular kernel k. In this section we will encounter the convolution product $*$ for the first time and give a detailed description of this product. We will also provide a working hypothesis on the regularity of the kernel k which will be used throughout the rest of the text.

(iii) In Section 4 we move to the case when the driving noise x of a Volterra path is only α-Hölder continuous with $\alpha \in (0, 1)$. We construct a generalized space of Volterra–Hölder paths, and give a pathwise construction of the Volterra process given by (1.3) sitting in this Volterra–Hölder space. Furthermore, we prove that the convolution product is well defined for any Volterra path. This results in the definition of a convolutional path (obtained as an extension of Lyon's concept of multiplicative paths) and then the creation of the Volterra signature from such paths. Both algebraic and analytic aspects of these objects are discussed.

(iv) Section 5.1 deals with the extension of the rough path theory to the Volterra equations case, through the introduction of the Volterra signature and the convolution product defined in Section 4. To this end we define a class of Volterra controlled paths, and prove that the Volterra integral and the operation of composition with regular functions are continuous operations on this class of functions. This is then used in Section 5.2 to show existence and uniqueness of Volterra integral equations on the form of (1.7) with singular kernel k and rough driving noise x.

2. Preliminary notions

This section is devoted to some preliminary notations and notions of classical rough paths, which will help to understand our considerations in the Volterra case. We start with some general notation in Section 2.1, and recall some notions of rough path analysis in Section 2.2.

2.1. General notation

We will frequently use Banach spaces E, V and H, and write $|\cdot| = \|\cdot\|_E$ as long as this does not leave any confusion. Throughout we will write $a \lesssim b$ meaning that there exists a constant $C > 0$ such that $a \leq Cb$. We will denote by $\Delta_n([a,b])$ the n-simplex over $[a,b]$ defined by

$$\Delta_n([a,b]) = \{(x_1,\dots,x_n) \in [a,b]^n \mid a \leq x_1 < \cdots < x_n \leq b\}, \tag{2.1}$$

and when the set $[a,b]$ is clear from context we will just write Δ_n.

The kernels involved in equations like (1.1) are closely related to fractional integral operators. We will mostly use the operator $I^\alpha : L^1([0,T];E) \to L^1([0,T];E)$, which is defined for a given $\alpha > 0$ and $(u,t) \in \Delta_2$ as follows:

$$I^\alpha_{u+}(f)(t) := \frac{1}{\Gamma(\alpha)} \int_u^t (t-r)^{\alpha-1} f(r)\, dr,$$

where Γ denotes the Gamma function. Fractional integrals have been widely studied in the literature, and we refer to [25] for a thorough account on the topic. However, we mention here a few properties of the operators I^α which will be frequently used. Most important is the convolution property; for $\alpha, \beta > 0$ and $(u,t) \in \Delta_2$:

$$I^\alpha_{u+}\left(I^\beta_{u+}(f)\right)(t) = I^{\alpha+\beta}_{u+}(f)(t). \tag{2.2}$$

We will also use the following action of I^α_{u+} on elementary functions

$$I^\alpha_{u+}(1)(t) = \frac{(t-u)^\alpha}{\Gamma(\alpha+1)}, \quad \text{and} \quad I^\alpha_{u+}((\cdot - u)^\beta)(t) = \frac{(t-u)^{\alpha+\beta}\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}, \tag{2.3}$$

where $\alpha, \beta > 0$. Throughout the article we will rely on partitions of intervals. A partition over an interval $[a,b]$ will be denoted by $\mathcal{P}[a,b]$. If the interval $[a,b]$ is clear from the context we may write $\mathcal{P}$. Throughout this article we will work with increments of functions, which for $(s,t) \in \Delta_2$ will be denoted by

$$f_{ts} = f_t - f_s. \tag{2.4}$$

We ask the reader to note that the order of t and s in f_{ts} is changed from the traditional notation used in rough path theory. This is to accommodate the algebraic side of the Volterra specific setting we will encounter in later sections. For $\alpha \in (0,1)$, we will denote by $\mathcal{C}^\alpha(I;E)$ the space of Hölder continuous functions from an interval I to a Banach space E. If I is reduced to a singleton $\{t\}$ then

$$\mathcal{C}^{\alpha}(\{t\}; E) := \left\{ f : V_t \to E \mid \sup_{s \in V_t} \frac{|f_{ts}|}{|t-s|^{\alpha}} < \infty \right\}, \tag{2.5}$$

where V_t stands for a neighbourhood of $\{t\}$. Furthermore, we will frequently use an operator δ well known in the theory rough paths, given by

$$\delta_u f_{ts} = f_{ts} - f_{tu} - f_{us}. \tag{2.6}$$

2.2. *Short introduction to rough path theory*

In this section we recall some basic notions about signatures of paths and related geometric structures appearing in rough path theory, which will make the generalization to Volterra type objects more natural. For a more extensive introduction to rough path theory, see e.g. [3,13,18].

2.2.1. *Signatures*

One natural way to introduce signatures of paths is to see how they arise from expansions of linear differential equations. Namely assume first the path $x : [0, T] \to E$ is smooth, where E is a given Banach space. Let V be another Banach space and consider the V-valued ODE

$$\dot{y}_t = A(\dot{x}_t)\, y_t, \quad y_0 = \xi \in V, \tag{2.7}$$

where A is a linear operator, namely $A \in \mathcal{L}(E, \mathcal{L}(V))$. Whenever x is smooth, a Picard type iteration yields the following expansion:

$$y_t = \xi \left(1 + \sum_{i=1}^{\infty} A^{\circ i} \left(\int_{\Delta_i([0,t])} dx_{r_1} \otimes \cdots \otimes dx_{r_i} \right) \right), \tag{2.8}$$

where $A^{\circ i}$ is the ith composition of the linear operator A which is given as a linear operator on $E^{\otimes i}$ defined from the action

$$A^{\circ i}(x_1 \otimes \cdots \otimes x_i) := A(x_1) \circ \cdots \circ A(x_i).$$

The expansion (2.8) reveals that y can be seen as a continuous function of the collection $\{\int_{\Delta_i([0,t])} dx_{r_1} \otimes \cdots \otimes dx_{r_i};\ i \geq 1\}$, which is called the signature of x.

In order to describe the algebraic structures behind the expansion (2.8), let us first give some definitions.

Definition 1. Let E be a real Banach space. For $l \in \mathbb{N}$, the truncated algebra $T^{(l)}$ is defined by $T^{(l)} = \bigoplus_{n=0}^{l} E^{\otimes n}$, with the convention $E^{\otimes 0} = \mathbb{R}$. The set $T^{(l)}$ is equipped with a straightforward vector space structure, plus an operation $\otimes$ defined by

$$[g \otimes h]^n = \sum_{k=0}^{l} g^{n-k} \otimes h^k, \qquad g, h \in T^{(l)}, \tag{2.9}$$

where g^n designates the projection on the nth tensor level for $n \leq l$.

Notice that $T^{(l)}$ should be denoted $T^{(l)}(E)$. We have dropped the dependence on E for notational sake. Also observe that with Definition 1 in hand, $(T^{(l)}, +, \otimes)$ is an associative algebra with unit element $\mathbf{1} \in E^{\otimes 0}$. The polynomial terms in the expansions which will be considered later on are contained in a subspace of $T^{(l)}$ that we proceed to define now.

Definition 2. The *free nilpotent Lie algebra* $\mathfrak{g}^{(l)}$ of order l is defined to be the graded sum

$$\mathfrak{g}^{(l)} \triangleq \bigoplus_{k=1}^{l} \mathcal{L}_k \subseteq T^{(l)}.$$

Here $\mathcal{L}_k$ is the space of homogeneous Lie polynomials of degree k given inductively by $\mathcal{L}_1 \triangleq E$ and $\mathcal{L}_k \triangleq [E, \mathcal{L}_{k-1}]$, where the Lie bracket is defined to be the commutator of the tensor product.

We now define some groups related to the algebras given in Definitions 1 and 2. To this aim, introduce the subspace $T_0^{(l)} \subseteq T^{(l)}$ of tensors whose scalar component is zero and recall that $\mathbf{1} \triangleq (1, 0, \dots, 0)$. For $u \in T_0^{(l)}$, one can define the inverse $(1+u)^{-1}$, the exponential $\exp(u)$ and the logarithm $\log(1+u)$ in $T^{(l)}$ by using the standard Taylor expansion formula with respect to the tensor product. For instance,

$$\exp(u) \triangleq \sum_{k=0}^{\infty} \frac{1}{k!} u^{\otimes k} \in T^{(l)}, \tag{2.10}$$

where the sum is indeed locally finite and hence well-defined. We can now introduce the following group.

Definition 3. The *free nilpotent Lie group* $G^{(l)}$ of order l is defined by

$$G^{(l)} \triangleq \exp(\mathfrak{g}^{(l)}) \subseteq T^{(l)}.$$

The exponential function is a diffeomorphism under which $\mathfrak{g}^{(l)}$ in Definition 2 is the Lie algebra of $G^{(l)}$.

As mentioned above, the link between free groups and differential equations like (2.7) is made through the notion of signature. Namely a continuous map $\mathbf{x} : \Delta_2 \to T^{(l)}$ is called a multiplicative functional if for $s < u < t$ one has $\mathbf{x}_{ts} = \mathbf{x}_{tu} \otimes \mathbf{x}_{us}$, where $\otimes$ is the operation introduced in Definition 1. A particular occurrence of this kind of map is given when one considers a smooth path w and sets for $(s,t) \in \Delta_2$,

$$\mathbf{w}_{ts}^{n} = \int_{t>r_n>\cdots>r_1>s} dw_{r_n} \otimes \cdots \otimes dw_{r_1}. \tag{2.11}$$

Then the so-called *signature* of w is the following object:

$$S_l(w) : \Delta_2([0,1]) \to T^{(l)}, \qquad (s,t) \mapsto S_l(w)_{ts} := 1 + \sum_{n=1}^{l} \mathbf{w}_{ts}^{n}. \tag{2.12}$$

It is worth mentioning that $S_l(w)$ will be our typical example of a multiplicative functional. In addition, signatures of paths belong to the group $G^{(l)}$ introduced in Definition 3 and in fact any element in $G^{(l)}$ can be written as the signature of a smooth path.

Another important property in the theory of signatures, originally proved by Chen [4], relates the multiplicative property to the signature of the concatenation of two paths. That is, if $x : [0,s] \to E$ and $y : [s,t] \to E$, we can define their concatenation $x \star y : [0,t] \to E$ by the mapping

$$[x \star y]_r = \begin{cases} x_r & r \in [0,s] \\ x_s + y_{rs} & r \in [s,t] \end{cases}. \tag{2.13}$$

Then if S_l is the truncated signature of a path as described in (2.12), we get the following relation, whose proof can be found e.g. in [3, Theorem 2.9]:

$$S_l(x \star y) = S_l(y) \otimes S_l(x). \tag{2.14}$$

One can now go back to the expansion (2.8), and realize that it can be expressed in terms of the signature of the path x. Whenever x is smooth, the terms $\mathbf{x}^n$ exhibit a factorial decay, which kill the possibly exponential growth from $A^{\otimes n}$. This fact is not obvious anymore in case of an irregular path x, which motivates the notion of rough path introduced below.

2.2.2. Rough path lift of a Hölder path

Let us now assume that the path x driving (2.7) is only α-Hölder continuous with $\alpha \in (0, 1)$. Then the iterated integrals appearing in the expansion in Eq. (2.8) are possibly not well defined. In particular when the continuity of the driving signal is of order $\alpha \leq \frac{1}{2}$, there is no canonical way of constructing such integrals. The seminal idea put forward by T. Lyons is that one can construct those iterated integrals by means of probabilistic tools, and then build a differential calculus with respect to x starting from the iterated integrals. Those considerations motivate the introduction of Hölder continuous multiplicative functionals.

Definition 4. Consider $\alpha \in (0, 1)$ and let $n = \lfloor \frac{1}{\alpha} \rfloor$. Let $x \in \mathcal{C}^\alpha([0, T]; E)$ be a Hölder path and assume there exists an object $\mathbf{x} : \Delta_2 \to G^{(n)}(E)$ defined through the mapping

$$(s, t) \mapsto \mathbf{x}_{ts} := \left(1, \mathbf{x}^1_{ts}, \mathbf{x}^2_{ts}, \dots, \mathbf{x}^n_{ts}\right),$$

where $\mathbf{x}^1_{ts} := x_t - x_s$ and where we recall that $G^{(n)}$ is introduced in Definition 3. In addition, we suppose that $\mathbf{x}$ enjoys the following two properties:

$$\mathbf{x}_{tu} \otimes \mathbf{x}_{us} = \mathbf{x}_{ts} \quad \text{(Multiplicative property)} \tag{2.15}$$

and

$$|\mathbf{x}^i_{ts}| \leq \|x^1\|^i_\alpha \frac{|t-s|^{i\alpha}}{\Gamma(i\alpha + 1)} \quad \text{for all } i \in \{1, \dots, n\} \qquad \text{(Analytic property).}$$

Here Γ is the Gamma function. Then we call $\mathbf{x}$ a rough path above x and we denote the space of all α-Hölder rough paths by $\mathscr{C}^\alpha([0, T]; E)$. Whenever the underlying domain and range of this space is clear from context, we simply write $\mathscr{C}^\alpha$.

Note that $\mathscr{C}^\alpha$ is not a vector space. Indeed, $\mathscr{C}^\alpha$ is not a linear space due to the fact that $G^{(n)}$ is not a linear space. However, we can equip $\mathscr{C}^\alpha$ with the following metric:

$$d_\alpha(\mathbf{x}, \mathbf{y}) := \sum_{i=1}^n \|\mathbf{x}^i - \mathbf{y}^i\|_{i\alpha}. \tag{2.16}$$

One can also consider a subspace of this space called the space of geometric rough paths and denoted by $\mathscr{C}^\alpha_g$, which is defined as the closure of all smooth rough paths with respect to the metric d_α given by (2.16). Otherwise stated, $\mathbf{x} \in \mathscr{C}^\alpha$ is a geometric rough path if there exists a sequence of smooth paths $\{\mathbf{x}^n\} : \Delta_2 \to G^{(n)}(E)$ such that $d_\alpha(\mathbf{x}^n, \mathbf{x})$ converges to 0.

The next theorem will give us a canonical extension of the rough path from the truncated space $T^{(n)}(E)$ to all the space $T(E)$. This extension is crucial in order to ensure the existence and uniqueness of linear differential equations controlled by irregular noise. The theorem and its proof can be found in [3, Theorem 3.7].

Theorem 5. *Let $\mathbf{x} \in \mathscr{C}^\alpha$ be a rough path of order $\alpha \in (0,1)$ and let $n = \lfloor \frac{1}{\alpha} \rfloor$. Then there exists a unique extension of $\mathbf{x}$ to the space $T(E)$ which satisfies the multiplicative and analytic property. That is, for all $m \geq n+1$ there exists an object $\mathbf{x}^m : \Delta_2 \to E^{\otimes m}$ such that*

$$\mathbf{x}_{ts}^m = \sum_{i=0}^{m} \mathbf{x}_{tu}^{m-i} \otimes \mathbf{x}_{us}^i,$$

and for all $(s,t) \in \Delta_2$ we have

$$|\mathbf{x}_{ts}^i| \leq \|\mathbf{x}^1\|_\alpha^i \frac{|t-s|^{i\alpha}}{\Gamma(i\alpha+1)} \quad \forall i \geq 1.$$

Notice that Theorem 5 tells us that in order to construct the solution to a rough differential equation in terms of its signature, we just need to give a probabilistic construction of the first $n = \lfloor \frac{1}{\alpha} \rfloor$ iterated integrals. Then we know that all higher order iterated integrals have a canonical (and deterministic) construction only depending on the lower order integrals. We will try to reproduce this mechanism in the Volterra context.

3. Volterra signatures

3.1. Definition and first properties

In this section we will define precisely what we mean by a Volterra signature over a smooth path. In this way the Volterra type integrals will be trivially defined and we can focus on their algebraic and analytic properties. This gives some insight on what can be expected in more irregular cases. First we need to present an elementary inequality we will use later (see e.g. [8, Lemma 4.4] for more details).

Lemma 6. *Let $\beta \in [0,1]$, $\gamma > 0$, and $0 \leq r \leq q \leq \tau \leq T$. Then the following inequality holds*

$$|(\tau-r)^{-\gamma} - (q-r)^{-\gamma}| \leq (\tau-q)^\beta (q-r)^{-\gamma-\beta}.$$

Our constructions will rely on specific assumptions about the power type singularity of the kernel k appearing in (1.1). Inspired by Lemma 6, the main hypothesis we shall use can be summarized as follows.

H: Let k be a kernel $k : \Delta_2 \to \mathbb{R}$. We assume that there exists $\gamma \in (0,1)$ such that for all $(s,r,q,\tau) \in \Delta_4([0,T])$ and $\eta, \beta \in [0,1]$ we have

$$|k(\tau,r)| \lesssim |\tau-r|^{-\gamma}, \tag{3.1}$$

$$|k(\tau,r) - k(q,r)| \lesssim |q-r|^{-\gamma-\eta}|\tau-q|^\eta, \tag{3.2}$$

$$|k(\tau,r) - k(\tau,s)| \lesssim |\tau-r|^{-\gamma-\eta}|r-s|^\eta, \tag{3.3}$$

$$|k(\tau,r) - k(q,r) - k(\tau,s) + k(q,s)| \lesssim |q-r|^{-\gamma-\beta}|r-s|^\beta, \tag{3.4}$$

$$|k(\tau,r) - k(q,r) - k(\tau,s) + k(q,s)| \lesssim |q-r|^{-\gamma-\eta}|\tau-q|^\eta. \tag{3.5}$$

Here all the inequalities $\lesssim$ are independent of the parameters γ, β and η. In the sequel a kernel fulfilling condition **(H)** will be called Volterra kernel of order $-\gamma$.

Remark 7. If a kernel k satisfies (**H**) then by the interpolation inequality $a \wedge b \leq a^{\theta} b^{1-\theta}$ for any $\theta \in [0,1]$ applied to the minimum of (3.4) and (3.5) it follows that for any $\beta, \eta \in [0,1]$ we have

$$|k(\tau, r) - k(q, r) - k(\tau, s) + k(q, s)| \lesssim |\tau - q|^{\eta} |q - r|^{-\beta-\gamma-\eta} |r - s|^{\beta}. \tag{3.6}$$

With those assumptions in hand we can now introduce the notion of iterated Volterra integral and Volterra signature, which parallel (2.11) and (2.12).

Definition 8. Let us consider a path $x \in C^1([0,T];E)$ and a Volterra kernel $k : \Delta_2 \to \mathbb{R}$ satisfying (**H**). The *iterated Volterra integral* of order n is a mapping $\mathbf{z}^n : \Delta_3 \to E^{\otimes n}$ given by

$$(s,t,\tau) \mapsto \mathbf{z}^{n,\tau}_{ts} = \int_{t>r_n>\cdots>r_1>s} k(\tau, r_n) \bigotimes_{j=1}^{n-1} k\left(r_{j+1}, r_j\right) dx_{r_j}. \tag{3.7}$$

We also consider the collection of iterated Volterra integrals as an element of the free algebra. Specifically, we define the element $\mathbf{z}^{\tau}_{ts} \in T^{(\infty)}(E)$ as follows:

$$\mathbf{z}^{\tau}_{ts} = \left(1, \mathbf{z}^{1,\tau}_{ts}, \ldots, \mathbf{z}^{n,\tau}_{ts}, \ldots\right),$$

where we recall that the spaces $T^{(\infty)}(E)$ is introduced in Definition 3.

Remark 9. As already highlighted in the introduction, notice that in the definition (3.7) the variable τ is considered as an additional parameter indexing z. While we might be mostly interested in the case $\tau = t$, this extra freedom will play an essential role in our considerations.

Remark 10. Observe that the Volterra integrals are denoted by $(s,t,\tau) \mapsto \mathbf{z}^{n,\tau}_{ts}$ as opposed to $(s,t) \mapsto \mathbf{z}^n_{st}$ in the regular rough path setting. This small modification will ease our notation when one has to deal with integrals of the form $\int_0^t k(t,r) f_r dx_r$.

Remark 11. A particularly important note is that the collection of Volterra iterated integrals $\mathbf{z} = (1, \mathbf{z}^1, \ldots)$ is not contained in the free nilpotent Lie group of G given in Definition 3. We expect that one needs a different algebraic approach to these integrals due to the kernels k involved in the integrals. Especially in the singular case it is quite intuitive that Volterra iterated integrals do not lie in the free nilpotent lie group, as there is no concept of integration by parts. That is, let x^i and x^j be two real valued smooth paths, and consider the second level $\mathbf{z}^2$. Then observe that a simple integration by parts would yield

$$\begin{aligned} \int_{t>r>u>s} k(t,r)k(r,u)dx^i_u dx^j_r &= \int_s^t k(t,r)dx^i_r \int_s^t k(t,r)dx^j_r \\ - \int_{t>r>u>s} k(t,r)k(r,r)dx^j_u dx^i_r &- \int_{t>r>u>s} k(t,r)\frac{d}{dr}k(r,u)dx^j_u dx^i_r. \end{aligned} \tag{3.8}$$

However, since k is singular we have $k(r,r) = \infty$, and the derivative $\frac{d}{dr}k(r,u)$ would no longer be integrable. This additional singularity prevents us to exhibit a bracket defined as the commutator of the tensor product in Definition 2 (here considered for the second level term). Therefore a deeper investigation into the algebraic properties of the Volterra iterated integrals given in (3.7) would be highly interesting, and we hope to tell more on this aspect in the future.

When x is a smooth function, iterated Volterra integrals enjoy a regularity property which is similar to the analytic property in Definition 4. This is labelled in the following proposition.

Proposition 12. *Let $k : \Delta_2([0,T]) \to \mathbb{R}$ be a Volterra kernel which satisfies* (**H**) *with $\gamma < 1$, and assume x is a continuously differentiable function. For $n \geq 1$, consider the path $\mathbf{z}^{n,\tau}$ defined by* (3.7)*. Then for $(s,t) \in \Delta_2([0,T])$ we have that*

$$|\mathbf{z}^{n,\tau}_{ts}| \leq \frac{\left(\|x\|_{\mathcal{C}^1}\Gamma(1-\gamma)\right)^n}{\Gamma\left(n\left(1-\gamma\right)\right)}\left(\tau-s\right)^{-\gamma}\left(t-s\right)^{(n-1)(1-\gamma)+1},$$

where the $\mathcal{C}^1$ norm of x is defined by $\|x\|_{\mathcal{C}^1} := \sup_{t\in[0,T]}(|x_t| + |\dot{x}_t|)$.

Proof. Starting from (3.7) and invoking the fact that x is a $\mathcal{C}^1$ function, we directly get

$$\begin{aligned}|\mathbf{z}^{n,\tau}_{ts}| &= \left|\int_{t>r_n>\cdots>r_1>s} k(\tau,r_n)\bigotimes_{j=1}^{n-1} k\left(r_{j+1},r_j\right)dx_{r_j}\right| \\ &\leq \int_{t>r_n>\cdots>r_1>s} |k\left(\tau,r_n\right)|\prod_{i=1}^{n-1}|k\left(r_{i+1},r_i\right)||\dot{x}_{r_1}|\otimes\cdots\otimes|\dot{x}_{r_n}|dr_1\cdots dr_n.\end{aligned}$$

Therefore hypothesis (**H**) on the kernel k entails

$$\begin{aligned}|\mathbf{z}^{n,\tau}_{ts}| &\leq \|x\|^n_{\mathcal{C}^1}\int_{t>r_n>\cdots>r_1>s}\left(\tau-r_n\right)^{-\gamma}\prod_{i=1}^{n-1}\left(r_{i+1}-r_i\right)^{-\gamma}dr_1\cdots dr_n \\ &= \|x\|^n_{\mathcal{C}^1}\Gamma\left(1-\gamma\right)^{n-1}\int_s^t\left(\tau-r\right)^{-\gamma}I^{(n-1)(1-\gamma)}_{s+}\left(1\right)\left(r\right)dr,\end{aligned}$$

where we have used the convolution property (2.2) of the Riemann–Liouville integral operator I^α described in Section 2.1. Furthermore, it follows from the identities in (2.3) that

$$\begin{aligned}&\int_s^t\left(\tau-r\right)^{-\gamma}I^{(n-1)(1-\gamma)}_{s+}\left(1\right)\left(r\right)dr = c_{n,\gamma}\int_s^t\left(\tau-r\right)^{-\gamma}\left(r-s\right)^{(n-1)(1-\gamma)}dr \\ &= c_{n,\gamma}\left(t-s\right)^{(n-1)(1-\gamma)+1}\left(\tau-s\right)^{-\gamma}\int_0^1\left(1-\theta\frac{t-s}{\tau-s}\right)^{-\gamma}\theta^{(n-1)(1-\gamma)-1}d\theta,\end{aligned} \tag{3.9}$$

where we have used the notation $c_{n,\gamma} = [\Gamma((n-1)(1-\gamma)+1)]^{-1}$ and the substitution $r = s+\theta\left(t-s\right)$. In addition, since $\tau \geq t$, it is clear that

$$\int_0^1\left(1-\theta\frac{t-s}{\tau-s}\right)^{-\gamma}\theta^{(n-1)(1-\gamma)-1}d\theta \leq B\left(1-\gamma,\left(n-1\right)\left(1-\gamma\right)\right), \tag{3.10}$$

where B is the Beta function. Observe that classical identities for Gamma and Beta functions, yields that

$$\frac{B\left(1-\gamma,\left(n-1\right)\left(1-\gamma\right)\right)}{\Gamma\left(\left(n-1\right)\left(1-\gamma\right)\right)} = \frac{\Gamma(1-\gamma)}{\Gamma((n-1)(1-\gamma)+1)} \tag{3.11}$$

plugging relation (3.11) into (3.10) and then (3.9) we have then obtained

$$\int_s^t\left(\tau-r\right)^{-\gamma}I^{(n-1)(1-\gamma)}_{s+}\left(1\right)\left(r\right)dr \leq \frac{\Gamma(1-\gamma)}{\Gamma\left((n-1)(1-\gamma)+1\right)}(\tau-s)^{-\gamma}(t-s)^{(n-1)(1-\gamma)+1},$$

which is our claim. □

3.2. Convolution product

We will now try to get an equivalent to the multiplicative property of the signature (2.15) in a Volterra context. Unfortunately this property does not hold directly for a Volterra rough path, due to the interaction between variables in the kernel k. However, we will show that if we modify the tensor product to be a type of convolution product, then we still get a concatenation type property under this product.

Proposition 13. *Let $(s,u,t) \in \Delta_3$. Consider two C^1 functions $x : [s,u] \to E$ and $y : [u,t] \to E$, and denote by $q = x \star y$ their concatenation. Let $\mathbf{z}^n$ be the nth Volterra integral of q on (s,t) as defined in (3.7), namely for all $(s,t,\tau) \in \Delta_3$ set*

$$\mathbf{z}_{ts}^{n,\tau} := \int_{t>r_n>\cdots>r_1>s} \bigotimes_{j=n}^{1} k\left(r_{j+1}, r_j\right) dq_{r_j},$$

with the convention that $r_{n+1} = \tau$. Then for $(s,u,t,\tau) \in \Delta_4$ we have

$$\mathbf{z}_{ts}^{n,\tau} = \sum_{i=0}^{n} \mathbf{z}_{tu}^{n-i,\tau} * \mathbf{z}_{us}^{i,\cdot}, \tag{3.12}$$

where the convolution product $*$ is defined as follows for all $0 \leq i \leq n$

$$\begin{aligned} &\mathbf{z}_{tu}^{n-i,\tau} * \mathbf{z}_{us}^{i,\cdot} \\ &:= \int_{t>r_n>\cdots>r_{i+1}>u} \bigotimes_{j=n}^{i+1} k\left(r_{j+1}, r_j\right) dy_{r_j} \otimes \int_{u>r_i>\cdots>r_1>s} k\left(r_{i+1}, r_i\right) \bigotimes_{j=i-1}^{1} k\left(r_{j+1}, r_j\right) dx_{r_j}. \end{aligned} \tag{3.13}$$

Here we have used the convention $\mathbf{z}^0 \equiv 1$ and $\mathbf{z}^n * 1 = 1 * \mathbf{z}^n = \mathbf{z}^n$.

Proof. This proof is left to the patient reader. The result is easily checked by splitting the domain

$$\Delta_n([s,t]) = \{(r_1,\ldots,r_n) \in [s,t] \mid t > r_n > \cdots > r_1 > s\}$$

into sub-domains

$$\Delta_{n,j} = \{(r_1,\ldots,r_n) \in [s,t] \mid t > \cdots > r_{j+1} > u > r_j > \cdots > s\}. \quad \square$$

Remark 14. In order to make formula (3.13) more concrete, let us explicitly compute the integrals we obtain for $n=2$. In this case relation (3.12) reads

$$\mathbf{z}_{ts}^{2,\tau} = \mathbf{z}_{tu}^{2,\tau} + \mathbf{z}_{tu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot} + \mathbf{z}_{us}^{2,\tau}, \tag{3.14}$$

and we observe that

$$\mathbf{z}_{tu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot} = \int_{t>r_2>u} k(\tau, r_2) dx_{r_2} \otimes \int_{u>r_1>s} k(r_2, r_1) dx_{r_1}, \tag{3.15}$$

where we note the common integration variable r_2 in the above product. In relation (3.13), we also notice that since the kernel k is smooth except on the diagonal, the function

$$l_{r_2} = \int_{u>r_1>s} k(r_2, r_1) dx_{r_1}$$

inherits the smoothness of k. Therefore the integral

$$\int_{t>r_2>u} k(\tau, r_2) dx_{r_2} \otimes l_{r_2},$$

which features in (3.15), can be interpreted as a Riemann–Stieltjes integral. One of our main tasks will then be to control possible singularities arising from k when x is no longer assumed to be smooth, but rather a Hölder path. We refer to Section 4 for a further analysis of this point.

Next we will present a technical lemma which will become useful in later analysis of the Volterra signature. It states that the convolution product $*$ behaves similarly to the tensor product $\otimes$ on small scales.

Lemma 15. *Let $\mathcal{P}$ be a partition of $[s,t]$ such that $|\mathcal{P}| \to 0$, and consider $\mathbf{z}^j$ for $j = 1, \dots, p$ as constructed in Eq. (3.7) with a continuously differentiable driving noise and a kernel k satisfying* (**H**) *with singularity of order $\gamma < \frac{1}{2}$. Then for $n, p \geq 1$ with $p - n \geq 1$, we have*

$$\lim_{|\mathcal{P}|\to 0} \left| \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{p-n,\tau} * \mathbf{z}_{us}^{n,\cdot} - \mathbf{z}_{vu}^{p-n,\tau} \otimes \mathbf{z}_{us}^{n,u} \right| = 0. \tag{3.16}$$

Proof. In order to study the left hand side of (3.16), let us set for $(u, v) \in \Delta_2$

$$D(u,v) = \mathbf{z}_{vu}^{p-n,\tau} * \mathbf{z}_{us}^{n,\cdot} - \mathbf{z}_{vu}^{p-n,\tau} \otimes \mathbf{z}_{us}^{n,u}. \tag{3.17}$$

Then according to Definition (3.13) it is readily checked that

$$D(u,v) = \int_{v>r_p>\cdots>r_{n+1}>u} \bigotimes_{i=p}^{n+1} k(r_{i+1}, r_i)\, dx_{r_i}$$
$$\otimes \int_{u>r_n>\cdots>r_1>s} \Big[k(r_{n+1}, r_n) - k(u, r_n)\Big] dx_{r_n} \bigotimes_{i=n-1}^{1} k(r_{i+1}, r_i)\, dx_{r_i},$$

where we have written $r_{n+1} = \tau$ for the sake of readability. We now proceed along the same lines as for Proposition 12. Namely if we assume that $\|\dot{x}\|_\infty \leq M$, we get

$$|D(u,v)| \leq M^p \int_{v>r_p>\cdots>r_{n+1}>u} \prod_{i=n+1}^{p} |k(r_{i+1}, r_i)|$$
$$\times \left(\int_{u>r_n>\cdots>r_1>s} |k(r_{n+1}, r_n) - k(u, r_n)| \prod_{i=1}^{n-1} |k(r_{i+1}, r_i)| dr_n \dots dr_1 \right) dr_p \dots dr_{n+1}.$$

Furthermore, from (**H**) we have $|k(r_{i+1}, r_i)| \lesssim |r_{i+1} - r_i|^{-\gamma}$, and any $\beta \in [0,1]$ we have

$$|k(r_{n+1}, r_n) - k(u, r_n)| \lesssim |r_{n+1} - u|^\beta |u - r_n|^{-\gamma-\beta}.$$

Thus restricting $\beta \in (0, 1-\gamma)$, we get

$$|D(u,v)| \leq M^p \int_{v>r_p>\cdots>r_{n+1}>u} (\tau - r_n)^{-\gamma} \prod_{i=n+1}^{p-1} |r_{i+1} - r_i|^{-\gamma} |r_{n+1} - u|^\beta dr_{n+1} \cdots dr_p$$
$$\times \int_{u>r_n>\cdots>r_1>s} |u - r_n|^{-\gamma-\beta} \prod_{i=1}^{n-1} |r_{i+1} - r_i|^{-\gamma} dr_1 \cdots dr_n.$$

Hence, integrating the outside integral over the simplex $u > r_n > \cdots > r_1 > s$, we end up with

$$\begin{aligned}
&|D(u,v)| \\
&\leq C_{\beta,\gamma,p,n} \int_{v>r_p>\cdots>r_{n+1}>u} \prod_{i=n+1}^{p} |r_{i+1}-r_i|^{-\gamma} (r_{n+1}-u)^{\beta} dr_{n+1}\ldots dr_p \times (u-s)^{n(1-\gamma)-\beta} \\
&\leq C_{\beta,\gamma,p,n} \int_u^v (\tau-r)^{-\gamma} (r-u)^{(p-n-2)(1-\gamma)+(\beta+1)} dr \times (u-s)^{n(1-\gamma)-\beta}, \qquad (3.18)
\end{aligned}$$

where $C_{\beta,\gamma,p,n} := \frac{\Gamma(1-\gamma-\beta)\Gamma(1-\gamma)^{n-1}M^p}{\Gamma(n(1-\gamma)-\beta+1)}$ and where we have used the convolution property (2.2) of the Riemann–Liouville fractional integral. Now we can do a change of variables $r = u+\theta(v-u)$ and find

$$\int_u^v (\tau-r)^{-\gamma}(r-u)^{(p-n-2)(1-\gamma)+\beta+1} dr = (\tau-u)^{-\gamma}(v-u)^{(p-n-2)(1-\gamma)+\beta+2} c_{\gamma,\tau,u,v},$$

where $c_{\gamma,\tau,u,v}$ is a function bounded by the Beta function, i.e.

$$\begin{aligned}
c_{\gamma,\tau,u,v} &= \int_0^1 \left(1+\theta\frac{v-u}{\tau-u}\right)^{-\gamma} \theta^{(p-n-2)(1-\gamma)+\beta+1} d\theta \\
&\leq B(1-\gamma,(p-n-2)(1-\gamma)+\beta+2) < \infty.
\end{aligned}$$

Plugging this identity into (3.18) and writing $C = C_{\beta,\gamma,p,n}$ for constants which may change from line to line, we get

$$|D(u,v)| \leq C(\tau-u)^{-\gamma}(v-u)^{(p-n-2)(1-\gamma)+\beta+2}(u-s)^{n(1-\gamma)-\beta}$$

Therefore it is readily checked that

$$\sum_{[u,v]\in\mathcal{P}} |D(u,v)| \leq C|\mathcal{P}|^{(p-n-2)(1-\gamma)+\beta+1} \times \int_s^t (\tau-u)^{-\gamma}(u-s)^{n(1-\gamma)-\beta} du,$$

where $|\mathcal{P}|$ denotes the size of the mesh of $\mathcal{P}$. Taking into account the definition of $D(u,v)$ in (3.17), this finishes the proof. □

4. Volterra rough paths

To begin the study of Volterra rough paths, we need to understand the structure and regularity which may be extracted from a Volterra path. As we have already seen, a Volterra path is really a two parameters function on a simplex Δ_2 taking values in some space E. A simple example of a function of this form could be the singular kernel

$$f_t^{\tau} := (\tau-t)^{-\gamma}, \qquad (4.1)$$

defined for $t \leq \tau$ and $\gamma \in (0,1)$. Note that for a function f given as in (4.1) and $(s,t,\tau) \in \Delta_3$ it follows from Lemma 6 that

$$|f_{ts}^{\tau}| \leq (\tau-t)^{-(\gamma+1)}(t-s).$$

This tells us that as long as $t < \tau$ then we have a Lipschitz bound on f^{τ}, i.e. for any $\epsilon > 0$ we have $f^{\tau} \in C_{\mathrm{Lip}}([0,\tau-\epsilon])$. Similarly one can consider the function

$$g_t^{\tau} = (\tau-t)^{\alpha},$$

for some $\alpha \in (0, 1)$ and $t \le \tau$. Then it is easy to see that globally, g is α-Hölder continuous in both variables. However, for any small $\epsilon > 0$ we have that $t \mapsto g_t^\tau$ is $C^\infty([0, \tau - \epsilon])$. Along the same lines, one can see that $\tau \mapsto g_t^\tau \in C^\infty([t + \epsilon, T])$. In the sequel we will generalize the above considerations to processes of the form

$$z_{ts} = \int_s^t k(t, r) dx_r, \tag{4.2}$$

where x is an α-Hölder path and k a possibly singular kernel of order $-\gamma$. This section is devoted to a definition and analysis of generic Volterra type rough paths like in (4.2).

4.1. Definition and sewing lemma

Let us go back for a moment to the increment defined in (4.2). One way to define the term $\int_s^t k(\tau, r)dx_r$ is to split the integral in the right hand side of (4.2) along a partition $\mathcal{P}$ of $[s, t]$,

$$\int_s^t k(t, r) dx_r = \sum_{[u,v]\in\mathcal{P}[s,t]} \int_u^v k(t, r) dx_r,$$

Then for each $[u, v] \in \mathcal{P}$ we have some regularity of $\int_u^v k(t, r) dx_r$ coming from the difference $v - u$ which is contributed by the driving noise, and some (possibly singular) regularity coming from the difference $t - v$. Much of the difficulty in the analysis of Volterra rough paths will be due to such considerations. In order to capture the different regularities discussed above, we will make use of three different quantities, which will later be used in the definition of various classes of Volterra Hölder functions. For two parameters $(\alpha, \gamma) \in (0, 1)^2$, we set $\rho = \alpha - \gamma$, and we will consider the semi-norms defined by

$$\|z\|_{(\alpha,\gamma),1} := \sup_{(s,t,\tau)\in\Delta_3} \frac{|z_{ts}^\tau|}{|\tau - t|^{-\gamma}|t - s|^\alpha \wedge |\tau - s|^\rho} \tag{4.3}$$

$$\|z\|_{(\alpha,\gamma),1,2} := \sup_{\substack{(s,t,\tau',\tau)\in\Delta_4 \\ \eta\in[0,1],\zeta\in[0,\rho)}} \frac{|z_{ts}^{\tau\tau'}|}{|\tau - \tau'|^\eta|\tau' - t|^{-\eta+\zeta}\left(|\tau' - t|^{-\gamma-\zeta}|t - s|^\alpha \wedge |\tau' - s|^{\rho-\zeta}\right)}, \tag{4.4}$$

with the convention $z_{ts}^\tau = z_t^\tau - z_s^\tau$ and $z_s^{\tau\tau'} = z_s^\tau - z_s^{\tau'}$. With these quantities at hand, let us define a space of functions which we will call Volterra paths.

Definition 16. Let $(\alpha, \gamma) \in (0, 1)^2$ and consider a function $z : \Delta_2 \to E$, such that $(t, \tau) \mapsto z_t^\tau$. We assume that for all $\tau \in [0, T]$ we have

$$t \mapsto z_t^\tau \in \mathcal{C}^{\alpha-\gamma}(\{\tau\}) \cap \mathcal{C}^\alpha([0, \tau)),$$

where we recall that the notation $\mathcal{C}^{\alpha-\gamma}(\{\tau\})$ has been introduced in (2.5). We also assume that for all $t \in [0, T]$ the following holds:

$$\tau \mapsto z_t^\tau \in \mathcal{C}^{\alpha-\gamma}(\{t\}) \cap \mathcal{C}^1((t, T]).$$

Then for such a function z, define

$$\|z\|_{(\alpha,\gamma)} := \|z\|_{(\alpha,\gamma),1} + \|z\|_{(\alpha,\gamma),1,2}, \tag{4.5}$$

where the norms are given as in (4.3) and (4.4). We define the space of Volterra paths $z : \Delta_2 \to E$ as all paths such that $z_0^\tau = z_0 \in E$ for all $\tau \in (0, T]$, and

$$\|z\|_{(\alpha,\gamma)} < \infty.$$

We denote this space by $\mathcal{V}^{(\alpha,\gamma)}(\Delta_2; E)$. In addition, under the mapping

$$z \mapsto |z_0| + \|z\|_{(\alpha,\gamma)},$$

the space $\mathcal{V}^{(\alpha,\gamma)}$ is a Banach space.

Remark 17. Conventionally, we will use the notation y_{ts}^τ to signify both functions with three arguments, and the increment of functions with two arguments, i.e. $y_{ts}^\tau = y^\tau(s,t)$ and $y_{ts}^\tau = y_t^\tau - y_s^\tau$. We hope the specific meaning will always be clear from the context. Moreover, we will use the same norms as those defined in (4.5) for three variable functions $y : \Delta_3 \to E$ given by $(s,t,\tau) \mapsto y_{ts}^\tau$.

Remark 18. The space $\mathcal{V}^{(\alpha,\gamma)}$ really captures three different regularities in different areas of $\Delta_2([0,T])$. On the diagonal line, (t,t) we clearly have that $z \in \mathcal{V}^{(\alpha,\gamma)}$ is of ρ-Hölder regularity in both variables, where $\rho = \gamma - \alpha$. However, at any point off the diagonal we have α-regularity in the lower variable and 1-regularity in the upper variable. The space could have therefore be defined more generally to capture three different regularities. However, for our purposes, under the assumption (**H**) and the fact that a Volterra path is of the form $z_t^\tau = \int_0^t k(\tau,r)\,dx_r$, we easily get the 1-regularity in the upper argument. This will play a central role throughout the analysis of such paths.

Remark 19. The reader might wonder about the introduction of an extra parameter ζ in the definition (4.4) of $\|z\|_{(\alpha,\gamma),1,2}$. In order to justify this new parameter, consider $z \in \mathcal{V}^{(\alpha,\gamma)}$ such that $z_0^\tau = 0$ for all τ. We wish to bound the diagonal difference $z_t^t - z_s^s$. To this aim, we decompose the difference as

$$z_t^t - z_s^s = z_{ts}^t + z_s^{ts} = z_{ts}^t + z_{s0}^{ts}. \tag{4.6}$$

Then the term z_{ts}^t in the right hand side of (4.6) is easily bounded by $|t-s|^\rho$ according to (4.3). In order to bound the term z_{s0}^{ts}, we resort to (4.4) and write

$$|z_{s0}^{ts}| \lesssim |t-s|^\eta |s-s|^{-\eta+\zeta} |s-0|^{\rho-\zeta}. \tag{4.7}$$

We now tune the parameter ζ in order to avoid the singularity in $|\cdot|^{-\eta}$ in (4.7). Namely, pick $\eta \le \rho$, and $\zeta = \eta$. This yields

$$|z_{s0}^{ts}| \lesssim |t-s|^\eta.$$

Plugging this information into (4.6) we end up with $|z_t^t - z_s^s| \lesssim |t-s|^\eta$ for any $\eta \le \rho$, which is the desired regularity on the diagonal.

Remark 20. The norms and spaces in Definition 16 can be easily generalized to increments of two variables, which yields the definition of a space $\mathcal{V}_2^{(\alpha,\gamma)}(\Delta_3, E)$. The norm on $\mathcal{V}_2^{(\alpha,\gamma)}(\Delta_3, E)$ is given by

$$\|z\|_{(\alpha,\gamma)} = \|z\|_{(\alpha,\gamma),1} + \|z\|_{(\alpha,\gamma),1,2}. \tag{4.8}$$

Those spaces will be used for the definition of convolutional controlled paths in Section 5.1.

Our construction of solutions to rough Volterra equations like (1.1) will hinge heavily on a Volterra version of the Sewing Lemma. We start by defining the class $\mathscr{V}^{(\alpha,\gamma)}$ of paths to which this Sewing Lemma will apply.

Definition 21. Let $\alpha \in (0,1)$, $\gamma \in (0,1)$ with $\alpha-\gamma>0$, $\kappa \in (0,\infty)$ and $\beta \in (1,\infty)$. Denote by $\mathscr{V}^{(\alpha,\gamma)(\beta,\kappa)}(\Delta_3[0;T];E)$, the space of all functions $\Xi : \Delta_3([0,T]) \to E$ such that

$$\|\Xi\|_{\mathscr{V}^{(\alpha,\gamma)(\beta,\kappa)}} = \|\Xi\|_{(\alpha,\gamma)} + \|\delta\Xi\|_{(\beta,\kappa)} < \infty, \tag{4.9}$$

where δ is the operator defined for any $s<u<t$ and a two variables function g by

$$\delta_u g_{ts} = g_{ts} - g_{tu} - g_{us}. \tag{4.10}$$

In (4.9), we also use the following convention: the norm $\|\Xi\|_{(\alpha,\gamma)}$ is given by (4.8), while we have

$$\|\delta\Xi\|_{(\alpha,\gamma)} = \|\delta\Xi\|_{(\alpha,\gamma),1} + \|\delta\Xi\|_{(\alpha,\gamma),1,2},$$

where the quantities $\|\delta\Xi\|_{(\beta,\gamma),1}$ and $\|\delta\Xi\|_{(\beta,\gamma),1,2}$ are slight modifications of (4.3) respectively defined by

$$\|\delta\Xi\|_{(\beta,\kappa),1} := \sup_{(s,m,t,\tau)\in\Delta_4} \frac{|\delta_m \Xi_{ts}^{\tau}|}{|\tau-t|^{-\kappa}|t-s|^{\beta} \wedge |\tau-s|^{\beta-\kappa}} \tag{4.11}$$

$$\|\delta\Xi\|_{(\beta,\kappa),1,2} := \sup_{\substack{(s,m,t,\tau',\tau)\in\Delta_5 \\ \eta\in[0,1],\,\zeta\in[0,\beta-\kappa)}} \frac{|\delta_m \Xi_{ts}^{\tau,\tau'}|}{|\tau-\tau'|^{\eta}|\tau'-t|^{-\eta+\zeta}\left(|\tau'-t|^{-\kappa-\zeta}|t-s|^{\beta} \wedge |\tau'-s|^{\beta-\kappa-\zeta}\right)}. \tag{4.12}$$

In the sequel the space $\mathscr{V}^{(\alpha,\gamma)(\beta,\kappa)}$ will be our space of abstract Volterra integrands.

We are now ready to state our Sewing Lemma adapted to Volterra integrands.

Lemma 22 (*Volterra Sewing Lemma*)**.** *Consider four exponents $\beta \in (1,\infty)$, $\kappa \in (0,1)$, $\alpha \in (0,1)$ and $\gamma \in (0,1)$ such that $\beta-\kappa \geq \alpha-\gamma > 0$. Let $\mathscr{V}^{(\alpha,\gamma)(\beta,\kappa)}$ and $\mathcal{V}^{(\alpha,\gamma)}$ be the spaces defined in Definitions 16 and 21 respectively. Then there exists a linear continuous map $\mathcal{I} : \mathscr{V}^{(\alpha,\gamma)(\beta,\kappa)}(\Delta_3;E) \to \mathcal{V}^{(\alpha,\gamma)}(\Delta_3;E)$ such that the following holds true:*

(i) The quantity $\mathcal{I}(\Xi^{\tau})_{ts} := \lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} \Xi_{vu}^{\tau}$ exists for all $(s,t,\tau)\in\Delta_3$, where $\mathcal{P}$ is a generic partition of $[s,t]$ and $|\mathcal{P}|$ denotes the mesh size of the partition.

(ii) For all $(s,t,\tau)\in\Delta_3$ we have that

$$|\mathcal{I}(\Xi^{\tau})_{ts} - \Xi_{ts}^{\tau}| \lesssim \|\delta\Xi\|_{(\beta,\kappa),1}\left(|\tau-t|^{-\kappa}|t-s|^{\beta} \wedge |\tau-s|^{\beta-\kappa}\right), \tag{4.13}$$

while for $(s,t,\tau',\tau)\in\Delta_4$ we get for any $\eta\in[0,1]$ and $\zeta\in[0,\beta-\kappa)$

$$\begin{aligned} &|\mathcal{I}(\Xi^{\tau\tau'})_{ts} - \Xi_{ts}^{\tau\tau'}| \\ &\lesssim \|\delta\Xi\|_{(\beta,\kappa),1,2}\left[|\tau-\tau'|^{\eta}|\tau'-t|^{-\eta+\zeta}\left(|\tau'-t|^{-\kappa-\zeta}|t-s|^{\beta} \wedge |\tau'-s|^{\beta-\kappa-\zeta}\right)\right]. \end{aligned} \tag{4.14}$$

Proof. This is an elaboration of [18, Lemma 4.2] and we give some details here for the sake of completeness. Specifically, we will focus on the convergence of Riemann type sums $\sum_{[u,v]\in\mathcal{P}} \Xi_{vu}^{\tau}$ along dyadic partitions. Referring to [18, Lemma 4.2], we leave to the patient reader the task of checking the convergence of $\sum_{[u,v]\in\mathcal{P}} \Xi_{vu}^{\tau}$ along a general partition whose mesh converges to 0, as well as the relation $\delta\mathcal{I}(\Xi) = 0$.

With those preliminaries in mind, let us consider the nth order dyadic partition $\mathcal{P}^n$ of $[s,t]$ where each set $[u,v]\subset\mathcal{P}^n$ is of length $2^{-n}|t-s|$. We define the nth order Riemann sum of Ξ^τ, denoted $\mathcal{I}^n\left(\Xi\right)_{ts}$, as follows

$$\mathcal{I}^n\left(\Xi^\tau\right)_{ts}=\sum_{[u,v]\in\mathcal{P}^n}\Xi^\tau_{vu}.$$

Our aim is to show that the sequence $\{\mathcal{I}^n\left(\Xi^\tau\right);n\geq 1\}$ converges to an element $\mathcal{I}(\Xi)$ which fulfills relation (4.13). To this aim we will analyse differences $\mathcal{I}^{n+1}(\Xi^\tau)-\mathcal{I}^n(\Xi^\tau)$ and prove the following bound

$$|\mathcal{I}^{n+1}(\Xi^\tau)-\mathcal{I}^n(\Xi^\tau)|\lesssim\frac{\|\delta\Xi\|_{(\beta,\kappa),1}}{2^{n(\beta-1)}}\left(|\tau-t|^{-\kappa}|t-s|^\beta\wedge|\tau-s|^{\beta-\kappa}\right). \tag{4.15}$$

In order to prove (4.15), observe that

$$\mathcal{I}^{n+1}\left(\Xi^\tau\right)_{ts}-\mathcal{I}^n\left(\Xi^\tau\right)_{ts}=\sum_{[u,v]\in\mathcal{P}^n}\delta_m\Xi^\tau_{vu}, \tag{4.16}$$

where we recall that δ is given by relation (4.10) and where we have set $m=\frac{u+v}{2}$. Plugging relation (4.11) into (4.16), it is thus readily checked that

$$|\sum_{[u,v]\in\mathcal{P}^n}\delta_m\Xi^\tau_{vu}|\lesssim\|\delta\Xi\|_{(\beta,\kappa)}\sum_{[u,v]\in\mathcal{P}^n}|\tau-v|^{-\kappa}|v-u|^\beta. \tag{4.17}$$

We will now upper bound the right hand side above. Invoking the fact that $\beta>1$ and $|v-u|=2^{-n}|t-s|$ for $u,v\in\mathcal{P}^n$ we write

$$\sum_{[u,v]\in\mathcal{P}^n}|\tau-v|^{-\kappa}|v-u|^\beta\leq 2^{-n(\beta-1)}|t-s|^{(\beta-1)}\sum_{[u,v]\in\mathcal{P}^n}|\tau-v|^{-\kappa}|v-u|. \tag{4.18}$$

Hence, some elementary considerations on the Riemann sums corresponding to the integral $\int_s^t|\tau-r|^{-\kappa}dr$ for a $t<\tau$ and parameter $\kappa\in(0,1)$ yield

$$\sum_{[u,v]\in\mathcal{P}^n}|\tau-v|^{-\kappa}|v-u|^\beta\lesssim 2^{-n(\beta-1)}|t-s|^{(\beta-1)}\int_s^t|\tau-r|^{-\kappa}dr. \tag{4.19}$$

In addition, some elementary calculations similar to those in Remark 7 show that for $\kappa\in(0,1)$ we have

$$\int_s^t|\tau-r|^{-\kappa}dr\lesssim(\tau-t)^{-\kappa}(t-s)\wedge(\tau-s)^{1-\kappa},$$

where we have used the fact that the integral $\int_s^t|\tau-r|^{-\kappa}dr$ is converging for $\kappa<1$. Putting this inequality into (4.19) we get

$$\sum_{[u,v]\in\mathcal{P}^n}|\tau-v|^{-\kappa}|v-u|^\beta\lesssim 2^{-n(\beta-1)}\left((\tau-t)^{-\kappa}(t-s)^\beta\wedge(\tau-s)^{\beta-\kappa}\right). \tag{4.20}$$

Inserting (4.20) into (4.18) and then into (4.17), our claim (4.15) is thus easily obtained. With relation (4.15) in hand, one immediately gets that the sequence $\{\mathcal{I}^n\left(\Xi^\tau\right)_{ts}\}_{n\geq 0}$ is a Cauchy sequence. It thus converges to a quantity $\mathcal{I}\left(\Xi^\tau\right)_{ts}$ which satisfies (4.13). As mentioned above, the remainder of the proof goes along the same lines as [18, Lemma 4.2]. We leave it to the patient reader for the sake of conciseness. This proves that the element $\mathcal{I}\left(\Xi^\tau\right)$ has finite $\|\cdot\|_{(\beta,\kappa),1}$ norm and that (4.13) holds. The next step will be to show that also the integral $\mathcal{I}(\Xi^{\tau\tau'})$ of the increment in the upper variable $\Xi^{\tau\tau'}_{ts}$ is finite in the $\|\cdot\|_{(\beta,\kappa),1,2}$ norm. Following the lines for the proof above, we can just change the integrand Ξ^τ_{ts} with $\Xi^{\tau\tau'}_{ts}$ and the norms accordingly. Thus, using exactly the same arguments as before, inequality (4.14) holds as well. This concludes the proof. □

In order to test the compatibility of our first definitions with the Sewing lemma, we will show that one can construct a Volterra path of the form $z_{ts}^{\tau} = \int_s^t k(\tau, r)\, dx_r$ in terms of Lemma 22.

Theorem 23. *Let $x \in \mathcal{C}^{\alpha}$ and k be a Volterra kernel of order $-\gamma$ satisfying* (**H**), *such that $\rho = \alpha - \gamma > 0$. We define an element $\Xi_{ts}^{\tau} = k(\tau, s)x_{ts}$. Then the following holds true:*

(i) There exists a $\beta > 1$ and $\kappa > 0$ with $\beta - \kappa = \alpha - \gamma$ such that $\Xi \in \mathscr{V}^{(\alpha,\gamma)(\beta,\kappa)}$, where $\mathscr{V}^{(\alpha,\gamma)(\beta,\kappa)}$ is given in Definition 21. Therefore the element $\mathcal{I}(\Xi^{\tau})$ obtained by applying Lemma 22 is well defined as an element of $\mathcal{V}^{(\alpha,\gamma)}$ and we set $z_{ts}^{\tau} \equiv \mathcal{I}(\Xi^{\tau})_{ts} = \int_s^t k(\tau, r)dx_r$.

(ii) There exists a strictly positive c such that for $(s, t, \tau) \in \Delta_3$ we have

$$|z_{ts}^{\tau} - k(\tau, s)x_{ts}| \leq c\left[(\tau - t)^{-\gamma}(t - s)^{\alpha} \wedge (\tau - s)^{\rho}\right], \tag{4.21}$$

and in particular z verifies $\|z\|_{(\alpha,\gamma),1} < \infty$.

(iii) For any $\eta \in [0, 1]$ and $\zeta \in [0, \rho)$ there exists a strictly positive constant c such that for any $(s, t, q, p) \in \Delta_4$ we have

$$|z_{ts}^{pq}| \leq c|p - q|^{\eta}|q - t|^{-\eta+\zeta}\left[|q - t|^{-\gamma-\zeta}|t - s|^{\alpha} \wedge |q - s|^{\rho-\zeta}\right], \tag{4.22}$$

where $z_{ts}^{pq} = z_t^p - z_t^q - z_s^p + z_s^q$.

Remark 24. According to the standard rules of algebraic integration we would be naturally prone to set $\Xi_{ts}^{\tau} = k(\tau, t)x_{ts}$. Here we have chosen to take $\Xi_{ts}^{\tau} = k(\tau, s)x_{ts}$, which will ease the treatment of the singularity of k on the diagonal. This small twist on the usual theory does not affect the fact that we are generalizing Volterra equations from the smooth to the rough case.

Proof. Recall that we have set $\Xi_{ts}^{\tau} = k(\tau, s)x_{ts}$. We will show that Lemma 22 may be applied to Ξ, which amounts to check that $\Xi \in \mathscr{V}_2^{(\alpha,\gamma)(\beta,\kappa)}$ with some parameters $\beta > 1$ and $\kappa > 0$ to be chosen later on. Furthermore, in order to show that $\|\Xi\|_{\mathscr{V}_2^{(\alpha,\gamma)(\beta,\kappa)}} < \infty$ we will focus on the norms $\|\delta\Xi\|_{(\beta,\kappa),1}$ and $\|\delta\Xi\|_{(\beta,\kappa),1,2}$ defined by (4.11) and (4.12), and we leave the proof of $\|\Xi\|_{(\alpha,\gamma)} < \infty$ to the reader for the sake of conciseness.

In order to check that $\|\delta\Xi\|_{(\beta,\kappa),1} < \infty$, we start by noting that the increment $\delta_m\Xi_{ts}^{\tau}$ can be written as $\delta_m\Xi_{ts}^{\tau} = [k(\tau, s) - k(\tau, m)]\, x_{tm}$, which stems from elementary algebraic manipulations. Therefore, according to (3.3) in Hypothesis (**H**) we have for an additional parameter $\nu \in [0, 1]$

$$|\delta_m\Xi_{ts}^{\tau}| \lesssim \|x\|_{\alpha}(\tau - m)^{-\gamma-\nu}(t - m)^{\alpha}(m - s)^{\nu}. \tag{4.23}$$

Next we pick our parameter ν such that the condition

$$\beta \equiv \nu + \alpha > 1 \tag{4.24}$$

is satisfied. As far as the singularity at τ is concerned, relation (4.22) asserts that in order to apply Lemma 22 item (ii) we get the restriction

$$\kappa \equiv \gamma + \nu < 1. \tag{4.25}$$

Note that if we put conditions (4.24) and (4.25) together, we get $1-\alpha<\nu<1-\gamma$ which can be fulfilled as long as $\alpha>\gamma$. Furthermore, it is immediate that $\beta-\kappa=\alpha-\gamma$. Then putting together (4.23) with (4.24) we get that $\|\delta\Xi\|_{(\beta,\kappa),1}<\infty$. Next we need to show that $\|\delta\Xi\|_{(\beta,\kappa),1,2}<\infty$. To this aim, define $g_p(q,s)=k(p,s)-k(q,s)$. Then combining (3.4) and (3.5) in assumption (**H**) there exist two parameters $\eta,\theta\in[0,1]$ such that for $p>q>t>m>s$ we have

$$|g_p(q,m)-g_p(q,s)|\lesssim(p-q)^{\eta}(q-m)^{-(\gamma+\theta+\eta)}(m-s)^{\theta}. \tag{4.26}$$

With this estimate in mind, let us now define a new abstract Volterra integrand $\Xi_{ts}^{pq}=g_p(q,s)x_{ts}$. Repeating the computations of step (i) with $(s,m,t,q,p)\in\Delta_5$, and applying (3.6) on g we end up with

$$|\delta_m\Xi_{ts}^{pq}|\lesssim(p-q)^{\eta}(q-t)^{-\eta+\zeta}(q-m)^{-(\gamma+\theta+\zeta)}(m-s)^{\theta}(t-m)^{\alpha}, \tag{4.27}$$

where $\eta,\theta\in[0,1]$. Observe that $(m-s)^{\theta}(t-m)^{\alpha}\leq(t-s)^{-\theta+\alpha}$. Thus for any $\zeta\in[0,\beta-\kappa)$ set $\kappa=\gamma+\theta+\zeta<1$ and $\beta=\theta+\alpha>1$ in the same way as in the previous step. Note that this is always possible due to the fact that $\beta-\kappa>0$. It follows that

$$\|\delta_m\Xi\|_{(\beta,\kappa),1,2}<\infty.$$

It is therefore clear that $\Xi\in\mathscr{V}^{(\alpha,\gamma),(\beta,\kappa)}$. An application of Lemma 22 now yields that $\mathcal{I}(\Xi)\in\mathcal{V}^{(\alpha,\gamma)}$ and that the inequalities in (ii)–(iii) holds. $\square$

Remark 25. Owing to Theorem 23, we now know that a typical example of a Volterra path in $\mathcal{V}^{(\alpha,\gamma)}$ is given by processes of the form $\int_s^t k(\tau,r)\,dx_r$. Having this large class of objects in hand, we will mostly focus on computations for general elements in $\mathcal{V}^{(\alpha,\gamma)}$ whenever it is not needed to explicitly state the kernel k or the driving noise x.

4.2. Convolution product in the rough case

As we have seen in Section 3.2, the equivalent of Chen's relation in our Volterra context involves convolution type integrals. In order to clarify this point, let us go back to Remark 14 concerning second order iterated integrals. One way to rephrase relation (3.14) with the operator δ introduced in (4.10) is the following

$$\delta_s\mathbf{z}_{t0}^{\tau,2}=\int_{t>r_2>s}k(\tau,r_2)\,dx_{r_2}\otimes\int_{s>r_1>0}k(r_2,r_1)\,dx_{r_1}, \tag{4.28}$$

In the right hand side of (4.28) we point out that the limits of the integration with respect to x_{r_1} are fixed; the only thing that is connecting the two integrals is the dependence on r_2 through the kernels. Thus the integral $\int_0^s k(r_2,r_1)\,dx_{r_1}$ can really be thought of as a re-scaling of the path x as r_2 moves from s to t. Our next step is to show that this operation is indeed valid for two generic Volterra paths y,z.

Theorem 26. *We consider two Volterra paths $z\in\mathcal{V}^{(\alpha,\gamma)}$ and $y\in\mathcal{V}^{(\alpha',\gamma')}$ as given in Definition 16, where we recall that $\alpha,\gamma,\alpha',\gamma'\in(0,1)$, and define $\rho\equiv\alpha-\gamma>0$ and $\rho'\equiv\alpha'-\gamma'>0$. Then the convolution product is a bilinear operation on $\mathcal{V}^{(\alpha,\gamma)}$ given by*

$$z_{tu}^{\tau}*y_{us}^{\cdot}=\int_{t>r>u}dz_r^{\tau}\otimes y_{us}^{r}:=\lim_{|\mathcal{P}|\to0}\sum_{[u',v']\in\mathcal{P}}z_{v'u'}^{\tau}\otimes y_{us}^{u'}. \tag{4.29}$$

The integral is understood as a Volterra–Young integral for all $(s,u,t,\tau)\in\Delta_4$. *Moreover, the following inequality holds true,*

$$|z^{\tau}_{tu} * y^{\cdot}_{us}| \lesssim \|z\|_{(\alpha,\gamma),1}\|y\|_{(\alpha',\gamma'),1,2}\left[(\tau-t)^{-\gamma}\,(t-s)^{\rho+\rho'+\gamma}\wedge(\tau-s)^{\rho+\rho'}\right]. \tag{4.30}$$

Proof. Define $\Xi^{\tau}_{r'r} := z^{\tau}_{r'r}\otimes y^{r}_{us}$, for $0\le s<u\le r\le m\le r'\le t$. In spirit of Lemma 22, we will show that

$$|\mathcal{I}(\Xi^{\tau})_{tu}-\Xi^{\tau}_{tu}| \lesssim \|z\|_{(\alpha,\gamma),1}\|y\|_{(\alpha',\gamma'),1,2}\left[(\tau-t)^{-\gamma}(t-s)^{\rho+\rho'+\gamma}\wedge(\tau-s)^{\rho+\rho'}\right].$$

Following the strategy outlined in the proof of Lemma 22, we know from (4.17) that we must show that the sum $\sum_{[r,r']\in\mathcal{P}^n[u,t]}|\delta_m\Xi^{\tau}_{r'r}|$ is converging (here $\mathcal{P}^n$ is the dyadic partition used in the proof of Lemma 22). Let us therefore consider the action of δ on Ξ. By simple algebraic manipulations we see that

$$\delta_m\Xi^{\tau}_{r'r} = -z^{\tau}_{r'm}\otimes y^{mr}_{us}. \tag{4.31}$$

Let us now analyse the right hand side of (4.31). The term $z^{\tau}_{r'm}$ can be bounded thanks to assumption (4.3). We get

$$|z^{\tau}_{r'm}| \le \|z\|_{(\alpha,\gamma),1}|\tau-r'|^{-\gamma}|r'-m|^{\alpha}. \tag{4.32}$$

As for the term y^{mr}_{us} we can use assumption (4.4) to write

$$|y^{mr}_{us}| \le \|y\|_{(\alpha',\gamma'),1,2}|m-r|^{\eta}|r-u|^{-\eta}|r-s|^{\rho'}, \tag{4.33}$$

for an arbitrary $\eta\in[0,1]$. Hence gathering (4.32) and (4.33) we bound (4.31) by

$$|z^{\tau}_{r'm}\otimes y^{mr}_{us}| \lesssim \|y\|_{(\alpha',\gamma'),1,2}\|z\|_{(\alpha,\gamma),1}(r-u)^{-\eta}(\tau-r')^{-\gamma}(r'-r)^{\alpha+\eta}\,(r-s)^{\rho'}, \tag{4.34}$$

where we have used the fact that $|r'-m|\lesssim|r'-r|$ and $|m-r|\lesssim|r'-r|$.

Combining (4.34) with (4.31) and summing over the points of the dyadic partition $\mathcal{P}^n$, we end up with

$$\begin{aligned}\sum_{[r,r']\in\mathcal{P}^n[u,t]}&|\delta_m\Xi^{\tau}_{r'r}|\\ &\lesssim \|y\|_{(\alpha',\gamma'),1,2}\|z\|_{(\alpha,\gamma),1}\sum_{[r,r']\in\mathcal{P}^n[u,t]}(r-u)^{-\eta}(\tau-r')^{-\gamma}(r'-r)^{\alpha+\eta}\,(r-s)^{\rho'}.\end{aligned} \tag{4.35}$$

Note that we have two separate possible singular points above, both when $r\to u$ and $r'\to\tau$. However, taking limits in the Riemann sums on the right hand side of (4.35), we know that we obtain a converging integral as long as $\eta+\alpha>1$ and $\eta<1$. Indeed, the right hand side of (4.35) is bounded (up to a multiplicative constant) by the integral $|\mathcal{P}^n|^{\alpha+\eta-1}\int_u^t(\tau-a)^{-\gamma}(a-u)^{-\eta}(a-s)^{\rho'}da$, and by doing a change of variables $a=u+\theta(t-u)$ as well as applying the inequality

$$\sup_{\theta\in[0,1]}(u-s+\theta(t-u))^{\rho'}\le(t-s)^{\rho'},$$

we find that

$$\int_u^t(\tau-a)^{-\gamma}(a-u)^{-\eta}(a-s)^{\rho'}da \le c_{\eta,\gamma}\,(\tau-u)^{-\gamma}(t-u)^{1-\eta}(t-s)^{\rho'}, \tag{4.36}$$

where $c_{\eta,\gamma} = B(1-\gamma, 1-\eta)$ and we recall that B stands for the Beta function as in the proof of Proposition 12. It follows that

$$\sum_{[r,r']\in\mathcal{P}^n[u,t]} |\delta_m \Xi^\tau_{r'r}| \lesssim |\mathcal{P}^n|^{\alpha+\eta-1} \|y\|_{(\alpha',\gamma'),1,2} \|z\|_{(\alpha,\gamma),1} (\tau-u)^{-\gamma}(t-u)^{1-\eta}(t-s)^{\rho'}. \quad (4.37)$$

Since we must choose $\eta > 1-\alpha$, let us choose $\eta = 1-\alpha+\epsilon$ for some small $\epsilon > 0$ satisfying $\rho - \epsilon > 0$. Then inequality (4.37) reads

$$\sum_{[r,r']\in\mathcal{P}^n[u,t]} |\delta_m \Xi^\tau_{r'r}| \lesssim |\mathcal{P}^n|^{\epsilon} \|y\|_{(\alpha',\gamma'),1,2} \|z\|_{(\alpha,\gamma),1} (\tau-u)^{-\gamma}(t-u)^{\alpha-\epsilon}(t-s)^{\rho'}. \quad (4.38)$$

Note that for the dyadic partition $\mathcal{P}^n$ we have $|\mathcal{P}^n|^\epsilon = 2^{-n\epsilon}(t-u)^\epsilon$, and observe that (4.38) is the equivalent of (4.20) in our current setting. Therefore, one can follow the same steps as in Lemma 22 in order to get the following relation, which is the analogue of (4.15):

$$|\mathcal{I}^{n+1}(\Xi^\tau) - \mathcal{I}^n(\Xi^\tau)| \lesssim \frac{\|y\|_{(\alpha',\gamma'),1,2}\|z\|_{(\alpha,\gamma),1}}{2^{n\epsilon}} (\tau-u)^{-\gamma}(t-u)^{\alpha}(t-s)^{\rho'}, \quad (4.39)$$

where we recall that $2\alpha - \gamma = 2\rho + \gamma$. We also let the patient reader check from (4.39) that

$$|\mathcal{I}^{n+1}(\Xi^\tau) - \mathcal{I}^n(\Xi^\tau)| \lesssim \frac{\|y\|_{(\alpha',\gamma'),1,2}\|z\|_{(\alpha,\gamma),1}}{2^{n\epsilon}} \left[(\tau-t)^{-\gamma}(t-s)^{\rho+\rho'+\gamma} \wedge (\tau-s)^{\rho+\rho'}\right], \quad (4.40)$$

where we have used that for $s \le u \le t \le \tau$ the following two inequalities holds:

$$(\tau-u)^{-\gamma}(t-u)^{\alpha}(t-s)^{\rho'} \le (\tau-s)^{\rho+\rho'} \quad \text{and} \quad (\tau-u)^{-\gamma}(t-u)^{\alpha}(t-s)^{\rho'} \le (\tau-t)^{-\gamma}(t-s)^{\rho+\rho'+\gamma}.$$

Putting together (4.39) and (4.40) and reasoning exactly as in Lemma 22 after (4.20), we obtain that $\mathcal{I}^n(\Xi^\tau)$ converges to an element $\mathcal{I}(\Xi^\tau)$ verifying

$$|\mathcal{I}(\Xi^\tau)_{tu} - \Xi^\tau_{tu}| \lesssim \|y\|_{(\alpha',\gamma'),1,2}\|z\|_{(\alpha,\gamma),1} \left[(\tau-t)^{-\gamma}(t-s)^{\rho+\rho'+\gamma} \wedge (\tau-s)^{\rho+\rho'}\right]. \quad (4.41)$$

We therefore define $z^\tau_{tu} * y^{\cdot}_{us} := \mathcal{I}(\Xi^\tau)_{tu}$, and one can directly see from (4.41) that $z^\tau_{tu} * y^{\cdot}_{us}$ satisfies the relation

$$|z^\tau_{tu} * y^{\cdot}_{us}| \lesssim \|y\|_{(\alpha',\gamma'),1,2}\|z\|_{(\alpha,\gamma),1} \left[(\tau-t)^{-\gamma}(t-s)^{\rho+\rho'+\gamma} \wedge (\tau-s)^{\rho+\rho'}\right].$$

This completes the proof. □

Our next step is to mimick Proposition 13 in a rough Volterra context. Specifically we would like to extend Theorem 26 in order to get a proper definition of the nth order convolution products for Volterra rough paths (where we recall that Volterra rough paths are introduced in Definition 16). For those nth order convolution rough paths, we also wish to get a multiplicative property similar to Proposition 13.

Observe that in order to properly define the aforementioned nth order convolution product, we will need to extend the domain of the definition of our convolution product $*$. Namely, we would like to define products of the form $z^{2,\tau}_{ts} * f^{\cdot,\cdot}_s$ for a generic function $(s, \tau_1, \tau_2) \mapsto f^{\tau_2,\tau_1}_s$. To motivate this construction, suppose $x : [0, T] \to \mathbb{R}$ is a smooth path, and $f : \Delta_2 \to \mathbb{R}$ is a smooth function. Furthermore, assume that there exists $f' : \Delta_3 \to \mathbb{R}$ and $R : \Delta_3 \to \mathbb{R}$ such that $f^\tau_{ts} = z^\tau_{ts} * f^{',\tau,\cdot}_s + R^\tau_{ts}$ for some smooth $z : \Delta_2 \to \mathbb{R}$. Consider the integral $\int_s^t k(\tau, r) f^r_r dx_r$. Inserting the relation on f inside the integral, we see that

$$\int_s^t k(\tau,r) f^r_r dx_r = \int_s^t k(\tau,r) f^r_s dx_r + \int_s^t k(\tau,r) z^r_{rs} * f^{',r,\cdot}_s dx_r + \int_s^t k(\tau,r) R^\tau_{rs} dx_r$$

If we assume that $z_t^\tau = \int_0^t k(\tau,r)dx_r$, we recognize that $\int_s^t k(\tau,r) f_s^r dx_r = z_{ts}^\tau * f_s^\cdot$, which is the first order convolution product. However, observe that for the second term we have (since all functions considered are smooth)

$$\int_s^t k(\tau,r) z_{rs}^r * f_s^{\prime,r,\cdot} dx_r = \int_s^t \int_s^r k(\tau,r)k(r,u) f_s^{\prime,r,u} dx_u dx_r. \tag{4.42}$$

Now observe that

$$\int_s^t \int_s^r k(\tau,r)k(r,u) dx_u dx_r = \mathbf{z}_{ts}^{2,\tau},$$

and so we are tempted to define

$$\int_s^t \int_s^r k(\tau,r)k(r,u) f_s^{\prime,r,u} dx_u dx_r = \mathbf{z}_{ts}^{2,\tau} * f_s^{\prime,\cdot_1,\cdot_2}.$$

However, in the current situation the convolution product is performed over two upper variables. Hence we need to extend the construction from Theorem 26 to this context. In subsequent sections we will give a proper definition of controlled rough Volterra paths, and will then see that this is exactly the type of relations that is needed in order to define rough integrals.

Let us first explain how a product like (4.42) behaves in case of a smooth path x with a Volterra kernel k. Namely in this situation, consider a smooth three variables function $f : \Delta_3 \to \mathcal{L}(E, \mathcal{L}(E))$. Then a natural way to define $z_{ts}^{2,\tau} * f_s^{\cdot,\cdot}$ is the following (the reason we assume f has two upper arguments will be discussed in detail in Section 5.1).

Definition 27. Let x be a continuously differentiable function and consider a Volterra kernel k which fulfills (**H**) with $\gamma < 1$. Let also $f : \Delta_3 \to \mathcal{L}(E, \mathcal{L}(E))$ be a smooth function. Then for $\tau \geq t > s \geq v$ the convolution $\mathbf{z}_{ts}^{2,\tau} * f_v^{\cdot_1,\cdot_2}$ is defined by

$$\mathbf{z}_{ts}^{2,\tau} * f_v^{\cdot_1,\cdot_2} = \int_{t>r>s} k(\tau,r) dx_r \otimes \int_{r>l>s} k(r,l) f_v^{r,l} dx_l, \tag{4.43}$$

where the notation $f_v^{\cdot_1,\cdot_2}$ is introduced to prevent ambiguities about the order of integration.

We now state an algebraic type lemma which will be useful in order to extend Definition 27 to rougher contexts.

Lemma 28. *Under the same conditions as in Definition 27, let $\mathbf{z}_{ts}^{2,\tau} * f_s^{\cdot_1,\cdot_2}$ be the increment given by (4.43). Consider $(s,t) \in \Delta_2$ and a generic partition $\mathcal{P}$ of $[s,t]$. Then we have*

$$\mathbf{z}_{ts}^{2,\tau} * f_s^{\cdot_1,\cdot_2} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{2,\tau} * f_s^{\cdot_1,\cdot_2} + \left(\delta_u \mathbf{z}_{vs}^{2,\tau}\right) * f_s^{\cdot_1,\cdot_2}. \tag{4.44}$$

Proof. Starting from (4.43), we first write

$$\mathbf{z}_{ts}^{2,\tau} * f_s^{\cdot_1,\cdot_2} = \sum_{[u,v]\in\mathcal{P}} \int_{v>r>u} k(\tau,r) dx_r \otimes \int_{r>l>s} k(r,l) f_s^{r,l} dx_l.$$

Then for each $[u,v] \in \mathcal{P}$, divide the region $\{v > r > u\} \cap \{r > l > s\}$ into

$$\{v > r > l > u\} \cup \{v > r > u > l > s\}.$$

This yields a decomposition of $\mathbf{z}_{ts}^{2,\tau} * f_s^{\cdot_1,\cdot_2}$ of the form

$$\mathbf{z}_{ts}^{2,\tau} * f_s^{\cdot_1,\cdot_2} = \sum_{[u,v]\in\mathcal{P}} A_{vu}^\tau + B_{vu}^\tau, \tag{4.45}$$

where A and B are respectively given by

$$A_{vu}^{\tau} = \int_{v>r>u} k(\tau, r) dx_r \otimes \int_{r>l>u} k(r, l) f_s^{r,l} dx_l$$
$$B_{vu}^{\tau} = \int_{v>r>u} k(\tau, r) dx_r \otimes \int_{u>l>s} k(r, l) f_s^{r,l} dx_l.$$

Now we immediately recognize the term A_{vu}^{τ} as the expression $\mathbf{z}_{vu}^{2,\tau} * f_s^{\cdot_1,\cdot_2}$ given by (4.43). Moreover, it is also readily checked that $B_{vu}^{\tau} = z_{vu}^{1,\tau} * \left(z_{us}^{1,\cdot_1} * f_s^{\cdot_1,\cdot_2}\right)$. Hence thanks to relation (3.12) for smooth paths we can also write

$$B_{vu}^{\tau} = \left(\delta_u \mathbf{z}_{vs}^{2,\tau}\right) * f_s^{\cdot_1,\cdot_2}.$$

Plugging this relation into (4.45) and gathering the information we have on the term A_{vu}^{τ}, our proof is complete. □

Remark 29. The identity (4.44) makes sense as long as one can define $\mathbf{z}_{ts}^{2,\tau} * f_s^{\cdot_1,\cdot_2}$ and if $\mathbf{z}^2$ verifies (3.12). This opens the way to a generalization to rougher situations, having Theorem 26 in mind for the equivalent of (3.12). These considerations motivate the definition in Theorem 33.

We now take another step towards a proper definition of general convolution products. To this aim, we will assume for a moment that our generic Volterra path z^{τ} gives raise to a stack $\{\mathbf{z}^{j,\tau}; j \leq n\}$ of iterated integrals. Specifically our standing assumption is the following:

H2: Let $z \in \mathcal{V}^{(\alpha,\gamma)}$ be a Volterra path, as introduced in Definition 16. For n such that $(n+1)\rho + \gamma > 1$, we assume that there exists a family $\{\mathbf{z}^{j,\tau}; j \leq n\}$ with $\mathbf{z}^1 = z$ satisfying

$$\delta_u \mathbf{z}_{ts}^{j,\tau} = \sum_{i=1}^{j-1} \mathbf{z}_{tu}^{j-i,\tau} * \mathbf{z}_{us}^{i,\cdot}, \tag{4.46}$$

where the convolution product is defined by the right hand side of (4.29). In addition, we suppose that for $j = 1, \ldots, n$ we have $\mathbf{z}^j \in \mathcal{V}^{(j\rho+\gamma,\gamma)}(\Delta_3, E)$.

Remark 30. The definition of the convolution product in (4.29) is currently only valid for one dimensional increments. Thus the true meaning of the convolution products between elements $\mathbf{z}^i$ and $\mathbf{z}^j$ for general $1 \leq i, j \leq 2$ will become apparent with Theorem 33, and specified further in Remarks 37 and 38.

Let us also specify the kind of norm we shall consider for processes with two upper variables of the form $y^{\cdot_1,\cdot_2}$.

Definition 31. Let y be a function from Δ_3 to V such that for any $(\tau_1, \tau_2) \in \Delta_2$ we have $y_0^{\tau_1,\tau_2} = y_0 \in V$, and such that

$$\|y^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1,2} := \|y^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1,2,>} + \|y^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1,2,<} < \infty \tag{4.47}$$

where the two norms $\|\cdot\|_{(\alpha,\gamma),2,>}$ and $\|\cdot\|_{(\alpha,\gamma),2,<}$ are small variations of (4.4), respectively defined by

$$\|y^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1,2,>} = \sup_{\substack{(s,t,r',r_1,r_2)\in\Delta_5 \\ \eta\in[0,1],\,\zeta\in[0,\alpha-\gamma)}} \frac{|y_{ts}^{r',r_2} - y_{ts}^{r',r_1}|}{|r_2 - r_1|^{\eta}|r_1 - t|^{-\eta+\zeta}\left[|r_1 - t|^{-\gamma-\zeta}|t-s|^{\alpha} \wedge |r_1 - s|^{\alpha-\gamma-\zeta}\right]} \tag{4.48}$$

$$\|y^{\cdot 1,\cdot 2}\|_{(\alpha,\gamma),1,2,<} = \sup_{\substack{(s,t,r_1,r_2,r')\in\Delta_5\\ \eta\in[0,1],\,\zeta\in[0,\alpha-\gamma)}} \frac{|y_{ts}^{r_2,r'} - y_{ts}^{r_1,r'}|}{|r_2-r_1|^{\eta}|r_1-t|^{-\eta+\zeta}\left[|r_1-t|^{-\gamma-\zeta}|t-s|^{\alpha}\wedge|r_1-s|^{\alpha-\gamma-\zeta}\right]}, \tag{4.49}$$

We denote the space of functions such that (4.47) is fulfilled by $\mathcal{V}^{\cdot 1,\cdot 2}_{(\alpha,\gamma)}$.

Remark 32. In the sequel we will need to estimate differences of functions $y^{\cdot 1,\cdot 2}:\Delta_3\to V$ the form $|y_s^{\tau,v}-y_s^{\tau,u}|\lesssim|v-u|^{\eta}|u-s|^{-\eta}$ uniformly over τ and s. However, if $y\in\mathcal{V}^{\cdot 1,\cdot 2}_{(\alpha,\gamma)}$, it is readily checked that

$$\begin{aligned}|y_s^{\tau,v}-y_s^{\tau,u}| &\le |y_0^{\tau,v}-y_0^{\tau,u}|+|y_{s0}^{\tau,v}-y_{s0}^{\tau,u}|\\ &\le \|y^{\cdot 1,\cdot 2}\|_{(\alpha,\gamma),1,2,>}|v-u|^{\eta}|u-s|^{-\eta}|s-0|^{\alpha-\gamma}\end{aligned}$$

where we have used that fact that since $y\in\mathcal{V}^{\cdot 1,\cdot 2}_{(\alpha,\gamma)}$ the difference $|y_0^{\tau,v}-y_0^{\tau,u}|=0$. Thus, we can use the norm in (4.48) to control the increments $y_s^{\tau,v}-y_s^{\tau,u}$. The same can of course be done for increments in the first variable, using the norm in (4.49).

Assuming Hypothesis (**H2**), and having Definition 31 in mind, we now state a general convolution result for functions defined on Δ_3.

Theorem 33. *Let $z\in\mathcal{V}^{(\alpha,\gamma)}$ with $\alpha,\gamma\in(0,1)$ satisfying $\rho=\alpha-\gamma>0$, as given in Definition 16. We assume that z fulfills hypothesis (**H2**). Consider a function $y:\Delta_3\to\mathcal{L}(E,V)$ such that y is in the space $\mathcal{V}^{\cdot 1,\cdot 2}_{(\alpha,\gamma)}$ given in Definition 31. Then we have for all fixed $(s,t,\tau)\in\Delta_3$ that*

$$\mathbf{z}_{ts}^{2,\tau}*y_s^{\cdot 1,\cdot 2}:=\lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}}\mathbf{z}_{vu}^{2,\tau}\otimes y_s^{u,u}+(\delta_u\mathbf{z}_{vs}^{2,\tau})*y_s^{\cdot 1,\cdot 2} \tag{4.50}$$

is a well defined Volterra–Young integral. It follows that $*$ is a well defined bilinear operation between the three parameters Volterra function $\mathbf{z}^2$ and a three parameters path y. Moreover, we have that

$$\begin{aligned}|\mathbf{z}_{ts}^{2,\tau}*y_s^{\cdot 1,\cdot 2}-\mathbf{z}_{ts}^{2,\tau}\otimes y_s^{s,s}| &\lesssim \|y^{\cdot 1,\cdot 2}\|_{(\alpha,\gamma),1,2}\left(\|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma),1}+\|\mathbf{z}^1\|_{(\alpha,\gamma),1,2}\|\mathbf{z}^1\|_{(\alpha,\gamma),1}\right)\\ &\times\left(|\tau-s|^{-\gamma}|t-s|^{2\rho+\gamma}\wedge|\tau-s|^{2\rho}\right).\end{aligned} \tag{4.51}$$

Remark 34. Our definition (4.50) for $\mathbf{z}_{ts}^{2,\tau}*y_s^{\cdot 1,\cdot 2}$ is obviously motivated by (4.44), which had been obtained for smooth Volterra paths. We are now extending this identity to a generic path in $\mathcal{V}^{(\alpha,\gamma)}$.

Remark 35. The term $(\delta_u\mathbf{z}_{vs}^{2,\tau})*y_s^{\cdot 1,\cdot 2}$ in (4.43) is defined in the following way: observe that according to (4.46) we have

$$\delta_u\mathbf{z}_{vs}^{2,\tau}=\mathbf{z}_{vu}^{1,\tau}*\mathbf{z}_{us}^{1,\cdot}. \tag{4.52}$$

Therefore we get

$$(\delta_u\mathbf{z}_{vs}^{2,\tau})*y_s^{\cdot 1,\cdot 2}=\mathbf{z}_{vu}^{1,\tau}*\mathbf{z}_{us}^{1,\cdot}*y_s^{\cdot 1,\cdot 2},$$

which is well defined from a successive application of Theorem 26. Indeed, the convolution $\mathbf{z}_{ts}^{1,p}*y_s^{r,\cdot}$ for $p\ge r$ can be constructed in the exact same way as we constructed $\mathbf{z}_{vu}^{1,\tau}*\mathbf{z}_{us}^{1,\cdot}$. Namely, $y_s^{\cdot 1,\cdot 2}$ has to be considered as a constant in the lower variable. However, in light of Remark 32, the $\|y\|_{(\alpha,\gamma),1,2}$ norm invoked in (4.30) will be changed to the regularity required in (4.47).

Proof of Theorem 33. Let us denote by $\mathcal{I}_{\mathcal{P}}$ the approximation of the right hand side of (4.50), that is

$$\mathcal{I}_{\mathcal{P}} := \sum_{[u,v]\in\mathcal{P}} \Xi^{\tau}_{vu} := \sum_{[u,v]\in\mathcal{P}} \mathbf{z}^{2,\tau}_{vu} \otimes y^{u,u}_s + (\delta_u \mathbf{z}^{2,\tau}_{vs}) * y^{\cdot 2,\cdot 1}_s. \tag{4.53}$$

Our goal is to apply Lemma 22 to the increment Ξ, and we must therefore check the regularity of the integrand under the action of δ. To this aim, two simple computations using that $\delta_r \mathbf{z}^{2,\tau}_{vu} = \mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru}$ reveal

$$\delta_r(\mathbf{z}^{2,\tau}_{vu} \otimes y^{u,u}_s) = -\mathbf{z}^{2,\tau}_{vr} \otimes (y^{r,r}_s - y^{u,u}_s) + \mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} \otimes y^{u,u}_s, \tag{4.54}$$

$$\delta_r((\delta_u \mathbf{z}^{2,\tau}_{vs}) * y^{\cdot 1,\cdot 2}_s) = -\mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} * y^{\cdot 1,\cdot 2}_s, \tag{4.55}$$

where we notice that (since we are computing $\delta_r \Xi^{\tau}_{vu}$) we have

$$\delta_r\left(\delta_u \mathbf{z}^{2,\tau}_{vs}\right) = \delta_u \mathbf{z}^{2,\tau}_{vs} - \delta_r \mathbf{z}^{2,\tau}_{vs} - \delta_u \mathbf{z}^{2,\tau}_{rs} = -\mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru},$$

where we invoked (4.52) for the last identity. Let us now analyse the regularities of the terms in (4.54)–(4.55), starting with the right hand side of (4.54). Namely we recall that we assume in hypothesis (**H2**) that $\mathbf{z}^2 \in \mathcal{V}^{(2\rho+\gamma,\gamma)}$, and we also have $\|y^{\cdot 1,\cdot 2}\|_{(\alpha,\gamma),1,2} < \infty$ according to (4.47). Therefore recalling (4.48), (4.49) and Remark 32, and also recalling that $u \le r \le v$ we have for all $\eta \in [0,1]$

$$|\mathbf{z}^{2,\tau}_{vu} \otimes (y^{r,r}_s - y^{u,u}_s)| \lesssim \|y^{\cdot 1,\cdot 2}\|_{(\alpha,\gamma),1,2} \|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma),1} |u-s|^{-\eta} |\tau - v|^{-\gamma} |v-u|^{2\rho+\gamma+\eta}, \tag{4.56}$$

We then choose η such that $2\rho + \gamma + \eta > 1$, at the same time as $\eta < 1$, which is always possible, since $\rho > 0$.

In order to treat the remaining terms in (4.54) and (4.55), observe that formula (4.29) trivially yields (recall again that $y^{u,u}_s$ has to be considered as a constant in the lower variable)

$$\mathbf{z}^{1,\tau}_{ts} * y^{u,u}_s = \mathbf{z}^{1,\tau}_{ts} \otimes y^{u,u}_s.$$

Therefore we can gather our two remaining terms into

$$\mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} \otimes y^{u,u}_s - \mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} * y^{\cdot 1,\cdot 2}_s = -\mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} * (y^{\cdot 1,\cdot 2}_s - y^{u,u}_s). \tag{4.57}$$

Now in the spirit of Theorem 26, Inequality (4.30) and using condition (4.47) as well as relation (4.56), we have

$$\begin{aligned} &|\mathbf{z}^{1,\tau}_{vr} * \mathbf{z}^{1,\cdot}_{ru} * (y^{\cdot 1,\cdot 2}_s - y^{u,u}_s)| \\ &\quad \lesssim \|y^{\cdot 1,\cdot 2}\|_{(\alpha,\gamma),1,2} \|\mathbf{z}^1\|_{(\alpha,\gamma),1} \|\mathbf{z}^1\|_{(\alpha,\gamma),1,2} |\tau - v|^{-\gamma} |v-u|^{2\rho+\gamma+\eta} |u-s|^{-\eta}. \end{aligned} \tag{4.58}$$

Notice that the regularity obtained in (4.58) is the same as for (4.56). Hence repeating the same arguments as after (4.56) and recalling (4.54) and (4.55), we have obtained that

$$|\delta_r \Xi^{\tau}_{vs}| \lesssim c_{y,\mathbf{z}} |\tau - v|^{-\gamma} |u-s|^{-\eta} |v-u|^{\mu},$$

where $\eta < 1$ and $\mu = 2\rho + \gamma + \eta > 1$, and where the constant $c_{y,\mathbf{z}}$ is the same as in the right hand side of (4.58).

We are now in a situation which is similar to the one we had encountered in the proof of Theorem 26 (see inequality (4.35) in particular). Thus along the same lines as Theorem 26, resorting to a slight modification of the Sewing Lemma 22 involving two possible singularities, we get that the Riemann sums defined by (4.53) converge as $|\mathcal{P}| \to 0$, and we define

$$\mathbf{z}^{2,\tau}_{ts} * y^{\cdot 1,\cdot 2} := \lim_{|\mathcal{P}|\to 0} \mathcal{I}_{\mathcal{P}}. \tag{4.59}$$

In order to check (4.51), let us apply inequality (4.13) to the increment Ξ^τ defined in Eq. (4.53). To this aim, observe that taking $v = t$ and $u = s$ in the definition of Ξ^τ we get $\delta_s \mathbf{z}^{2,\tau}_{ts} = 0$, and thus $\Xi^\tau_{ts} = \mathbf{z}^{2,\tau}_{ts} \otimes y^{s,s}_s$. In addition, we have just seen in (4.59) that $\mathcal{I}(\Xi^\tau)_{ts} = \mathbf{z}^{2,\tau}_{ts} * y^{\cdot_1,\cdot_2}$, and thus

$$\mathcal{I}(\Xi^\tau)_{ts} - \Xi^\tau_{ts} = \mathbf{z}^{2,\tau}_{ts} * y^{\cdot_1,\cdot_2} - \mathbf{z}^{2,\tau}_{ts} \otimes y^{s,s}_s.$$

Our claim (4.51) is then a direct application of Lemma 22, together with the inequality estimates (4.56) and (4.58). □

Remark 36. The general convolution $\mathbf{z}^{2,\tau}_{ts} * y^{\cdot_1,\cdot_2}_s$ given in (4.50), for a path y defined on Δ_3, will be invoked for our rough path constructions in the remainder of the article. If one wishes to consider the convolution restricted to a path $y^{\cdot}_u$ defined on Δ_2, a natural way to proceed is to define

$$\mathbf{z}^{2,\tau}_{ts} * y^{\cdot}_s := \mathbf{z}^{2,\tau}_{ts} * \hat{y}^{\cdot_1,\cdot_2}_s, \quad \text{with} \quad \hat{y}^{r_1,r_2}_s = y^{r_2}_s.$$

Namely the path $\hat{y}$ has no dependence in r_1. We let the patient reader check the norm identity $\|\hat{y}^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1,2} \simeq \|y\|_{(\alpha,\gamma),1,2}$, where $\|\hat{y}^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1,2}$ is given as in (4.47) and $\|y\|_{(\alpha,\gamma),1,2}$ is introduced in (4.4).

Remark 37. As a special case of Remark 36, we can define the convolution $\mathbf{z}^{2,\tau}_{tu} * \mathbf{z}^{1,\cdot}_{us}$ by setting $y^r_u = \mathbf{z}^{1,r}_{us}$. Then y trivially satisfies $\|y\|_{(\alpha,\gamma),1,2} < \infty$ if $\mathbf{z}^1 \in \mathcal{V}^{(\alpha,\gamma)}$, which ensures a proper definition of $\mathbf{z}^{2,\tau}_{tu} * \mathbf{z}^{1,\cdot}_{us}$. Moreover, a direct application of Theorem 33 yields

$$|\mathbf{z}^{2,\tau}_{tu} * \mathbf{z}^{1,\cdot}_{us}| \lesssim |\tau - t|^{-\gamma} |t - s|^{3\rho+\gamma} \wedge |\tau - s|^{3\rho}.$$

Remark 38. In our applications to rough Volterra equations we will consider the case $\rho = \alpha - \gamma \in (1/3, 1/2]$, and therefore it is sufficient to show that the convolution product $*$ can be performed on the first and second level of a Volterra rough path. Indeed, whenever $\rho > 1/3$, the convolution product for third or higher order terms in the Volterra rough path are of regularity 3ρ which is greater than 1. Therefore the higher order convolutions $\mathbf{z}^{n,\tau}$ introduced in (**H2**) may be constructed as a classical Riemann integral. For a general $\rho \in (0, 1)$, it is easily conceived that one could extend the construction of the convolution product given in Theorem 33 to any order Volterra rough path $\mathbf{z}^n$ satisfying (4.46). This can be done by induction on n, and one first need to give a proper definition of the convolution product up to order $k = [1/\rho]$. The convolution product between elements $\mathbf{z}^K$ of order $K \geq k + 1$ is then constructed canonically through Riemann integration, together with (4.46). We defer this extension to a forthcoming paper [19].

4.3. Volterra convolutional functionals

With the preliminary notions of Section 4.2 in hand, we are now ready to generalize the notion of multiplicative functional (as introduced by Lyons et al. in [3]) to a Volterra context. The basic definition of Volterra convolutional functional is the following.

Definition 39. Let $n \geq 1$, and recall that $T^{(n)} = T^{(n)}(E)$ has been introduced in Definition 1. Consider a continuous map

$$\mathbf{z} : \Delta_3 \to T^{(n)}, \qquad (s, t, \tau) \mapsto \mathbf{z}^\tau_{ts} = \left(1, \mathbf{z}^{1,\tau}_{ts}, \ldots, \mathbf{z}^{n,\tau}_{ts}\right).$$

We call this mapping a Volterra convolutional functional if for all $(s,u,t,\tau) \in \Delta_4$ it satisfies

$$\mathbf{z}^{\tau}_{ts} = \mathbf{z}^{\tau}_{tu} * \mathbf{z}^{\cdot}_{us}.$$

The convolution product on elements in $T^{(n)}$ should here be interpreted component-wise, in the sense that for all $1 \le p \le n$ the p'th element $\left(\mathbf{z}^{\tau}_{tu} * \mathbf{z}^{\cdot}_{us}\right)^{p}$ in the right hand side above is defined by

$$\left(\mathbf{z}^{\tau}_{tu} * \mathbf{z}^{\cdot}_{us}\right)^{p} = \sum_{i=0}^{p} \mathbf{z}^{p-i,\tau}_{tu} * \mathbf{z}^{i,\cdot}_{us}, \tag{4.60}$$

and where the convolution in the right hand side of (4.60) is understood as in (4.29) or (4.50).

Remark 40. The Chen relation for Volterra convolutional functionals should be understood in terms of a convolution of an arbitrary Volterra path. That is, given any $f : \Delta_{p+1} \to E$, it holds that

$$\mathbf{z}^{p,\tau}_{ts} * f^{\cdot_1,\ldots,\cdot_p}_{u} = \sum_{i=0}^{p} [\mathbf{z}^{p-i,\tau}_{tu} * \mathbf{z}^{i,\cdot}_{us}] * f^{\cdot_1,\ldots,\cdot_p}_{u}.$$

The double convolution on the right hand side needs some extra explanation; given three functions f, g, h it should always be interpreted (after taking limits over a sequence of smooth functions) as

$$[f^{\tau}_{t} * g^{\cdot}_{u}] * h^{\cdot}_{s} = \int_0^t df^{\tau}_{r_1} \int_0^s dg^{r_1}_{r_2} h^{r_2}_{s}.$$

It is then clear (still taking limits over a sequence of smooth functions) that the double convolution satisfies

$$[f^{\tau}_{t} * g^{\cdot}_{u}] * h^{\cdot}_{s} = f^{\tau}_{t} * [g^{\cdot}_{u} * h^{\cdot}_{s}].$$

Remark 41. In order to define (4.60), we need in fact an extension (4.50) to higher order integrals of the form $\mathbf{z}^{j,\tau}$. As mentioned in Remark 38, we defer this extension to a future article. Notice however that, thanks to our restriction to $\rho > \frac{1}{3}$, we mostly need $p-i$ and $i \le 2$ in (4.60). This case is covered by (4.29) or (4.50).

Proceeding as in [3], we will now define some Hölder type norms adapted to our Volterra convolutional functionals.

Definition 42. For $\alpha, \gamma \in (0,1)$ with $\rho := \alpha - \gamma > 0$, consider a Volterra convolutional functional $\mathbf{z}$ of degree $n = \lfloor \rho^{-1} \rfloor$ as given in Definition 39. Let us assume that for $1 \le j \le n$ the component $\mathbf{z}^{j}$ of $\mathbf{z}$ satisfies $\mathbf{z}^{j} \in \mathcal{V}_2^{(j\rho+\gamma,\gamma)}$ where the space $\mathcal{V}^{(\alpha,\gamma)}$ has been introduced in Definition 16. In addition we suppose that

$$\|\mathbf{z}^{j}\|_{(j\rho+\gamma,\gamma),1} \lesssim \frac{M^{j}}{\Gamma(j\rho+1)} \quad \text{and} \quad \|\mathbf{z}^{j}\|_{(j\rho+\gamma,\gamma),1,2} \lesssim \frac{M^{j}}{\Gamma(j\rho+1)}, \tag{4.61}$$

for all $1 \le j \le n$, where M is a constant such that $\|z^{1}\|_{(\alpha,\gamma)} \le M$. Then we say that $\mathbf{z}$ is a Volterra rough path, and we denote the space of Volterra rough paths of regularity (α,γ) by $\mathscr{V}^{(\alpha,\gamma)}(\Delta_2([0,T]);E)$.

Remark 43. All rough paths (in the classical framework recalled in Section 2.2) are also Volterra rough paths with Volterra kernel $k = 1$, i.e. $x^{1}_{t} := \int_0^t 1 dx_r$. Thus the definition of Volterra rough paths is truly extending the definition of a rough path, and the convolutional product $*$ is extending the usual truncated tensor product by coupling the product through the integration of kernels.

By definition we can see that a Volterra rough path is a continuous mapping from $\Delta_3([0,T])$ to $T^{(\lfloor\rho^{-1}\rfloor)}(E)$. We will also find it useful to equip the space with a metric generalizing (2.16). Let us therefore define a metric for two Volterra rough paths $\mathbf{z}$ and $\mathbf{y}$ in $\mathscr{V}^{(\alpha,\gamma)}$ where $\rho=\alpha-\gamma$ by

$$d_{(\alpha,\gamma)}(\mathbf{z},\mathbf{y})=|z_0-y_0|+\sum_{m=1}^{\lfloor\rho^{-1}\rfloor}\|z^m-y^m\|_{(m\rho+\gamma,\gamma)}. \tag{4.62}$$

Definition 44. We define the space of geometric Volterra paths as the closure of smooth Volterra paths (i.e. paths in $\mathcal{V}^{(1,\gamma)}$) in the rough path metric from Eq. (4.62). The space of all geometric Volterra rough paths is denoted by $\mathscr{G}\mathscr{V}^{(\alpha,\gamma)}$.

Remark 45. Note that the geometric Volterra paths are not contained in a free-nilpotent Lie group, as is the case for regular rough paths. Indeed, there exists no concept of integration by parts in general for Volterra paths due to the possible singularities, and thus the notion of geometric Volterra paths cannot be seen as an object in the space $G^{(l)}$ given in Definition 3.

The following is an equivalent of the extension theorem for multiplicative functionals to a Volterra context. It can also be seen as an extension of Propositions 12 and 13 to a rough context.

Theorem 46. *Let $n=\lfloor\rho^{-1}\rfloor$ for $\rho=\alpha-\gamma>0$ and assume that $\mathbf{z}\in\mathscr{V}^{(\alpha,\gamma)}$ is an nth order Volterra rough path with values in $T^{(n)}(E)$ according to Definition 42. Then there exists a unique extension of $\mathbf{z}$ to $T(E)$. In particular, for all $m\geq n+1$ there exists a unique element $\mathbf{z}^m\in E^{\otimes m}$ such that for any $u\in[s,t]$ the following algebraic property is satisfied*

$$\mathbf{z}_{ts}^{m,\tau}=\sum_{i=0}^{m}\mathbf{z}_{tu}^{m-i,\tau}*\mathbf{z}_{us}^{i,\cdot}, \tag{4.63}$$

where we have used the convention $\mathbf{z}^0\equiv 1$ and $\mathbf{z}^j*1=1*\mathbf{z}^j=\mathbf{z}^j$. In addition the bound (4.61) can be extended to $\mathbf{z}$. Namely for $m\geq n+1$ we have for a constant $M>0$ such that $\|\mathbf{z}^1\|_{(\alpha,\gamma)}\leq M$ the following properties

$$\|\mathbf{z}^m\|_{(m\rho+\gamma,\gamma),1}\lesssim\frac{M^m}{\Gamma(m\rho+1)},\quad\text{and}\quad\|\mathbf{z}^m\|_{(m\rho+\gamma,\gamma),1,2}\lesssim\frac{M^m}{\Gamma(m\rho+1)}, \tag{4.64}$$

for any $\beta\in[0,1]$. It follows that there exists a unique Volterra signature with respect to the nth order Volterra rough path.

Proof. We will divide the proof into several steps.

Step 1: Uniqueness. The uniqueness problem will be addressed by induction. Indeed, for $m=n+1$ relation (4.63) reads

$$\delta_u\mathbf{z}_{ts}^{\tau}=\sum_{i=1}^{n}\mathbf{z}_{tu}^{n+1-i,\tau}*\mathbf{z}_{us}^{i,\cdot}. \tag{4.65}$$

The right hand side of (4.65) only depends on the stack $\{\mathbf{z}^j\,|\,1\leq j\leq n\}$, and is therefore uniquely defined thanks to our assumptions. Now consider $\tilde{\mathbf{z}}^m$ and $\bar{\mathbf{z}}^m$ two candidates for $\mathbf{z}^m$

with $m = n + 1$, and define $\psi^{\tau}_{ts} = \tilde{\mathbf{z}}^{m,\tau}_{ts} - \bar{\mathbf{z}}^{m,\tau}_{ts}$. Then according to (4.65) and relation (4.64) we have

$$\delta\psi^{\tau} = 0, \qquad \text{and} \qquad |\psi^{\tau}_{ts}| \lesssim |\tau - s|^{-\gamma}|t - s|^{(n+1)\rho+\gamma}. \tag{4.66}$$

In particular ψ is an additive functional with regularity greater than 1. It is thus readily seen that $\psi = 0$, which proves the uniqueness for $m = n + 1$. Once the uniqueness is shown for the levels $k = n + 1, \ldots, m$, an induction procedure similar to what lead to (4.66) also shows uniqueness for $k = m + 1$.

Step 2: Existence. The existence will be proved again based on induction. We will first show that an $(m = n+1)$th order Volterra rough path can be constructed purely based on the information of $\mathbf{z}^1, \ldots, \mathbf{z}^n$. To this aim note that if there exists a lift $\mathbf{z}^m$, then it must satisfy for any partition $\mathcal{P}$ of $[s,t]$

$$\mathbf{z}^{m,\tau}_{ts} = \sum_{[u,v]\in\mathcal{P}} \left(\mathbf{z}^{m,\tau}_{vu} + \delta_u \mathbf{z}^{m,\tau}_{vs}\right). \tag{4.67}$$

We will now take limits in (4.67) as $|\mathcal{P}| \to 0$.

To this aim, notice that according to (4.64) we have

$$|\mathbf{z}^{m,\tau}_{vu}| \lesssim |\tau - u|^{-\gamma}|v - u|^{m\rho+\gamma},$$

Hence, since $m\rho > 1$ we easily check that $\lim_{|\mathcal{P}|\to 0}\sum_{[u,v]\in\mathcal{P}} \mathbf{z}^{m,\tau}_{vu} = 0$. In particular we obtain

$$L^{m,\tau}_{ts} \equiv \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \left(\mathbf{z}^{m,\tau}_{vu} + \delta_u \mathbf{z}^{m,\tau}_{vs}\right) = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \delta_u \mathbf{z}^{m,\tau}_{vs}. \tag{4.68}$$

In addition $\mathbf{z}^{m,\tau}$ is required to satisfy (4.63). Thus recalling that $m = n + 1$ we have

$$L^{\tau}_{ts} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \Xi^{\tau}_{vu} \qquad \text{where} \qquad \Xi^{\tau}_{vu} = \sum_{i=1}^{n} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i,\cdot}_{us}. \tag{4.69}$$

Our strategy is now to prove that L^{τ}_{ts} exists by applying the Sewing Lemma 22 to the increment Ξ. The main assumption to check in order to apply Lemma 22 concerns $\delta\Xi$, and thus we obtain

$$|\delta_r \Xi^{\tau}_{vu}| = \left|\sum_{i=1}^{n} \mathbf{z}^{m-i,\tau}_{vr} * \mathbf{z}^{i,\cdot}_{ru}\right| \lesssim M^m \sum_{i=1}^{n} \frac{|\tau - r|^{-\gamma}|v - r|^{(m-i)\rho+\gamma}|r - u|^{i\rho}}{\Gamma\left((m-i)\rho + 1\right)\Gamma\left(i\rho + 1\right)}, \tag{4.70}$$

where the first identity is obtained thanks to an elementary computation of $\delta_r(\mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i,\cdot}_{us})$. Also note that the second inequality in (4.70) directly stems from the assumption (4.61), which stipulates that

$$\|\mathbf{z}^{j}\|_{(j\rho+\gamma,\gamma),1} \leq M^{j}\,\Gamma(j\rho + 1)^{-1}. \tag{4.71}$$

One can improve (4.70) in the following way: applying the neo-classical inequality from [3, Lemma 3.8], we know that there exists a $C > 0$ such that

$$\sum_{i=1}^{m-1} \frac{|v - r|^{(m-i)\rho+\gamma}|r - u|^{i\rho}}{\Gamma\left((m-i)\rho + 1\right)\Gamma\left(i\rho + 1\right)} \leq C\frac{|v - u|^{m\rho+\gamma}}{\Gamma\left(m\rho + 1\right)}.$$

Plugging this information into (4.70), we conclude that $\delta\Xi$ satisfies

$$|\delta_r \Xi^{\tau}_{vu}| \lesssim M^m \frac{|\tau - v|^{-\gamma}|v - u|^{m\rho+\gamma}}{\Gamma\left(m\rho + 1\right)}. \tag{4.72}$$

With (4.72) in hand, we can apply Lemma 22 to the increment Ξ. We get that the limit L^{τ}_{ts} defined by (4.68) exists, and we set $\mathcal{I}(\Xi^{\tau})_{ts} = L^{\tau}_{ts} = \mathbf{z}^{m,\tau}_{ts}$ for $m = n+1$. Moreover, a direct application of (4.13) together with the fact that $\Xi^{\tau}_{ts} = 0$ yield

$$|\mathbf{z}^{m,\tau}_{ts}| \lesssim M^m \frac{\left(|\tau-t|^{-\gamma}|t-s|^{m\rho+\gamma}\right) \wedge |\tau-s|^{m\rho}}{\Gamma(m\rho+1)}. \tag{4.73}$$

It now follows that

$$\|\mathbf{z}^m\|_{(m\rho+\gamma,\gamma),1} \lesssim M^m \Gamma(m\rho+1)^{-1}. \tag{4.74}$$

We also let the patient reader check that a simple induction procedure allows to generalize all our considerations until (4.74) for a generic $m \geq n+1$.

We will now prove that

$$\|\mathbf{z}^m\|_{(m\rho+\gamma,\gamma),1,2} \lesssim M^m \Gamma(m\rho+1)^{-1}. \tag{4.75}$$

To this aim, we need to repeat the procedure of Steps 1–2 for $\mathbf{z}^{m,\tau\tau'}_{ts} = \mathbf{z}^{m,\tau}_{ts} - \mathbf{z}^{m,\tau'}_{ts}$. In particular, the equivalent of the incremental Ξ^{τ} defined in (4.69) will be

$$\Xi^{\tau\tau'}_{ts} = \sum_{i=1}^{m-1} \mathbf{z}^{m-i,\tau\tau'}_{vu} * \mathbf{z}^{i,\cdot}_{us}.$$

With this increment in hand, relation (4.75) is proved along the same lines as (4.74). Details are omitted for the sake of conciseness. The norm $\|\mathbf{z}^m\|_{(m\rho+\gamma,\gamma),2}$ can also be estimated with the same kind of argument. Hence gathering (4.74) and (4.75), we have obtained that $\mathbf{z}^m \in \mathcal{V}^{(m\rho+\gamma,\gamma)}$ where $\mathcal{V}^{(\alpha,\gamma)}$ is given in Definition 16.

Step 3: Convolutional property. It remains to be proven that $\mathbf{z}^m$ is a convolutional functional in terms of Definition 39, i.e. that for $m \geq n+1$ and $(s,r,t,\tau) \in \Delta_4$ it satisfies

$$\mathbf{z}^{m,\tau}_{ts} = \sum_{i=0}^{m} \mathbf{z}^{m-i,\tau}_{tr} * \mathbf{z}^{i,\cdot}_{rs}. \tag{4.76}$$

In order to prove identity (4.76), recall that (4.69) can be read as

$$\mathbf{z}^{m,\tau}_{ts} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \sum_{i=1}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i,\cdot}_{us}.$$

Let us now divide a typical partition $\mathcal{P}$ into $\mathcal{P} \cap [s,r]$ and $\mathcal{P} \cap [r,t]$. This yields

$$\begin{aligned}\mathbf{z}^{m,\tau}_{ts} &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}\cap[s,r]} \sum_{i=1}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i,\cdot}_{us} + \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}\cap[r,t]} \sum_{i=1}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i,\cdot}_{us} \\ &= \mathbf{z}^{m,\tau}_{rs} + \hat{L}^{\tau}_{trs},\end{aligned} \tag{4.77}$$

where we have invoked (4.69) again for the second identity and where we have set

$$\hat{L}^{\tau}_{trs} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}\cap[r,t]} \sum_{i=1}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i,\cdot}_{us}.$$

As in the previous steps we now proceed by induction. Namely assume that (4.63) holds for $k = 1, \ldots, m-1$, and let us propagate the relation until $k = m$. Then applying the identity $\mathbf{z}^{i,\tau}_{ts} = \sum_{j=0}^{i} \mathbf{z}^{i-j,\tau}_{tu} * \mathbf{z}^{j,\cdot}_{us}$, which is valid for all $l < m$, we get

$$\hat{L}^{\tau}_{trs} = \hat{L}^{1,\tau}_{trs} + \hat{L}^{2,\tau}_{trs}, \tag{4.78}$$

where we define

$$\hat{L}^{1,\tau}_{trs} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}\cap[r,t]} \sum_{i=1}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i,\cdot}_{ur}$$

$$\hat{L}^{2,\tau}_{trs} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}\cap[r,t]} \sum_{i=1}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \left[\sum_{j=1}^{i} \mathbf{z}^{i-j,\cdot}_{ur} * \mathbf{z}^{j,\cdot}_{rs}\right].$$

Next, another application of (4.69) enables us to obtain directly

$$\hat{L}^{1,\tau}_{trs} = \mathbf{z}^{m,\tau}_{tr}. \tag{4.79}$$

In order to handle the term $\hat{L}^{2,\tau}_{trs}$, let us change the order of the sums with respect to i, j and invoke the associativity of the convolution product $*$, as described in Remark 40. We get

$$\begin{aligned}\hat{L}^{2,\tau}_{trs} &= \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}\cap[r,t]} \sum_{j=1}^{m-1}\sum_{i=j}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i-j,\cdot}_{ur} * \mathbf{z}^{j,\cdot}_{rs} \\ &= \sum_{j=1}^{m-1}\left[\lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}\cap[r,t]} \sum_{i=j}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i-j,\cdot}_{ur}\right] * \mathbf{z}^{j,\cdot}_{rs}.\end{aligned} \tag{4.80}$$

Now an elementary change of variable and (4.63) yield

$$\sum_{i=j}^{m-1} \mathbf{z}^{m-i,\tau}_{vu} * \mathbf{z}^{i-j,\cdot}_{ur} = \sum_{k=0}^{m-1-j} \mathbf{z}^{m-j-k,\tau}_{vu} * \mathbf{z}^{k,\cdot}_{ur} = \mathbf{z}^{m-j,\tau}_{vr} - \mathbf{z}^{m-j,\tau}_{ur} = \mathbf{z}^{m-j,\tau}_{vu} + \delta_u \mathbf{z}^{m-j,\tau}_{vr}.$$

Plugging this information into (4.80) and invoking (4.67), we end up with

$$\hat{L}^{2,\tau}_{trs} = \sum_{j=1}^{m-1}\left[\lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}\cap[r,t]} \mathbf{z}^{m-j,\tau}_{vu} + \delta_u \mathbf{z}^{m-j,\tau}_{vr}\right] * \mathbf{z}^{j,\cdot}_{rs} = \sum_{j=1}^{m-1} \mathbf{z}^{m-j,\tau}_{tr} * \mathbf{z}^{j,\cdot}_{rs}. \tag{4.81}$$

Let us summarize our considerations so far: gathering (4.81) and (4.79) into (4.78), and then inserting (4.78) into (4.77) we have obtained that

$$\mathbf{z}^{m,\tau}_{ts} = \mathbf{z}^{m,\tau}_{tr} + \mathbf{z}^{m,\tau}_{rs} + \sum_{j=1}^{m-1} \mathbf{z}^{m-i,\tau}_{tr} * \mathbf{z}^{i,\cdot}_{rs} = \sum_{j=0}^{m} \mathbf{z}^{m-i,\tau}_{tr} * \mathbf{z}^{i,\cdot}_{rs}.$$

This concludes our induction procedure, and thus (4.63) holds for all $m \geq 1$. □

Remark 47. Theorem 46 tells us that the Volterra signature associated to a Volterra path is uniquely determined from the Volterra rough path introduced in Definition 42. That is, once we have constructed a truncated Volterra rough path (remember that this object is by no means unique) then there exists a unique extension with respect to the full Volterra rough path.

5. Non-linear Volterra integral equations driven by rough noise

In this section we will see how we can substitute the conventional tensor product from rough path theory with the convolution product defined in Section 4 in order to show existence and uniqueness of Volterra equations with singular kernels. Similarly to the theory of controlled rough path introduced by Gubinelli in [14], we define a class Volterra controlled paths. The composition of the Volterra controlled paths with the Volterra rough path from Definition 42 gives an abstract Riemann integrand such that we may construct a Volterra integral by application of the Volterra Sewing Lemma 22. This abstract integration step is then the key in order to define and solve Volterra type equations.

5.1. Volterra controlled processes and rough Volterra integration

As many of the results here are extensions of classical texts on rough path such as [18] or [14], we will try to keep the proofs as concise as possible. The reader is sent to the aforementioned references for further information on the results and properties of controlled rough paths and solutions to non-linear differential equations driven by rough paths. We will first give a definition of another modification of the Volterra–Hölder spaces given in Definition 16 in order to give a precise analysis of Volterra-controlled paths.

Definition 48. Let $\mathcal{W}_2^{(\alpha,\gamma)}$ denote the space of functions $u:\Delta_3\to V$ such that $(p,q,s)\mapsto u_s^{p,q}\in V$ and

$$\|u^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma)}:=\|u^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1}+\|u^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1,2}<\infty \tag{5.1}$$

where we define the norm (recall the convention $\rho=\alpha-\gamma$ below)

$$\|u^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1}:=\sup_{(s,t,\tau)\in\Delta_3}\frac{\left|u_{ts}^{\tau,\tau}\right|}{|\tau-t|^{-\gamma}|t-s|^{\alpha}\wedge|\tau-s|^{\rho}}, \tag{5.2}$$

and the norm $\|u^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma),1,2}$ is given as in Definition 31.

Remark 49. Note in particular that the definition of the space $\mathcal{W}_2^{(\alpha,\gamma)}$ does not involve a norm similar to (4.4). Although the definition of $\|u^{\cdot_1,\cdot_2}\|_{(\alpha,\gamma)}$ is a slight abuse of notation, we believe that it will be clear from the superscripts of u what norm we apply.

We now turn to the definition of controlled Volterra paths, which is crucial for a proper definition of rough Volterra equations.

Definition 50. Let $z\in\mathcal{V}^{(\alpha,\gamma)}(E)$ for some $\rho=\alpha-\gamma>0$. We assume that there exists two functions $y:\Delta_2\to V$ and $y':\Delta_3\to\mathcal{L}(E,V)$, such that $y_0^\tau=y_0\in E$ for any $\tau\in[0,T]$ and $y_0^{\prime,p,q}=y_0'\in E$ for any $(q,p)\in\Delta_2$, and satisfying the relation

$$y_{ts}^\tau=z_{ts}^\tau*y_s^{\prime,\tau,\cdot}+R_{ts}^\tau, \tag{5.3}$$

where $R\in\mathcal{V}_2^{(2\alpha,2\gamma)}(V)$ and $y'\in\mathcal{W}_2^{(\alpha,\gamma)}$. (Recall that the spaces $\mathcal{V}_2^{(2\alpha,2\gamma)}$ and $\mathcal{W}_2^{(\alpha,\gamma)}$ are respectively introduced in Remark 20 and Definition 48). Whenever (y,y') satisfies relation (5.3) we say that (y,y') is a Volterra path controlled by z (or controlled Volterra path in general) and we write $(y,y')\in\mathscr{D}_z^{(\alpha,\gamma)}(\Delta_2;V)$. We equip this space with a semi-norm $\|\cdot\|_{z,(\alpha,\gamma)}$ given by

$$\|y,y'\|_{z,(\alpha,\gamma)}=\|y^{\prime,\cdot_1,\cdot_2}\|_{(\alpha,\gamma)}+\|R\|_{(2\alpha,2\gamma)}. \tag{5.4}$$

Under the mapping $(y,y')\mapsto|y_0|+|y_0'|+\|y,y'\|_{z,(\alpha,\gamma)}$ the space $\mathscr{D}_z^{(\alpha,\gamma)}(\Delta_2;V)$ is a Banach space. The remainder term R in (5.3) with respect to a Volterra path $(y,y')\in\mathscr{D}_z^{(\alpha,\gamma)}$ will typically be denoted by R^y.

Remark 51. We call the function y' the Volterra–Gubinelli derivative, and emphasize that this function is evaluated on Δ_3, where it has *two* upper arguments. This is denoted by $\Delta_3 \ni (s,p,q) \mapsto y_s^{\prime,q,p}$ as opposed to the increment of a path y in the upper variable denoted by $\Delta_3 \ni (s,p,q) \mapsto y_s^{qp}$.

Remark 52. For a controlled Volterra path the regularity of y in the upper argument is inherited from the regularity of the upper argument of the driving noise z, Gubinelli derivative and remainder term R^y. That is, it is implied from relation (5.3) that for $(y, y') \in \mathscr{D}_z^{(\alpha,\gamma)}(\Delta_2; V)$ we have

$$y_{ts}^{qp} = z_{ts}^{qp} * y_s^{\prime,p,\cdot 2} + z_{ts}^{q} * y_s^{\prime,qp,\cdot} + R_{ts}^{qp}. \tag{5.5}$$

Our next step is to show that we may construct the Volterra rough integral in a very similar way to the classical rough path integral, but changing $\otimes$ for $*$ as well as applying the Volterra Sewing Lemma 22. It follows that the Volterra integral of a controlled path with respect to a driving Hölder noise $x \in \mathcal{C}^\alpha$ is again a controlled Volterra path.

Theorem 53. *Let $x \in \mathcal{C}^\alpha$ and k be a Volterra kernel satisfying* (**H**) *with a parameter γ such that $\rho = \alpha - \gamma > \frac{1}{3}$. Thanks to Theorem 23, define $z_t^\tau = \int_0^t k(\tau, r)\, dx_r$ and assume there exists a second order Volterra rough path $\mathbf{z} \in \mathscr{V}^{(\alpha,\gamma)}(\Delta_2; E)$ built from z according to Definition 42. Additionally, suppose both components of $\mathbf{z}$ are uniformly bounded. Namely, we assume there exists an $M > 0$ such that*

$$\|\mathbf{z}\|_{(\alpha,\gamma)} := \|\mathbf{z}^1\|_{(\alpha,\gamma)} + \|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma)} \leq M, \tag{5.6}$$

where the two norm quantities correspond to the norms given in Definition 16 and Remark 20. We now consider a controlled Volterra path $(y, y') \in \mathscr{D}_{\mathbf{z}^1}^{(\alpha,\gamma)}(\Delta_2; \mathcal{L}(E, V))$. Then the following holds true:

(i) *The following limit exists for all $(s,t,\tau) \in \Delta_3$,*

$$w_{ts}^\tau = \int_s^t k(\tau, r) y_r^r dx_r := \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}_{vu}^{1,\tau} * y_u^{\cdot} + \mathbf{z}_{vu}^{2,\tau} * y_u^{\prime,\cdot 1,\cdot 2}. \tag{5.7}$$

(ii) *Let w be defined by (5.7). There exists a constant $C = C_{M,\alpha,\gamma}$ such that for all $(s,t) \in \Delta_2$ we have*

$$\begin{aligned}&\left| w_{ts}^\tau - \mathbf{z}_{ts}^{1,\tau} * y_s^{\cdot} - \mathbf{z}_{ts}^{2,\tau} * y_s^{\prime,\cdot 1,\cdot 2} \right| \\ &\quad \leq C \|y, y'\|_{z,(\alpha,\gamma)} \|\mathbf{z}\|_{(\alpha,\gamma)} \left[|\tau - t|^{-\gamma} |t-s|^{3\rho+\gamma} \wedge |\tau - s|^{3\rho} \right].\end{aligned} \tag{5.8}$$

(iii) *For all $(s,t,p,q) \in \Delta_4$ and $\eta \in [0,1]$ and $\zeta \in [0,\rho)$ we have*

$$\begin{aligned}&\left| w_{ts}^{qp} - \mathbf{z}_{ts}^{1,qp} * y_s^{\cdot} - \mathbf{z}_{ts}^{2,qp} * y_s^{\prime,\cdot 1,\cdot 2} \right| \\ &\quad \leq C \|y, y'\|_{z,(\alpha,\gamma)} \|\mathbf{z}\|_{(\alpha,\gamma)} |p-q|^\eta |q-t|^{-\eta+\zeta} \left[|q-t|^{-\gamma-\zeta} |t-s|^{3\rho+\gamma} \wedge |q-s|^{3\rho-\zeta} \right].\end{aligned} \tag{5.9}$$

(iv) *The couple (w, w') is a controlled Volterra path in $\mathscr{D}_{\mathbf{z}^1}^{(\alpha,\gamma)}(\Delta_2, V)$, where we recall that w is defined by (5.7) and $w_t^{\prime,\tau,p} = y_t^p$.*

Remark 54. According to our computations (see in particular (5.14)) we believe that Theorem 53 should hold true under the condition $3\rho + \gamma > 1$ (vs. $3\rho > 1$). We have sticked to the more restrictive assumption $3\rho > 1$ in order to be compatible with Definition 42 for $n = 2$.

Proof of Theorem 53. We define $\Xi^{\tau}_{vu} = \mathbf{z}^{1,\tau}_{vu} * y^{\cdot}_{u} + \mathbf{z}^{2,\tau}_{vu} * y'^{,\cdot 1,\cdot 2}_{u}$, where $\mathbf{z}^{2,\tau}_{vu} * y'^{,\cdot 1,\cdot 2}_{u}$ is understood according to Theorem 33. Namely, it is readily checked, whenever $(y, y') \in \mathscr{D}^{(\alpha,\gamma)}_{\mathbf{z}^1}$ that $y' \in \mathcal{V}^{\cdot 1,\cdot 2}_{(\alpha,\gamma)}$ where $\mathcal{V}^{\cdot 1,\cdot 2}_{(\alpha,\gamma)}$ is given in Definition 31. Therefore Theorem 33 enables to define

$$\mathbf{z}^{2,\tau}_{ts} * y'^{,\cdot 1,\cdot 2}_{s} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} \mathbf{z}^{2,\tau}_{vu} \otimes y'^{,u,u}_{s} + \delta_u \mathbf{z}^{2,\tau}_{vs} * y'^{,\cdot 1,\cdot 2}_{s}.$$

Now that Ξ is properly defined, our next step is to invoke Lemma 22 in order to define

$$w^{\tau}_{ts} = \int_s^t k(\tau, r)\, y^r_r dx_r = \mathcal{I}(\Xi)_{ts}.$$

To this aim, similarly to the proof of Theorem 46, we need to check that $\delta\Xi$ is sufficiently regular. This is what we proceed to do below in order to obtain (5.7).

We first compute $\delta\Xi^{\tau}$, where we recall that $\Xi^{\tau}_{vu} = \mathbf{z}^{1,\tau}_{vu} * y^{\cdot}_{u} + \mathbf{z}^{2,\tau}_{vu} * y'^{,\cdot 1,\cdot 2}_{u}$. That is, combining elementary algebraic properties of the operator δ and relation (4.60) read for $p = 1, 2$ we get the following relation for $(u, m, v, \tau) \in \Delta_4$,

$$\delta_m \Xi^{\tau}_{vu} = -\mathbf{z}^{1,\tau}_{vm} * y^{\cdot}_{mu} - \mathbf{z}^{2,\tau}_{vm} * y'^{,\cdot 1,\cdot 2}_{mu} + \mathbf{z}^{1,\tau}_{vm} * \mathbf{z}^{1,\cdot}_{mu} * y'^{,\cdot,\cdot}. \tag{5.10}$$

Now we resort to the fact that y satisfies (5.3) in order to write

$$\mathbf{z}^{1,\tau}_{vm} * y^{\cdot}_{mu} = \mathbf{z}^{1,\tau}_{vm} * \left(\mathbf{z}^{1,\cdot}_{mu} * y'^{,\cdot 1,\cdot 2}_{u}\right) + \mathbf{z}^{1,\tau}_{vm} * R^{\cdot}_{mu}.$$

Plugging this into (5.10) we obtain

$$\delta_m \Xi^{\tau}_{vu} = -\mathbf{z}^{2,\tau}_{vm} * y'^{,\cdot 1,\cdot 2}_{mu} - \mathbf{z}^{1,\tau}_{vm} * R^{\cdot}_{mu}. \tag{5.11}$$

Thanks to relation (5.11), we can now analyse the regularity of $\delta\Xi^{\tau}$. Indeed, invoking Theorem 33 we get

$$|\mathbf{z}^{2,\tau}_{vm} * y'^{,\cdot 1,\cdot 2}_{mu}| \le \|y'^{,\cdot 1,\cdot 2}\|_{(\alpha,\gamma),1,2} \|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma)} |u-m|^{\rho} |\tau - m|^{-\gamma} |v-m|^{2\rho+\gamma} \tag{5.12}$$

and similarly

$$|\mathbf{z}^{1,\tau}_{vm} * R^{\cdot}_{mu}| \le \|R\|_{(2\rho+\gamma,\gamma)} \|\mathbf{z}^1\|_{(\alpha,\gamma)} |\tau - m|^{-\gamma} |v-m|^{\alpha} |u-m|^{2\rho}. \tag{5.13}$$

Gathering (5.12) and (5.13) into (5.11) and recalling that $\tau > v > m > u$, we thus obtain that

$$|\delta_m \Xi^{\tau}_{vu}| \lesssim \|y, y'\|_{\mathbf{z}^1,(\alpha,\gamma)} \|\mathbf{z}\|_{(\alpha,\gamma)} |\tau - v|^{-\gamma} |v-u|^{3\rho+\gamma}. \tag{5.14}$$

Since $3\rho + \gamma > 1$, we can apply the Volterra Sewing Lemma 22 and define $w^{\tau}_{ts} := \mathcal{I}(\Xi^{\tau})_{ts}$. This achieves the proof of (5.7) and relation (5.8).

Next, we shall prove Inequality (5.9). Start to set $\Xi^{qp}_{ts} = \mathbf{z}^{1,qp}_{ts} * y^{\cdot}_{s} + \mathbf{z}^{2,qp}_{ts} * y'^{,\cdot 1,\cdot 2}_{s}$, and observe that by the exact same computations as above (remember that $u \mapsto \delta_u$ acts on the lower argument of a function f^{τ}_t) we obtain

$$\delta_u \Xi^{qp}_{ts} = -\mathbf{z}^{2,qp}_{vm} * y'^{,\cdot 1,\cdot 2}_{mu} - \mathbf{z}^{1,qp}_{vm} * R^{\cdot}_{mu}.$$

Thus, the regularity $\delta_u \Xi^{p,q}_{ts}$ follows from the assumption (4.61) of regularity on the Volterra rough path $\mathbf{z}$ and the controlled path (y, y') together with equivalent bounds as in (5.12) and (5.13), taking into account the increment in the upper parameters. We therefore obtain for $(s, u, t, p, q) \in \Delta_5$ and $\eta \in [0, 1]$ and $\zeta \in [0, \rho)$

$$|\delta_u \Xi^{qp}_{ts}| \le \left(\|y'^{,\cdot 1,\cdot 2}\|_{(\alpha,\gamma),1,2} \|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma)} + \|R\|_{1,(\alpha,\gamma)} \|\mathbf{z}^1\|_{(\alpha,\gamma)}\right)$$

$$\times|q-p|^{\eta}|p-t|^{-\eta+\zeta}|p-t|^{-\gamma-\zeta}|t-s|^{3\rho+\gamma}. \tag{5.15}$$

Applying again the Volterra Sewing Lemma 22, we now easily conclude that (5.9) holds. □

Remark 55. The definition of a controlled Volterra rough path tells us that $y' : \Delta_3 \to V$, i.e. it takes three ordered time variables as input. However, the computations of Theorem 53 reveal that when $(y, y') \in \mathscr{D}_{\mathbf{z}^1}^{(\alpha,\gamma)}(\mathcal{L}(E, V))$, the controlled derivative of $w_t^{\tau} = \int_0^t k(\tau, r)\, y_r^r dx_r$ only depends on two variables. Specifically we have $w_t^{\prime,\tau,q} = w_t^{\prime,q} \equiv y_t^q$, which is seen from item (iv) in Theorem 53. One can thus refine Theorem 53 and state that the Volterra rough integration sends $(y, y') \in \mathscr{D}_z^{(\alpha,\gamma)}$ to a controlled process $(w, w') \in \hat{\mathscr{D}}_z^{(\alpha,\gamma)}$ where the space $\hat{\mathscr{D}}_z^{(\alpha,\gamma)}$ is defined by

$$\hat{\mathscr{D}}_z^{(\alpha,\gamma)}(\Delta_2; V) := \{(w, w') \in \mathscr{D}_z^{(\alpha,\gamma)} \mid w_s^{\prime,\tau,p} = w_s^{\prime,p}\}. \tag{5.16}$$

The space $\hat{\mathscr{D}}_z^{(\alpha,\gamma)}$ will be used in the composition step below.

Proposition 56. *Let $f \in C_b^3(V)$ and assume $(y, y') \in \hat{\mathscr{D}}_z^{(\alpha,\gamma)}(V)$. Then the composition $(\varphi, \varphi') := (f(y), y'f'(y))$ is a controlled Volterra path in $\mathscr{D}_z^{(\alpha,\gamma)}(V)$, where the derivative $\varphi' : \Delta_3 \to V$ is given by*

$$\Delta_3 \ni (t, p, q) \mapsto y_t^{\prime,p} f'\left(y_t^q\right). \tag{5.17}$$

Moreover, there exists a constant $C = C_{M,\alpha,\gamma,\|f\|_{C_b^3}} > 0$ such that

$$\|\varphi, \varphi'\|_{z;(\alpha,\gamma)} \leq C\left(1 + \|z\|_{(\alpha,\gamma)}\right)^2 \left[\left(|y_0'| + \|y, y'\|_{z,(\alpha,\gamma)}\right) \vee \left(|y_0'| + \|y, y'\|_{z,(\alpha,\gamma)}\right)^2\right] \tag{5.18}$$

Proof. Let us first prove the algebraic part of the proposition, namely relation (5.17). We start to decompose the increment $f(y^q)_{ts}$ into

$$f(y^q)_{ts} = y_{ts}^{\tau} f'(y_s^q) + \left[f(y^q)_{ts} - y_{ts}^{\tau} f'(y_s^q)\right].$$

We then resort to relation (5.3) in order to write

$$f(y^q)_{ts} = z_{ts}^q * y_s^{\prime,q,\cdot} f'(y_s^q) + R_{ts}^{f(y),q},$$

where we have set R^y to be the remainder of y in (5.3), and

$$R_{ts}^{f(y),q} = \left[f(y^q)_{ts} - y_{ts}^{\tau} f'(y_s^q)\right] + R_{ts}^{y,q} f'(y_s^q). \tag{5.19}$$

In addition, recalling that $(y, y') \in \hat{\mathscr{D}}^{(\alpha,\gamma)}$ the path $y^{\prime,q,\cdot}$ does not depend on q. Hence we get

$$f(y^q)_{ts} = z_{ts}^q * y_s^{\prime,\cdot} f'(y_s^q) + R_{ts}^{f(y),q}. \tag{5.20}$$

We now set $\varphi_s^{\prime,q,p} = y_s^{\prime,p} f'(y_s^q)$. With relation (4.29) in mind it is readily checked that (5.20) can be recast as

$$f(y^q)_{ts} = z_{ts}^q * \varphi_s^{\prime,q,\cdot} + R_{ts}^{f(y),q},$$

which corresponds to our claim in (5.17).

Let us now focus on Inequality (5.18). To this end, recall that the norm $\|\varphi, \varphi'\|_{z;(\alpha,\gamma)}$ is defined by (5.4). Thus we have

$$\|\varphi, \varphi'\|_{z;(\alpha,\gamma)} = \|f(y), y'f(y)\|_{z,(\alpha,\gamma)} = \|y^{\prime,\cdot 2} f(y^{\cdot 1})\|_{(\alpha,\gamma)} + \|R^{f(y)}\|_{(2\rho+\gamma,\gamma)}. \tag{5.21}$$

We shall analyse the two terms in the right hand side of (5.21) separately. We start with the derivative φ', for which we will bound the two norms given by (5.2) and (4.47). Specifically, observe first that the difference in the lower variable for the derivative φ' is given by

$$\left(y' f(y)\right)_{ts}^{q,p} = y_t^{\prime,p} f\left(y_t^q\right) - y_s^{\prime,p} f\left(y_s^q\right),$$

and thus by addition and subtraction of $y_s^{\prime,p} f\left(y_t^q\right)$ it is readily checked that the following bound is satisfied

$$\|y^{\prime,\cdot 2} f(y^{\cdot 1})\|_{(\alpha,\gamma),1} \lesssim \|f\|_{\mathcal{C}_b^1}\left(\|y^{\prime,\cdot 2}\|_{(\alpha,\gamma)} + \|y\|_{(\alpha,\gamma)}\right). \tag{5.22}$$

Let us now consider the quantity $\|y^{\prime,\cdot 2} f(y^{\cdot 1})\|_{(\alpha,\gamma),1,2}$. To this end, we will in two stages encounter first order Taylor expansions, and thus we recall that for a differentiable function f on V we have for $a, b \in V$

$$g(a) - g(b) = L(a,b), \quad \text{where} \quad L(a,b) = \int_0^1 Df(\theta a + (1-\theta)b)d\theta(a-b). \tag{5.23}$$

We now need to control the simultaneous increment in the upper and lower variables according to (4.47). Let us first consider $y_t^{\prime,p} f\left(y_t^q\right)$ with fixed p and increments in the variables t and q. By a simple addition and subtraction argument, we obtain the identity

$$\left(y_\cdot^{\prime,p} f\left(y_\cdot^{\cdot}\right)\right)_{ts}^{qr} = y_t^{\prime,p} L(y_t^q, y_t^r) y_t^{qr} - y_s^{\prime,p} L(y_s^q, y_s^r) y_s^{qr}, \tag{5.24}$$

where L is given as above. Observe now that by adding and subtracting the quantity $y_s^{\prime,p} L(y_t^q, y_t^r) y_t^{qr}$ to the right hand side in of (5.24), we obtain that

$$y_t^{\prime,p} L(y_t^q, y_t^r) y_t^{qr} - y_s^{\prime,p} L(y_s^q, y_s^r) y_s^{qr} = y_{ts}^{\prime,p} F_t^{qr} + y_s^{\prime,p}(F_t^{qr} - F_s^{qr}), \tag{5.25}$$

where $F_t^{qr} := L(y_t^q, y_t^r) y_t^{qr}$. Due to the boundedness assumption on f and its derivatives, it is clear that $|F_t^{qr}| \lesssim \|f\|_{\mathcal{C}_b^1}\|y\|_{(\alpha,\gamma),1,2}|q-r|^{\eta}|r-t|^{-\eta}$. Furthermore, it is readily seen that

$$F_t^{qr} - F_s^{qr} = \mathbf{L}(y_t^q, y_t^r, y_s^q, y_s^r) y_t^{qr} + L(y_t^q, y_t^r) y_{ts}^{qr}, \tag{5.26}$$

Where $\mathbf{L}(a,b,a',b')$ is given as the remainder of a first order two-variable Taylor approximation of L, in the sense that $L(a,b) - L(a',b') = \mathbf{L}(a,b,a',b')$, which is explicitly given by

$$\begin{aligned}
&\mathbf{L}(a,b,a',b') \\
&\quad = \int_0^1\int_0^1 D^2 g(\theta'(\theta a + (1-\theta)b) + (1-\theta')(\theta a' + (1+\theta)b'))d\theta'(\theta(a-a') \\
&\qquad + (1-\theta)(b-b'))d\theta.
\end{aligned}$$

Again, due to the boundedness of f and its derivatives, for any $\eta \in [0,1]$ we obtain that

$$|F_t^{qr} - F_s^{qr}| \lesssim \|f\|_{\mathcal{C}_b^2}\|y\|_{(\alpha,\gamma)1,2}|q-r|^{\eta}|r-t|^{-\eta}\left[|r-t|^{-\gamma}|t-s|^{\alpha} \wedge |r-s|^{\rho}\right]. \tag{5.27}$$

Inserting relation (5.26) into (5.25), and invoking the bound in (5.27) as well as the regularity of y' and y, we obtain that

$$\|y^{\prime,\cdot 2} f(y^{\cdot 1})\|_{(\alpha,\gamma),1,2,<} \leq \|f\|_{\mathcal{C}_b^2}\left(\|y^{\prime,\cdot 1,\cdot 2}\|_{(\alpha,\gamma)} + \|y\|_{(\alpha,\gamma)}\right). \tag{5.28}$$

A similar argument can now be used to also show that $\|y^{\prime,\cdot 2} f(y^{\cdot 1})\|_{(\alpha,\gamma),1,2,>} < \infty$, and thus it follows by (4.47) that

$$\|y^{\prime,\cdot 2} f(y^{\cdot 1})\|_{(\alpha,\gamma),1,2} \leq \|f\|_{\mathcal{C}_b^2}\left(\|y^{\prime,\cdot 1,\cdot 2}\|_{(\alpha,\gamma)} + \|y\|_{(\alpha,\gamma)}\right). \tag{5.29}$$

Let us now handle the term $\|R^{f(y)}\|_{(2\rho+\gamma,\gamma)}$ in Eq. (5.21). More precisely recalling that the norm $\|\cdot\|_{(2\rho+\gamma,\gamma)}$ is given by (4.8), let us first bound the quantity $\|R^{f(y)}\|_{(2\rho+\gamma,\gamma),1}$. Towards this aim, we go back to the definition (5.19) of $R^{f(y)}$ and apply Taylor's expansion in a standard way. That is, define $c^\tau_{t,s}(a) = ay^\tau_s + (1-a)y^\tau_t$, and observe that

$$R^{f(y),\tau}_{ts} = R^{y,\tau}_{ts} f'\left(y^\tau_s\right) + \frac{1}{2}\left(y^\tau_{ts}\right)^{\otimes 2} \int_0^1 f''\left(c^\tau_{t,s}(a)\right) da. \tag{5.30}$$

The regularity of $R^{f(y)}$ for the $\|\cdot\|_{(2\rho+\gamma,\gamma),1}$ norm now follows from the boundedness of the second derivative of f, the squared regularity of the increment of y and the regularity of R^y.

Next, we will compute the regularity in the upper argument for $R^{f(y)}$, which corresponds to the semi-norm $\|\cdot\|_{(2\rho+\gamma,\gamma),1,2}$ in (4.4). In particular, we will consider the increment

$$\begin{aligned} R^{qp,f(y)}_{ts} &= R^{p,y}_{ts} f'(y_s)^{qp} + R^{qp,y}_{ts} f'\left(y^p_s\right) \\ &\quad + \frac{1}{2}\left(y^p_{ts}\right)^{\otimes 2}\int_0^1 f''(c_{t,s}(a))^{qp} da + \frac{1}{2}((y^q_{ts})^{\otimes 2} - \left(y^p_{ts}\right)^{\otimes 2})\int_0^1 f''(c^p_{t,s}(a)). \end{aligned} \tag{5.31}$$

Using that $f \in C^3_b$ and $a^2 - b^2 = (a+b)(a-b)$, it follows from a combination of (5.31) and (5.30) that

$$\begin{aligned} \|R^{f(y)}\|_{(2\rho+\gamma,\gamma)} &= \|R^{f(y)}\|_{(2\rho+\gamma,\gamma),1} + \|R^{f(y)}\|_{(2\rho+\gamma,\gamma)1,2} \\ &\le \|f\|_{C^3_b}\left(\|R^y\|_{(2\rho+\gamma,\gamma)} + \|y\|^2_{(\rho+\gamma,\gamma)}\right). \end{aligned}$$

We now use the fact that the regularity of the controlled Volterra path is inherited by the noise, as discussed in Remark 52 and see that

$$\|y\|_{(\alpha,\gamma)} \le \left(|y'_0| + \|y^{\prime,\cdot 2}\|_{(\alpha,\gamma)}\right)(\|z\|_{(\alpha,\gamma)} + \|R^y\|_{(2\rho+\gamma,\gamma)}). \tag{5.32}$$

Combining the information from (5.32), (5.30), and (5.22) yields (5.18). Namely, it follows that

$$\begin{aligned} &\|f(y), f(y)y'\|_{z,(\alpha,\gamma)} \\ &\quad \le C_{\|f\|_{C^3_b},\alpha,\gamma}\left(1+\|z\|_{(\alpha,\gamma)}\right)^2\left[\left(|y'_0| + \|y,y'\|_{z,(\alpha,\gamma)}\right)^2 \vee \left(|y'_0| + \|y,y'\|_{z,(\alpha,\gamma)}\right)\right]. \quad \square \end{aligned}$$

Remark 57. We point out that we require $f \in C^3_b$ in order to compose f with a controlled Volterra path $(y, y') \in \hat{\mathscr{D}}^{(\alpha,\gamma)}_z$. This requirement is one degree of differentiation more than what is standard in classical rough path theory (see e.g. [18, Section 7]). The reason for this comes from the fact that we also need regularity in the upper argument of the controlled Volterra paths, and thus we see that in order to bound the term $f''(c)^{pq}$ in (5.31), we need $f \in C^3_b$.

5.2. Rough Volterra equations

Based on the concept of controlled Volterra paths and Volterra integration introduced in Section 5.1, we are now ready to prove existence and uniqueness of non-linear Volterra equations. As we have seen so far, the results that we obtain are directly comparable to those known from the classical setting under substitution of the tensor product with the convolution product.

Theorem 58. *Let $\mathbf{z} \in \mathscr{V}^{(\alpha,\gamma)}(E)$ with $\alpha - \gamma > \frac{1}{3}$. Assume that $\mathbf{z}$ satisfies the same hypothesis as in Theorem 53 and suppose $f \in \mathcal{C}^4_b(V; \mathcal{L}(E,V))$. Then there exists a unique Volterra solution in $\hat{\mathscr{D}}^{(\alpha,\gamma)}_{\mathbf{z}^1}(V)$ to the equation*

$$y^\tau_t = y_0 + \int_0^t k(\tau,r) f\left(y^r_r\right) dx_r, \quad (t,\tau) \in \Delta_2([0,T]), \quad y_0 \in E, \tag{5.33}$$

where the integral is understood as a rough Volterra integral given in Theorem 53.

Proof. The parameter (s, τ) we consider in this proof sits in a small variation of the simplex Δ_2 defined by (2.1). Namely we define the trapezoid

$$\Delta_2^T\left([a,b]\right) = \{(s,\tau) \in [a,b] \times [0,T] \,|\, a \le s \le \tau \le T\}, \tag{5.34}$$

and note that the first component of $(s,\tau) \in \Delta_2^T([a,b])$ is restricted to $[a,b]$ and the second component to $[0,T]$. For simplicity, assume that $\|\mathbf{z}\|_{(\alpha,\gamma)} \le M \in \mathbb{R}_+$. Furthermore, throughout the proof we will consider a subset of $\hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right); V\right)$ of paths (y, y') starting in $(y_0, f(y_0))$. With a slight abuse of notation we still denote this subset by $\hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right); V\right)$.

We start by considering $\bar{T}, \beta$ such that $0 < \bar{T} \le T$ and $\beta < \alpha$ and $\beta - \gamma > \frac{1}{3}$ (note that this is made possible thanks to the fact that $\alpha - \gamma > \frac{1}{3}$). With Definition 50 and our notation (5.34) in mind, we introduce a mapping

$$\mathcal{M}_{\bar{T}} : \hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right); V\right) \to \hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right); V\right) \tag{5.35}$$

such that for all $\left(y, y'\right) \in \hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}(V)$ we have

$$\mathcal{M}_{\bar{T}}\left(y, y'\right) = \left\{ \left(y_0 + \int_0^t k(\tau, r) f\left(y_r^r\right) dx_r,\ f\left(y_t^\tau\right) \right) \,\middle|\, (t,\tau) \in \Delta_2^T\left([0,\bar{T}]\right) \right\}.$$

Our aim is to prove that if $\bar{T}$ is chosen to be small enough, then $\mathcal{M}_{\bar{T}}$ is a contraction. A first step in this direction is obtained by a direct application of Theorem 53, where the norms are restricted to $\Delta_2^T([0,\bar{T}])$. With the additional notation

$$(s,t,\tau) \mapsto \left(w_{ts}^\tau, w_{ts}^{\prime,\tau}\right) = \mathcal{M}_{\bar{T}}\left(y, y'\right)_{ts}^\tau, \tag{5.36}$$

we easily get

$$\|w, w'\|_{\mathbf{z}^1;(\beta,\gamma)} \le \|f(y), f(y) f'(y)\|_{\mathbf{z}^1,(\beta,\gamma)} \|\mathbf{z}\|_{(\alpha,\gamma)} \bar{T}^{\beta-\gamma},$$

where we recall our notation (5.6) for $\|\mathbf{z}\|_{(\alpha,\gamma)}$. Furthermore, it follows from the fact that any composition of a C_b^3 function with a controlled Volterra path in $\hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}$ is again a Volterra path (see Proposition 56) that

$$\|w, w'\|_{(\beta,\gamma)} \le C\left[\left(|y_0'| + \|y, y'\|_{(\beta,\gamma),\mathbf{z}^1}\right)^2 \vee \left(|y_0'| + \|y, y'\|_{(\beta,\gamma),\mathbf{z}^1}\right)\right] \|\mathbf{z}\|_{(\alpha,\gamma)} \bar{T}^{\beta-\alpha}, \tag{5.37}$$

where we recall that we assume $\|\mathbf{z}\|_{(\alpha,\gamma)} \le M$.

Next we will show that there exists a ball of radius 1 centred at a trivial element in $\hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}(\Delta_2^T([0,\bar{T}]); V)$, which is left invariant by $\mathcal{M}_{\bar{T}}$, provided that $\bar{T}$ is small enough. Namely consider the trivial path $(t,\tau) \mapsto (c_t^\tau, c_t^{\prime,\tau,\cdot})$ defined in the following way

$$\left(c_t^\tau, c_t^{\prime,\tau,\cdot}\right) = \left(y_0 + \mathbf{z}_{t0}^{1,\tau} f(y_0),\ f(y_0)\right),$$

where we recall that y_0 is the element in V such that $y_0^\tau = y_0$ for all $\tau \in [0,T]$. Note that this element satisfies $\|c, c'\|_{\mathbf{z}^1,(\beta,\gamma)} = 0$, due to invariance of Hölder norms to translations by constants, and that $R_{ts}^{c,\tau} = 0$ for all $(s,t,\tau) \in \Delta_3([0,T])$. Next consider the unit ball $\mathcal{B}_{\bar{T}}$ centred at the element (c, c') of $\hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}(\Delta_2^T([0,\bar{T}]); V)$ defined by

$$\mathcal{B}_{\bar{T}} = \left\{ \left(y, y'\right) \in \hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}\left(\Delta_2^T\left([0,\bar{T}]\right); V\right) \,\middle|\, y_0^\tau = y_0, \text{ and } y_0^{\prime,\tau,\cdot} = f(y_0),\right.$$

$$\text{with} \quad \|y-c, y'-c'\|_{\mathbf{z}^1,(\beta,\gamma)} \leq 1 \Big\}. \tag{5.38}$$

Again we observe that, thanks to the invariance of Hölder norms by translations by constants and according to the fact that $R^{c,\tau}_{ts} = 0$ for all $(s, t, \tau) \in \Delta_3([0, T])$, we have

$$\|y, y'\|_{\mathbf{z}^1,(\beta,\gamma)} = \|y-c, y'-c'\|_{\mathbf{z}^1,(\beta,\gamma)}$$

for all $(y, y') \in \mathcal{B}_{\bar{T}}$ defined as in (5.38).

Consider now $(y, y') \in \mathcal{B}_{\bar{T}}$ and define (w, w') as in (5.36). Thanks to the fact that $y'^{,\cdot}_0 = f(y_0)$, together with the assumption that f is bounded (recall that $f \in C^4_b$), relation (5.37) can be read as

$$\|w, w'\|_{\mathbf{z}^1,(\beta,\gamma)} \leq C \left(1 + \|y, y'\|_{\mathbf{z}^1,(\beta,\gamma)}\right)^2 \|\mathbf{z}\|_{(\alpha,\gamma)} \bar{T}^{\alpha-\beta}. \tag{5.39}$$

Moreover, since $\|y-c, y'-c'\|_{\mathbf{z}^1,(\beta,\gamma)} \leq 1$, we easily get

$$\|\mathcal{M}_{\bar{T}}\left(y, y'\right)\|_{\mathbf{z}^1,(\beta,\gamma);\Delta_2^T([0,\bar{T}])} \leq C\|\mathbf{z}\|_{(\alpha,\gamma)} \bar{T}^{\alpha-\beta}.$$

We now choose $\bar{T}$ satisfying $C\|\mathbf{z}\|_{(\alpha,\gamma)}\bar{T}^{\alpha-\beta} = \frac{1}{2}$, and we obtain that (w, w') is an element of $\mathcal{B}_{\bar{T}}$. Summarizing our considerations so far, we end up with the relation

$$C\|\mathbf{z}\|_{(\alpha,\gamma)} \bar{T}^{\alpha-\beta} = \frac{1}{2} \qquad \Longrightarrow \qquad \mathcal{B}_{\bar{T}} \text{ is left invariant by } \mathcal{M}_{\bar{T}}. \tag{5.40}$$

Notice the condition on $\bar{T}$ in relation (5.40) does not depend on the initial condition y_0.

Next, we will prove that $\mathcal{M}_{\bar{T}}$ is a contraction on $\hat{\mathscr{D}}^{(\alpha,\gamma)}_{\mathbf{z}^1}(\Delta_2^T([0,\bar{T}]); V)$, i.e. we will prove that for two controlled Volterra paths (y, y') and $(\tilde{y}, \tilde{y}')$ in $\hat{\mathscr{D}}^{(\beta,\gamma)}_{\mathbf{z}^1}(\Delta_2^T([0,\bar{T}]); V)$ there exists a $q \in (0, 1)$ such that

$$\|\mathcal{M}_{\bar{T}}\left(y-\tilde{y}, y'-\tilde{y}'\right)\|_{\mathbf{z}^1,(\beta,\gamma);\Delta_2^T([0,\bar{T}])} \leq q\|y-\tilde{y}, y'-\tilde{y}'\|_{\mathbf{z}^1,(\beta,\gamma);\Delta_2^T([0,\bar{T}])}. \tag{5.41}$$

Without loss of generality, and with a slight abuse of notation, we will from now denote by $\mathscr{D}^{(\beta,\gamma)}_{\mathbf{z}^1}(\Delta_2^T([0,\bar{T}]); V)$ the space of controlled Volterra paths starting from the point $y_0 \in V$. Thus, the two paths (y, y') and $(\tilde{y}, \tilde{y}') \in \mathscr{D}^{(\beta,\gamma)}_{\mathbf{z}^1}(\Delta_2^T([0,\bar{T}]); V)$ share the same initial value. Since $\mathscr{D}^{(\beta,\gamma)}_{\mathbf{z}^1}(\Delta_2^T([0,\bar{T}]); V)$ is a linear space, we may define

$$(F, F') = \left(f(y) - f(\tilde{y}), f'(y^{\cdot 2}) f(y^{\cdot 1}) - f'(\tilde{y}^{\cdot 2}) f(\tilde{y}^{\cdot 1})\right), \tag{5.42}$$

where (F, F') has to be seen as an element of $\hat{\mathscr{D}}^{(\beta,\gamma)}_{\mathbf{z}^1}(\Delta_2^T([0,\bar{T}]); V)$. Thus we have

$$\mathcal{M}_{\bar{T}}(y-\tilde{y}, y'-\tilde{y}')^\tau_t = \int_0^t k(\tau, r) F^r_r dx_r, \tag{5.43}$$

where we observe that the initial condition is now 0. In order to bound the right hand side of (5.43) we now apply Theorem 53 (in particular Eq. (5.8)), which yields

$$\|\mathcal{M}_{\bar{T}}\left(y-\tilde{y}, y'-\tilde{y}'\right)\|_{\mathbf{z}^1,(\beta,\gamma)} \leq \|F\|_{(\beta,\gamma)} + \|F'\|_{\infty}\|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma)}\bar{T}^{2(\alpha-\beta)}$$
$$+C\|F, F'\|_{\mathbf{z}^1,(\beta,\gamma)}\left(\|\mathbf{z}^1\|_{(\alpha,\gamma)} + \|\mathbf{z}^2\|_{(2\rho+\gamma,\gamma)}\right)\bar{T}^{3\alpha-\gamma-2\beta}, \tag{5.44}$$

where we have used that $\rho=\alpha-\gamma$. In (5.44) notice that the quantity $\|F\|_{(\beta,\gamma)}$ comes from the term $\|y^{\prime,\cdot 1,\cdot 2}\|_{(\alpha,\gamma)}$ in the definition (5.4) of the norm $\|y, y'\|_{\mathbf{z}^1;(\alpha,\gamma)}$, together with the fact that

$$\left[\mathcal{M}_{\bar{T}}(y-\tilde{y}, y'-\tilde{y}')\right]^{\prime,\cdot 1,\cdot 2} = F^{\cdot 2}. \tag{5.45}$$

Also observe that the other terms in the right hand side of (5.44) correspond to the evaluation of the remainder for $\mathcal{M}_{\bar{T}}(y-\tilde{y}, y'-\tilde{y}')$, which is obtained by invoking relation (5.8).

Let us now describe how to get the contraction term $\bar{T}^{\alpha-\beta}$ in front of the $\|F\|_{(\beta,\gamma)}$ term in (5.44). Indeed, even though we consider (y, y') and $(\tilde{y}, \tilde{y}')$ as elements of $\hat{\mathscr{D}}_{\mathbf{z}^1}^{(\beta,\gamma)}$, our decomposition (5.3) reveals that their Hölder regularity is dictated by $\mathbf{z}^1$ (see also Remark 52 for a similar observation). Therefore using the expression (5.42) for F and arguments similar to Proposition 56, we get

$$\|F^{\cdot 2}\|_{(\beta,\gamma)} \leq C\|y-\tilde{y}\|_{(\beta,\gamma)}\|\mathbf{z}^1\|_{(\alpha,\gamma)}\bar{T}^{\alpha-\beta} \tag{5.46}$$

Combining (5.44) and (5.46) we can see that

$$\|\mathcal{M}_{\bar{T}}\left(y-\tilde{y}, y'-\tilde{y}'\right)\|_{\mathbf{z}^1,(\beta,\gamma)} \leq C_{M,\alpha,\beta,\gamma}\left[\|y-\tilde{y}\|_{(\beta,\gamma)} + \|F, F'\|_{\mathbf{z}^1,(\beta,\gamma)}\right]\bar{T}^{\alpha-\beta}. \tag{5.47}$$

The dependence on $\bar{T}$ on the left hand side will later allow us to use this parameter to create a constant $q \in (0,1)$ such that (5.41) holds, similar to the argument for the invariance property of the unit ball. Next we will prove that

$$\|F, F'\|_{\mathbf{z}^1,(\beta,\gamma)} \lesssim \|y-\tilde{y}, y'-\tilde{y}'\|_{\mathbf{z}^1,(\alpha,\gamma)}. \tag{5.48}$$

We will mainly focus on the term $\|F^{\prime,\cdot 1,\cdot 2}\|_{(\beta,\gamma)}$, the remainder R^F being treated similarly. Now recall from (5.42) that $F^{\prime,\cdot 1,\cdot 2}(y, y') = f'(y^{\cdot 2})f(y^{\cdot 1}) - f'(\tilde{y}^{\cdot 2})f(\tilde{y}^{\cdot 1})$. To be able to treat the fact that we have two upper variables to take care of, we do a simple addition and subtraction to see that

$$F^{\prime,\cdot 1,\cdot 2}(y, y') = \left(f'(y^{\cdot 2}) - f'(\tilde{y}^{\cdot 2})\right)f(y^{\cdot 1}) + f'(\tilde{y}^{\cdot 2})\left(f(y^{\cdot 1}) - f(\tilde{y}^{\cdot 1})\right). \tag{5.49}$$

By invoking the fact that $f \in C_b^4$, let g and h denote the remainders from a first order Taylor expansion of the differences $f(y)-f(\tilde{y})$ and $f'(y)-f'(\tilde{y})$. Note in particular that this implies that $g \in C_b^3$ and $h \in C_b^2$, and we have that $\|g\|_{C_b^3} \vee \|h\|_{C_b^2} \leq \|f\|_{C_b^4}$. Then it follows from (5.49) that for any $t \in [0,\bar{T}]$ we have

$$F^{\prime,\cdot 1,\cdot 2}(y, y')_t = g(y_t, \tilde{y}_t)\left(y_t^{\cdot 2} - \tilde{y}_t^{\cdot 2}\right)f(y_t^{\cdot 1}) + f'(\tilde{y}_t^{\cdot 2})h(y_t^{\cdot 1}, \tilde{y}_t^{\cdot 1})\left(y_t^{\cdot 1} - \tilde{y}_t^{\cdot 1}\right)$$
$$=: I_t^{1,\cdot 1,\cdot 2} + I_t^{2,\cdot 1,\cdot 2}. \tag{5.50}$$

Let us now consider the increment $I_{ts}^{1,\tau,\tau}$. By elementary addition of subtraction of terms coming from g and f, we obtain that

$$|I_{ts}^{1,\tau,\tau}| \leq C_{\|f\|_{C_b^3}}\|y-\tilde{y}\|_{(\beta,\gamma),1}\left(|\tau-t|^{-\gamma}|t-s|^{\beta} \wedge |\tau-s|^{\beta-\gamma}\right), \tag{5.51}$$

from which it follows that $\|I^1\|_{(\beta,\gamma),1} < \infty$. A similar argument can be used to show that also $\|I^2\|_{(\beta,\gamma),1} < \infty$, however in this case we get dependence on the norm $\|f\|_{C_b^4}$ in the bounding constant. Putting the two terms together, and invoking the relation in (5.50), we observe that

$$\|F^{\prime,\cdot 1,\cdot 2}\|_{(\beta,\gamma),1} \lesssim \|y-\tilde{y}, y'-\tilde{y}'\|^2_{\mathbf{z}^1,(\beta,\gamma)} \lesssim \|y-\tilde{y}, y'-\tilde{y}'\|_{\mathbf{z}^1,(\beta,\gamma)}, \tag{5.52}$$

where we have invoked the fact that $(y, y'), (\tilde{y}, \tilde{y}') \in \mathcal{B}_{\bar{T}}$ for the second inequality. The quantity $\|F^{\prime,\cdot 1,\cdot 2}\|_{(\beta,\gamma),1,2}$ can be bounded using a similar argument, and we leave this component for the patient reader, for conciseness of the proof. It follows that

$$\|F^{\prime,\cdot 1,\cdot 2}\|_{(\beta,\gamma)} \lesssim \|y-\tilde{y}, y'-\tilde{y}'\|_{\mathbf{z}^1,(\beta,\gamma)}, \tag{5.53}$$

and our claim (5.48) is now proved.

In conclusion of this step, we are ready to state the desired contraction property on a small interval $[0, \bar{T}]$. Indeed, plugging (5.48) into (5.47) we obtain

$$\|\mathcal{M}_{\bar{T}}\left(y-\tilde{y}, y'-\tilde{y}'\right)\|_{\mathbf{z}^1,(\beta,\gamma)} \le C\|y-\tilde{y}, y'-\tilde{y}'\|_{\mathbf{z}^1,(\beta,\gamma)}\bar{T}^{\alpha-\beta}.$$

By choosing $\bar{T}$ small enough, it is clear that there exists a $q \in (0,1)$ such that (5.41) holds. It follows that $\mathcal{M}_{\bar{T}}$ admits fixed point in $\hat{\mathscr{D}}^{(\beta,\gamma)}_{\mathbf{z}^1}(\Delta_2^T([0,\bar{T}]); V)$, and thus existence and uniqueness of Eq. (5.33) on $\Delta_2^T([0,\bar{T}])$ is established. Next we want to extend the solution to all of Δ_2, which we do by constructing a solution on all intervals of length $\bar{T}$. That is, we construct a solution to (5.33) on $\Delta_2^T\left([\bar{T}, 2\bar{T}]\right)$ using the terminal value of the solution created on $\Delta_2^T([0,\bar{T}])$. Note that for any $(t,\tau) \in \Delta_2^T([k\bar{T}, (k+1)\bar{T}]) \subset \Delta_2$ for some $k \ge 1$ we formally have that

$$y_t^\tau = y_{k\bar{T}}^\tau + \int_{k\bar{T}}^t k(\tau, r) f(y_r^r) dx_r.$$

It follows, similarly as in the classical results on existence and uniqueness of SDEs, that there exists a solution on all subintervals of length $\bar{T}$, i.e. all intervals $[a, a+\bar{T}] \subset [0,T]$ for some $a \ge 0$. All these solutions are connected on the boundaries, and thus we use that a function which is Hölder on any subinterval $[a, a\bar{T}] \subset [0,T]$ of length $\bar{T}$ is also Hölder continuous on $[0,T]$ (see e.g. [18], exercise 4.24), which applies to the Hölder continuity in both variables. Here notice that the time step $\bar{T}$ can be made constant thanks to the fact that f is a bounded function (see relation (5.39)).

We can conclude that there exists a unique global solution to Eq. (5.33) in the space $\hat{\mathscr{D}}^{(\beta,\gamma)}_{\mathbf{z}^1}(\Delta_2; V)$ for $\beta < \alpha$. Actually, by (5.5) it is clear that the solution inherits the regularity of the controlling noise, and thus, the solution is in $\hat{\mathscr{D}}^{(\alpha,\gamma)}_{\mathbf{z}^1}(\Delta_2; V)$. □

Remark 59. We would like to point out that the existence and uniqueness of Eq. (5.33) requires one more degree of regularity on the diffusion coefficient f than what is standard for regular Rough differential equations (see e.g. [18] section 8). This higher regularity requirement comes from the fact that we need control of the Hölder regularity of the upper argument when composing a function with a controlled Volterra path, as seen in (5.31). This is in contrast to [23] where the authors only need a C_b^3 diffusion coefficients. However, [23] is restricted to the case of a coefficient f such that $f(0) = 0$ and to Volterra equations with kernels which can be written as $k(t,s) = k(t-s)$.

Remark 60. Although Eq. (5.33) is a two parameter object, we can study the solution on the diagonal of Δ_2 to obtain the classical type of one parameter Volterra equations. The Hölder continuity on the diagonal is already guaranteed by the Hölder topologies used on the space of controlled paths. In particular, there exists a unique solution to the equation

$$y_t \equiv y_t^t = y_0 + \int_0^t k(t,r) f(y_r^r) dx_r, \quad y_0 \in V.$$

One can easily check that $t \mapsto y_t \in \mathcal{C}^\rho$ for $\rho = \alpha - \gamma$, where $\mathcal{C}^\rho$ denotes the classical Hölder spaces of order ρ.

5.3. Discussion

Theorem 58 tells us that for any $T > 0$ there exists a solution to Eq. (5.33) on $[0, T]$ for any singular Volterra kernel satisfying (**H**) (in particular, as mentioned in Remark 59, we do not require a convolutional type of kernel like in [23]). Furthermore, since the extension developed here is fully based on the framework of classical rough path, one can also construct solutions to equations driven by lower regularity noise (i.e. with $\rho = \alpha - \gamma$ positive but lower than $1/3$). In fact, let n be the whole number part of $1/\rho$. One can extend Definition 50 to any regularity α by considering a formal expansion of a path to degree n such that the jth Volterra–Gubinelli derivative is convoluted with the $(j+1)$th term in the Volterra rough path, namely

$$y_{ts}^\tau = \sum_{j=1}^{n-1} \mathbf{z}_{ts}^{j,\tau} * y_s^{j,\tau,\cdot_j,\ldots,\cdot_1} + R_{ts}^\tau,$$

where $R \in \mathcal{V}_2^{(n-1)\rho+\gamma,\gamma}$, and each derivative $y^j \in \mathcal{V}^{(\alpha,\gamma)}$ for $j = 1, \ldots, n-1$. We will perform this construction more explicitly in the forthcoming paper [19].

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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