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Weak approximation of a fractional SDE

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Abstract

In this note, a diffusion approximation result is shown for stochastic differential equations driven by a (Liouville) fractional Brownian motion B with Hurst parameter $H \in (1/3, 1/2)$. More precisely, we resort to the Kac–Stroock type approximation using a Poisson process studied in Bardina et al. (2003) [4] and Delgado and Jolis (2000) [9], and our method of proof relies on the algebraic integration theory introduced by Gubinelli in Gubinelli (2004) [14].

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1. Introduction

After a decade of efforts [2,8,14,21,22,28,29], it can arguably be said that the basis of the stochastic integration theory with respect to a rough path in general, and with respect to a fractional Brownian motion (fBm) in particular, has been now settled in a rather simple and secure way. This allows in particular to define rigorously and solve equations on an arbitrary

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interval $[0, T]$ with $T > 0$, of the form:

$$\mathrm{d}y_t = \sigma(y_t)\,\mathrm{d}B_t + b(y_t)\,\mathrm{d}t, \qquad y_0 = a \in \mathbb{R}^n, \tag{1}$$

where $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $b : \mathbb{R}^n \to \mathbb{R}^n$ are two bounded and smooth functions, and B stands for a d-dimensional fBm with Hurst parameter $H > 1/4$. A question which arises naturally in this context is then to try to establish some of the basic properties of the process y defined by (1), and this global program has already been started as far as moments estimates [16], large deviations [20,24], or properties of the law [6,26] are concerned (let us mention at this point that the forthcoming book [12] will give a detailed account on most of these topics).

In the current note, we wish to address another natural problem related to the fractional diffusion process y defined by (1). Indeed, in the case where B is an ordinary Brownian motion, one of the most popular methods in order to simulate y is the following: approximate B by a sequence of smooth or piecewise linear functions, say $(X^\varepsilon)_{\varepsilon>0}$, which converges in law to B, e.g. an interpolated and rescaled random walk. Then see if the process y^ε solution of Eq. (1) driven by X^ε converges in law, as a process, to y. This kind of result, usually known as diffusion approximation, has been thoroughly studied in the literature (see e.g. [17,32,33]), since it also shows that equations like (1) may emerge as the limit of a noisy equation driven by a fast oscillating function. The diffusion approximation program has also been taken up in the fBm case by Marty in [23], with some random wave problems in mind, but only in the cases where $H > 1/2$ or the dimension d of the fBm is 1. Also note that, in a more general context, strong and weak approximations to Gaussian rough paths have been studied systematically by Friz and Victoir in [11]. Among other results, the following is proved in this latter reference: let $(X^\varepsilon)_{\varepsilon>0}$ be a sequence of d-dimensional centered Gaussian processes with independent components and covariance function R^ε. Let X be another d-dimensional centered Gaussian processes with independent components and covariance function R. Assume that all those processes admit a rough path of order 2, that R^ε converges pointwise to R, and that R^ε is suitably dominated in p-variation norm for some $p \in [1, 2)$. Then the rough path associated to X^ε also converges weakly, in $2p$-variation norm, to the rough path associated to X.

This result does not close the diffusion approximation problem for solutions of SDEs like (1). Indeed, for computational and implementation reasons, the most typical processes taken as approximations to B are non-Gaussian, and more specifically, are usually based on random walks [19,33,30] or the Kac–Stroock type [4,9,18,31] approximations. However, the issue of diffusion approximations in a non-Gaussian context has hardly been addressed in the literature, and we are only aware of the aforementioned reference [23], as well as the recent preprint [7] (which deals with Donsker's theorem in the rough path topology) for significant results on the topic. The current article proposes then a natural step in this direction, and studies diffusion approximations to (1) based on the Kac–Stroock approximation to white noise.

Let us be more specific about the kind of result we will obtain. First of all, we consider in the sequel the so-called d-dimensional Liouville fBm B, with Hurst parameter $H \in (1/3, 1/2)$, as the driving process of Eq. (1). This is convenient for computational reasons (especially for the bounds we use on integration kernels), and is harmless in terms of generality, since the difference between the usual fBm and Liouville's one is a finite variation process (as shown in [3]). More precisely, we assume that B can be written as $B = (B^1, \ldots, B^d)$, where the B^i's are d independent centered Gaussian processes of the form

$$B^i_t = \int_0^t (t-r)^{H-\frac{1}{2}}\,\mathrm{d}W^i_r,$$

for a d-dimensional Wiener process $W = (W^1, \ldots, W^d)$. As an approximating sequence of B, we shall choose $(X^\varepsilon)_{\varepsilon>0}$, where $X^{\varepsilon,i}$ is defined as follows, for $i = 1, \ldots, d$:

$$X^{i,\varepsilon}(t) = \int_0^t (t+\varepsilon-r)^{H-\frac{1}{2}} \theta^{\varepsilon,i}(r) \mathrm{d}r, \tag{2}$$

where

$$\theta^{\varepsilon,i}(r) = \frac{1}{\varepsilon}(-1)^{N^i(\frac{r}{\varepsilon})}, \tag{3}$$

for N^i, $i = 1, \ldots, d$, some independent standard Poisson processes. Let us then consider the process y^ε solution to Eq. (1) driven by X^ε, namely:

$$\mathrm{d}y_t^\varepsilon = \sigma\left(y_t^\varepsilon\right)\mathrm{d}X_t^\varepsilon + b\left(y_t^\varepsilon\right)\mathrm{d}t, \qquad y_0^\varepsilon = a \in \mathbb{R}^n, \quad t \in [0, T]. \tag{4}$$

Then our main result is as follows:

Theorem 1.1. *Assume that $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ is a bounded C^2 function having bounded derivatives, and $b : \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz and bounded function. Let $(y^\varepsilon)_{\varepsilon>0}$ be the family of processes defined by (4), and let $1/3 < \gamma < H$, where H is the Hurst parameter of B. Then, as $\varepsilon \to 0$, y^ε converges in law to the process y obtained as the solution to (1), where the convergence takes place in the Hölder space $\mathcal{C}^\gamma([0,T];\mathbb{R}^n)$.*

Observe that we have only considered the case $H > 1/3$ in the last result. This is of course for computational and notational sake, but it should also be mentioned that some of our kernel estimates, needed for the convergence in law, heavily rely on the assumption $H > 1/3$. On the other hand, the case $H > 1/2$ follows easily from the results contained in [9], and the case $H = 1/2$ is precisely Stroock's result [31]. This is why our future computations focus on the case $1/3 < H < 1/2$.

The general strategy we shall follow in order to get our main result is rather natural in the rough path context: it is a well-known fact that the solution y to (1) is a continuous function of B and of the Lévy area of B (which will be called $\mathbf{B^2}$), considered as elements of some suitable Hölder (or p-variation) spaces. Hence, in order to obtain the convergence $y^\varepsilon \to y$ in law, it will be sufficient to check the convergence of the corresponding approximations X^ε and $\mathbf{X}^{\mathbf{2},\varepsilon}$ in their respective Hölder spaces (observe however that $\mathbf{X}^{\mathbf{2},\varepsilon}$ is not needed, in principle, for the definition of y^ε). Then the two main technical problems we will have to solve are the following:

(1) First of all, we shall use the *simplified* version of the rough path formalism, called algebraic integration, introduced by Gubinelli in [14], which will be summarized in the next section. In the particular context of weak approximations, this allows us to deal with approximations of B and $\mathbf{B^2}$ directly, without recurring to discretized paths as in [8]. However, the algebraic integration formalism relies on some space $\mathcal{C}_k^\gamma$, where k stands for a number of variables in $[0, T]$, and γ for a Hölder type exponent. Thus, an important step will be to find a suitable tightness criterion in these spaces. For this point, we refer to Section 4.
(2) The convergence of finite-dimensional distributions ("fdd" in the sequel) for the Lévy area $\mathbf{B^2}$ will be proved in Section 5, and will be based on some sharp estimates concerning the Kac–Stroock kernel (3) that are performed in Section 6. Indeed, this latter section is mostly devoted to quantify the distance between $\int_0^T f(u)\theta^\varepsilon(u)\mathrm{d}u$ and $\int_0^T f(u)\mathrm{d}W_u$ for a smooth enough function f, in the sense of characteristic functions. This constitutes a generalization of [9], which is interesting in its own right.

Here is how our paper is structured: in Section 2, we shall recall the main notions of the algebraic integration theory. Then Section 3 will be devoted to the weak convergence, divided into the tightness result (Section 4) and the fdd convergence (Section 5). Finally, Section 6 contains the technical lemmas of the paper.

2. Background on algebraic integration and fractional SDEs

This section contains a summary of the algebraic integration introduced in [14], which was also used in [26,25] in order to solve and analyze fractional SDEs. We recall its main features here, since our approximation result will also be obtained in this setting.

Let x be a Hölder continuous $\mathbb{R}^d$-valued function of order γ, with $1/3 < \gamma \le 1/2$, and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$, $b : \mathbb{R}^n \to \mathbb{R}^n$ be two bounded and smooth functions. We shall consider in the sequel the n-dimensional equation

$$\mathrm{d}y_t = \sigma(y_t)\,\mathrm{d}x_t + b(y_t)\,\mathrm{d}t, \qquad y_0 = a \in \mathbb{R}^n,\ t \in [0,T]. \tag{5}$$

In order to define rigorously and solve this equation, we will need some algebraic and analytic notions which are introduced in the next subsection.

2.1. Increments

We first present the basic algebraic structures which will allow us to define a pathwise integral with respect to irregular functions. For an arbitrary real number $T > 0$, a vector space V and an integer $k \ge 1$ we denote by $\mathcal{C}_k(V)$ the set of functions $g : [0,T]^k \to V$ such that $g_{t_1\cdots t_k} = 0$ whenever $t_i = t_{i+1}$ for some $i \le k-1$. Such a function will be called a $(k-1)$-*increment*, and we will set $\mathcal{C}_*(V) = \cup_{k\ge 1}\mathcal{C}_k(V)$. An important elementary operator is defined by

$$\delta : \mathcal{C}_k(V) \to \mathcal{C}_{k+1}(V), \qquad (\delta g)_{t_1\cdots t_{k+1}} = \sum_{i=1}^{k+1}(-1)^{k-i} g_{t_1\cdots\hat{t}_i\cdots t_{k+1}}, \tag{6}$$

where $\hat{t}_i$ means that this particular argument is omitted. A fundamental property of δ, which is easily verified, is that $\delta\delta = 0$, where $\delta\delta$ is considered as an operator from $\mathcal{C}_k(V)$ to $\mathcal{C}_{k+2}(V)$. We will denote $\mathcal{ZC}_k(V) = \mathcal{C}_k(V) \cap \mathrm{Ker}\delta$ and $\mathcal{BC}_k(V) = \mathcal{C}_{k+1}(V) \cap \mathrm{Im}\delta$.

Some simple examples of actions of δ are obtained for $g \in \mathcal{C}_1(V)$ and $h \in \mathcal{C}_2(V)$. Then, for any $s, u, t \in [0,T]$, we have

$$(\delta g)_{st} = g_t - g_s, \quad \text{and} \quad (\delta h)_{sut} = h_{st} - h_{su} - h_{ut}. \tag{7}$$

Furthermore, it is easily checked that $\mathcal{ZC}_{k+1}(V) = \mathcal{BC}_k(V)$ for any $k \ge 1$. In particular, the following basic property holds:

Lemma 2.1. *Let $k \ge 1$ and $h \in \mathcal{ZC}_{k+1}(V)$. Then there exists a (nonunique) $f \in \mathcal{C}_k(V)$ such that $h = \delta f$.*

Observe that Lemma 2.1 implies that all elements $h \in \mathcal{C}_2(V)$ with $\delta h = 0$ can be written as $h = \delta f$ for some (nonunique) $f \in \mathcal{C}_1(V)$. Thus we get a heuristic interpretation of $\delta|_{\mathcal{C}_2(V)}$: it measures how much a given 1-increment is far from being an exact increment of a function, i.e., a finite difference.

Note that our further discussion will mainly rely on k-increments with $k \le 2$. For the simplicity of the exposition, we will assume from now that $V = \mathbb{R}^d$. We measure the size of these increments by Hölder norms, which are defined in the following way: for $f \in \mathcal{C}_2(V)$ let

$$\|f\|_\mu = \sup_{s,t\in[0,T]} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu(V) = \left\{f \in \mathcal{C}_2(V);\, \|f\|_\mu < \infty\right\}.$$

Obviously, the usual Hölder spaces $\mathcal{C}_1^\mu(V)$ are determined in the following way: for a continuous function $g \in \mathcal{C}_1(V)$ simply set

$$\|g\|_\mu = \|\delta g\|_\mu, \tag{8}$$

and we will say that $g \in \mathcal{C}_1^\mu(V)$ iff $\|g\|_\mu$ is finite. Note that $\|\cdot\|_\mu$ is only a semi-norm on $\mathcal{C}_1(V)$, but we will work in general on spaces of the type

$$\mathcal{C}_{1,a}^\mu(V) = \left\{g : [0,T] \to V;\ g_0 = a,\ \|g\|_\mu < \infty\right\}, \tag{9}$$

for a given $a \in V$, on which $\|g\|_\mu$ is a norm. For $h \in \mathcal{C}_3(V)$ set in the same way

$$\|h\|_{\gamma,\rho} = \sup_{s,u,t\in[0,T]} \frac{|h_{sut}|}{|u-s|^\gamma |t-u|^\rho} \tag{10}$$

$$\|h\|_\mu = \inf\left\{\sum_i \|h_i\|_{\rho_i,\mu-\rho_i};\, h = \sum_i h_i,\, 0 < \rho_i < \mu\right\},$$

where the infimum is taken over all sequences $\{h_i \in \mathcal{C}_3(V)\}$ such that $h = \sum_i h_i$ and for all choices of the numbers $\rho_i \in (0,\mu)$. Then $\|\cdot\|_\mu$ is easily seen to be a norm on $\mathcal{C}_3(V)$, and we set

$$\mathcal{C}_3^\mu(V) := \left\{h \in \mathcal{C}_3(V);\ \|h\|_\mu < \infty\right\}.$$

Eventually, let $\mathcal{C}_3^{1+}(V) = \cup_{\mu>1} \mathcal{C}_3^\mu(V)$, and note that the same kind of norms can be considered on the spaces $\mathcal{Z}\mathcal{C}_3(V)$, leading to the definition of the spaces $\mathcal{Z}\mathcal{C}_3^\mu(V)$ and $\mathcal{Z}\mathcal{C}_3^{1+}(V)$.

With these notations in mind, the crucial point in the current approach to pathwise integration of irregular paths is that the operator δ can be inverted under mild smoothness assumptions. This inverse is called Λ. The proof of the following proposition may be found in [14], and in a more elementary form in [15]:

Proposition 2.2. *There exists a unique linear map $\Lambda : \mathcal{Z}\mathcal{C}_3^{1+}(V) \to \mathcal{C}_2^{1+}(V)$ such that*

$$\delta\Lambda = Id_{\mathcal{Z}\mathcal{C}_3^{1+}(V)} \quad \textit{and} \quad \Lambda\delta = Id_{\mathcal{C}_2^{1+}(V)}.$$

In other words, for any $h \in \mathcal{C}_3^{1+}(V)$ such that $\delta h = 0$ there exists a unique $g = \Lambda(h) \in \mathcal{C}_2^{1+}(V)$ such that $\delta g = h$. Furthermore, for any $\mu > 1$, the map Λ is continuous from $\mathcal{Z}\mathcal{C}_3^\mu(V)$ to $\mathcal{C}_2^\mu(V)$ and we have

$$\|\Lambda h\|_\mu \le \frac{1}{2^\mu - 2}\|h\|_\mu, \quad h \in \mathcal{Z}\mathcal{C}_3^\mu(V). \tag{11}$$

Moreover, Λ has a nice interpretation in terms of generalized Young integrals:

Corollary 2.3. *For any* 1*-increment* $g \in \mathcal{C}_2(V)$ *such that* $\delta g \in \mathcal{C}_3^{1+}(V)$ *set* $\delta f = (Id - \Lambda\delta)g$. *Then*

$$(\delta f)_{st} = \lim_{|\Pi_{ts}|\to 0} \sum_{i=0}^{n} g_{t_i\, t_{i+1}},$$

where the limit is over any partition $\Pi_{st} = \{t_0 = s, \ldots, t_n = t\}$ *of* $[s,t]$, *whose mesh tends to zero. Thus, the* 1*-increment* δf *is the indefinite integral of the* 1*-increment* g.

2.2. Weakly controlled paths

This subsection is devoted to the definition of generalized integrals with respect to a rough path of order 2, and to the resolution of Eq. (5). Notice that, in the sequel of our paper, we will use both the notations $\int_s^t f \mathrm{d}g$ or $\mathcal{J}_{st}(f\, \mathrm{d}g)$ for the integral of a function f with respect to a given increment $\mathrm{d}g$ on the interval $[s,t]$. The second notation $\mathcal{J}_{st}(f\, \mathrm{d}g)$ will be used to avoid some cumbersome notations in our computations. Observe also that the drift term b is generally harmless if one wants to solve the Eq. (5). See e.g. Remark 3.14 in [27]. Hence, we will simply deal with an equation of the form

$$\mathrm{d}y_t = \sigma(y_t)\,\mathrm{d}x_t, \quad t \in [0,T], \text{ with } y_0 = a \tag{12}$$

in the remainder of this section.

Before going into the technical details, let us make some heuristic considerations about the properties that a solution of Eq. (5) should have. Set $\hat{\sigma}_t = \sigma(y_t)$, and suppose that y is a solution of (12), with $y \in \mathcal{C}_1^{\kappa}$ for a given $1/3 < \kappa < \gamma$. Then the integral form of our equation can be written as

$$y_t = a + \int_0^t \hat{\sigma}_u \mathrm{d}x_u, \quad t \in [0,T]. \tag{13}$$

Our approach to generalized integrals induces us to work with increments of the form $(\delta y)_{st} = y_t - y_s$ instead of (13). However, it is easily checked that one can decompose (13) into

$$(\delta y)_{st} = \int_s^t \hat{\sigma}_u \mathrm{d}x_u = \hat{\sigma}_s(\delta x)_{st} + \rho_{st}, \quad \text{with } \rho_{st} = \int_s^t (\hat{\sigma}_u - \hat{\sigma}_s)\mathrm{d}x_u,$$

if our integral is linear. We thus have obtained a decomposition of y of the form $\delta y = \hat{\sigma}\delta x + \rho$. Let us see, still at a heuristic level, which regularity we can expect for $\hat{\sigma}$ and r. If σ is a C_b^1-function, we have that $\hat{\sigma}$ is bounded and

$$|\hat{\sigma}_t - \hat{\sigma}_s| \le \|\nabla\sigma\|_\infty \|y\|_\kappa |t-s|^\kappa,$$

where $\|y\|_\kappa$ denotes the Hölder norm of y defined by (8). Hence we have that $\hat{\sigma}$ belongs to $\mathcal{C}_1^\kappa$ and is bounded. As far as ρ is concerned, it should inherit both the regularities of $\delta\hat{\sigma}$ and x, provided that the integral $\int_s^t (\hat{\sigma}_u - \hat{\sigma}_s)\mathrm{d}x_u = \int_s^t (\delta\hat{\sigma})_{su}\mathrm{d}x_u$ is well defined. Thus, one should expect that $\rho \in \mathcal{C}_2^{2\kappa}$, and even $\rho \in \mathcal{C}_2^{\kappa+\gamma}$. To summarize, we have found that a solution δy of the equation should be decomposable into

$$\delta y = \hat{\sigma}\delta x + \rho, \quad \text{with } \hat{\sigma} \in \mathcal{C}_1^\gamma \text{ bounded and } \rho \in \mathcal{C}_2^{2\kappa}. \tag{14}$$

This is precisely the structure we will demand for a possible solution of (12):

Definition 2.4. Let z be a path in $\mathcal{C}_1^\kappa(\mathbb{R}^k)$ with $\kappa \le \gamma$ and $2\kappa + \gamma > 1$. We say that z is a controlled path based on x, if $z_0 = a$, which is a given initial condition in $\mathbb{R}^k$, and $\delta z \in \mathcal{C}_2^\kappa(\mathbb{R}^k)$ can be decomposed into

$$\delta z = \zeta \delta x + r, \quad \text{i.e. } (\delta z)_{st} = \zeta_s (\delta x)_{st} + \rho_{st}, s, t \in [0, T], \tag{15}$$

with $\zeta \in \mathcal{C}_1^\kappa(\mathbb{R}^{k\times d})$ and ρ is a regular part belonging to $\mathcal{C}_2^{2\kappa}(\mathbb{R}^k)$. The space of controlled paths will be denoted by $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$, and a path $z \in \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ should be considered in fact as a couple (z, ζ). The natural semi-norm on $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ is given by

$$\mathcal{N}[z; \mathcal{Q}_{\kappa,a}(\mathbb{R}^k)] = \mathcal{N}[z; \mathcal{C}_1^\kappa(\mathbb{R}^k)] + \mathcal{N}[\zeta; \mathcal{C}_1^b(\mathbb{R}^{k,d})] + \mathcal{N}[\zeta; \mathcal{C}_1^\kappa(\mathbb{R}^{k,d})] + \mathcal{N}[\rho; \mathcal{C}_2^{2\kappa}(\mathbb{R}^k)]$$

with $\mathcal{N}[g; \mathcal{C}_1^\kappa(V)]$ defined by (8) and $\mathcal{N}[\zeta; \mathcal{C}_1^b(V)] = \sup_{0\le s\le T} |\zeta_s|_V$.

Having defined our algebraic and analytic framework, we now can give a sketch of the strategy used in [14] in order to solve Eq. (12):

1. Verify the stability of $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ under a smooth map $\varphi : \mathbb{R}^k \to \mathbb{R}^n$.
2. Define rigorously the integral $\int z_u \mathrm{d}x_u = \mathcal{J}(z\mathrm{d}x)$ for a controlled path z and computed its decomposition (15).
3. Solve Eq. (12) in the space $\mathcal{Q}_{\kappa,a}(\mathbb{R}^k)$ by a fixed point argument.

Actually, for the second point one has to assume a priori the following hypothesis on the driving rough path, which is standard in rough path type considerations:

Hypothesis 2.5. The $\mathbb{R}^d$-valued γ-Hölder path x admits a Lévy area, that is a process $\mathbf{x^2} = \mathcal{J}(\mathrm{d}x\mathrm{d}x) \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ satisfying

$$\begin{aligned}&\delta \mathbf{x^2} = \delta x \otimes \delta x, \quad \text{i.e. } \left[(\delta \mathbf{x^2})_{sut}\right](i, j) = [\delta x^i]_{su}[\delta x^j]_{ut},\\ &\quad s, u, t \in [0, T],\ i, j \in \{1, \dots, d\}.\end{aligned}$$

Then the following result is proved in [14], using the strategy sketched above:

Theorem 2.6. *Let x be a process satisfying Hypothesis 2.5 and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ be a C^2 function, which is bounded together with its derivatives. Then*

(1) *Eq. (12) admits a unique solution y in $\mathcal{Q}_{\kappa,a}(\mathbb{R}^n)$ for any $\kappa < \gamma$ such that $2\kappa + \gamma > 1$.*
(2) *The mapping $(a, x, \mathbf{x^2}) \mapsto y$ is continuous from $\mathbb{R}^n \times \mathcal{C}_1^\gamma(\mathbb{R}^d) \times \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ to $\mathcal{Q}_{\kappa,a}(\mathbb{R}^n)$.*

We shall see in the next subsection that this general theorem can be applied in the fBm context.

2.3. Application to the fBm

Let $B = (B^1, \dots, B^d)$ be a d-dimensional Liouville fBm of Hurst index $H \in (\frac{1}{3}, \frac{1}{2})$, that is $B^1, \dots, B^d$ are d independent centered Gaussian processes of the form

$$B_t^i = \int_0^t (t - r)^{H-\frac{1}{2}} \mathrm{d}W_r^i,$$

where $W = (W^1, \dots, W^d)$ is a d-dimensional Wiener process. The next lemma, whose proof is straightforward (see [5] page 7), will be useful all along the paper.

Lemma 2.7. *There exists a positive constant c, depending only on H, such that*

$$E|B_t^i - B_s^i|^2 = \int_0^s \left[(t-r)^{H-\frac{1}{2}} - (s-r)^{H-\frac{1}{2}}\right]^2 \mathrm{d}r + \int_s^t (t-r)^{2H-1}\mathrm{d}r \\ \leq c|t-s|^{2H} \tag{16}$$

for all $t > s \geq 0$.

Let $\mathcal{E}$ be the set of step functions on $[0, T]$ with values in $\mathbb{R}^d$. Consider the Hilbert space $\mathcal{H}$ defined as the closure of $\mathcal{E}$ with respect to the scalar product induced by

$$\left\langle (\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_d]}), (\mathbf{1}_{[0,s_1]}, \ldots, \mathbf{1}_{[0,s_d]}) \right\rangle_{\mathcal{H}} = \sum_{i=1}^d R(t_i, s_i),$$
$$s_i, t_i \in [0, T], \; i = 1, \ldots, d,$$

where $R(t, s) := E[B_t^i B_s^i]$. Then a natural representation of the inner product in $\mathcal{H}$ is given via the operator $\mathscr{K}$, defined from $\mathcal{E}$ to $L^2([0, T])$, by:

$$\mathscr{K}\varphi(t) = (T-t)^{H-\frac{1}{2}}\varphi(t) - \left(\frac{1}{2} - H\right)\int_t^T [\varphi(r) - \varphi(t)](r-t)^{H-\frac{3}{2}}\mathrm{d}r,$$

and it can be checked that $\mathscr{K}$ can be extended as an isometry between $\mathcal{H}$ and the Hilbert space $L^2([0, T]; \mathbb{R}^d)$. Thus the inner product in $\mathcal{H}$ can be defined as:

$$\langle \varphi, \psi \rangle_{\mathcal{H}} \triangleq \langle \mathscr{K}\varphi, \mathscr{K}\psi \rangle_{L^2([0,T];\mathbb{R}^d)}.$$

The mapping $(\mathbf{1}_{[0,t_1]}, \ldots, \mathbf{1}_{[0,t_d]}) \mapsto \sum_{i=1}^d B_{t_i}^i$ can also be extended into an isometry between $\mathcal{H}$ and the first Gaussian chaos $H_1(B)$ associated with $B = (B^1, \ldots, B^d)$. We denote this isometry by $\varphi \mapsto B(\varphi)$, and $B(\varphi)$ is called the Wiener–Itô integral of φ. It is shown in [10, page 284] that $\mathcal{C}_1^\gamma(\mathbb{R}^d) \subset \mathcal{H}$ whenever $\gamma > 1/2 - H$, which allows to define $B(\varphi)$ for such kind of functions.

We are now ready to prove that Theorem 2.6 can be applied to the Liouville fBm, which amounts to check Hypothesis 2.5.

Proposition 2.8. *Let B be a d-dimensional Liouville fBm, and suppose that its Hurst parameter satisfies* $H \in (1/3, 1/2)$. *Then almost all sample paths of B satisfy Hypothesis 2.5, with any Hölder exponent* $1/3 < \gamma < H$, *and a Lévy area given by*

$$\mathbf{B}_{st}^{\mathbf{2}} = \int_s^t \mathrm{d}B_u \otimes \int_s^u \mathrm{d}B_v, \quad \text{i.e. } \mathbf{B}_{st}^{\mathbf{2}}(i,j) = \int_s^t \mathrm{d}B_u^i \int_s^u \mathrm{d}B_v^j,\ i, j \in \{1, \ldots, d\},$$

for $0 \leq s < t \leq T$. *Here, the stochastic integrals are defined as Wiener–Itô integrals when* $i \neq j$, *while, when* $i = j$, *they are simply given by*

$$\int_s^t \mathrm{d}B_u^i \int_s^u \mathrm{d}B_v^i = \frac{1}{2}\left(B_t^i - B_s^i\right)^2.$$

Proof. First of all, it is a classical fact that $B \in \mathcal{C}_1^\gamma(\mathbb{R}^d)$ for any $1/3 < \gamma < H$, when B is a Liouville fBm with $H > 1/3$ (indeed, combine the Kolmogorov–Čentsov theorem with Lemma 2.7). Furthermore, we have already mentioned that $\mathcal{C}_1^\gamma(\mathbb{R}^d) \subset \mathcal{H}$ for any $\gamma > 1/2 - H$. In particular, if $H > \gamma > 1/3$, the condition $\gamma > 1/2 - H$ is satisfied and, conditionally to B^j,

$\int_s^t \mathrm{d}B_u^i \int_s^u \mathrm{d}B_v^j$ is well defined for $i \neq j$, as a Wiener–Itô integral with respect to B^i, of the form $B^i(\varphi)$ for a well-chosen φ. Hence, $\mathbf{B^2}$ is almost surely a well-defined element of $\mathcal{C}_2(\mathbb{R}^{d\times d})$.

Now, simple algebraic computations immediately yield that $\delta\mathbf{B^2} = \delta B \otimes \delta B$. Furthermore, Lemma 6.4 yields

$$E\left[|\mathbf{B}^{\mathbf{2}}_{st}(i,j)|^2\right] \leq c|t-s|^{4H}.$$

Invoking this inequality and thanks to the fact that $\mathbf{B^2}$ is a process in the second chaos of B, on which all L^p norms ($p>1$) are equivalent, we get that

$$E\left[|\mathbf{B}^{\mathbf{2}}_{st}(i,j)|^p\right] \leq c_p|t-s|^{2pH}.$$

This allows to conclude, thanks to an elaboration of Garsia's lemma which can be found in [14, Lemma 4] (and will be recalled at (30)), that $\mathbf{B^2} \in \mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ for any $\gamma < 1/3$. This ends the proof. □

With all these results in hand, we have obtained a reasonable definition of diffusion processes driven by a fBm, and we can now proceed to their approximation in law.

3. Approximating sequence

In this section, we will introduce our smooth approximation of B, namely X^ε, which shall converge *in law* to B. This will allow to interpret Eq. (4) in the usual Lebesgue–Stieltjes sense. We will then study the convergence in law of the process y^ε solution to (4) towards the solution y of (1).

As mentioned in the introduction, the approximation of B we shall deal with is defined as follows, for $i = 1,\ldots,d$:

$$X^{i,\varepsilon}(t) = \int_0^t (t+\varepsilon-r)^{H-\frac{1}{2}}\theta^{\varepsilon,i}(r)\mathrm{d}r, \tag{17}$$

where

$$\theta^{\varepsilon,i}(r) = \frac{1}{\varepsilon}(-1)^{N^i(\frac{r}{\varepsilon})},$$

for N^i, $i = 1,\ldots,d$, some independent standard Poisson processes. Furthermore, we have recalled in Theorem 2.6 that the solution y to (1) is a continuous function of $(a, B, \mathbf{B^2})$, considered respectively as elements of $\mathbb{R}^d$, $\mathcal{C}_1^\gamma(\mathbb{R}^d)$ and $\mathcal{C}_2^{2\gamma}(\mathbb{R}^{d\times d})$ for $1/3 < \gamma < H$. Thus our approximation Theorem 1.1 can be easily deduced from the following result:

Theorem 3.1. *For any $\varepsilon > 0$, let $\mathbf{X}^{\mathbf{2},\varepsilon} = (\mathbf{X}^{\mathbf{2},\varepsilon}_{st}(i,j))_{s,t\geq 0;\, i,j=1,\ldots,d}$ be the natural Lévy's area associated to X^ε, defined by*

$$\mathbf{X}^{\mathbf{2},\varepsilon}_{st}(i,j) = \int_s^t (X_u^{j,\varepsilon} - X_s^{j,\varepsilon})\mathrm{d}X_u^{i,\varepsilon}, \tag{18}$$

where the integral is understood in the usual Lebesgue–Stieltjes sense. Then, as $\varepsilon \to 0$,

$$(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon}) \xrightarrow{\text{Law}} (B, \mathbf{B^2}), \tag{19}$$

where $\mathbf{B^2}$ denotes the Lévy area defined in Proposition 2.8, and where the convergence in law holds in spaces $\mathcal{C}_1^\mu(\mathbb{R}^d) \times \mathcal{C}_2^{2\mu}(\mathbb{R}^{d\times d})$, for any $\mu < H$.

The remainder of our work is devoted to the proof of Theorem 3.1. As usual in the context of weak convergence of stochastic processes, we divide the proof into the weak convergence for finite-dimensional distributions (Section 5) and a tightness type result (Section 4).

Remark 3.2. A natural idea for the proof of Theorem 3.1 could be to use the methodology initiated by Kurtz and Protter in [19]. But the problem, here, is that the quantities we are dealing with are not "close enough" to a martingale.

4. Tightness in Theorem 3.1

From now, we write $\mathcal{C}_1^{\mu}$ (resp. $\mathcal{C}_2^{2\mu}$) instead of $\mathcal{C}_1^{\mu}(\mathbb{R}^d)$ (resp. $\mathcal{C}_2^{2\mu}(\mathbb{R}^{d\times d})$). We first need a general tightness criterion in the Hölder spaces $\mathcal{C}_1^{\mu}$ and $\mathcal{C}_2^{2\mu}$.

Lemma 4.1. *Let $\mathcal{E}^{\gamma}$ denote the set of $(x, \mathbf{x^2}) \in \mathcal{C}_1^{\gamma} \times \mathcal{C}_2^{2\gamma}$ verifying $x_0 = 0$ and*

$$\forall s, t \geq 0,\ \forall i, j = 1, \ldots, d: \quad \mathbf{x}^{\mathbf{2}}_{st}(i, j) = \mathbf{x}^{\mathbf{2}}_{0t}(i, j) - \mathbf{x}^{\mathbf{2}}_{0s}(i, j) - x_s^i(x_t^j - x_s^j). \tag{20}$$

Let μ such that $0 \leq \mu < \gamma$. Then, any bounded subset $\mathcal{Q}$ of $\mathcal{E}^{\gamma}$ is precompact in $\mathcal{C}_1^{\mu} \times \mathcal{C}_2^{2\mu}$.

Proof. Let $(x^n, \mathbf{x}^{\mathbf{2},\boldsymbol{n}})$ be a sequence of $\mathcal{Q}$. By assumption, $(x^n, \mathbf{x}^{\mathbf{2},\boldsymbol{n}}_{0\cdot})$ is bounded and equicontinuous. Then, Ascoli's theorem applies and, at least along a subsequence, which may also be called $(x^n, \mathbf{x}^{\mathbf{2},\boldsymbol{n}}_{0\cdot})$, it converges uniformly to $(x, \mathbf{x}^{\mathbf{2}}_{0\cdot})$. Using (20), we obtain in fact that $(x^n, \mathbf{x}^{\mathbf{2},\boldsymbol{n}})$ converges uniformly to $(x, \mathbf{x^2})$. Moreover, since we obviously have

$$\|x\|_{\mu} \leq \liminf_{n\to\infty} \|x^n\|_{\mu} \quad \text{and} \quad \|\mathbf{x^2}\|_{2\mu} \leq \liminf_{n\to\infty} \|\mathbf{x}^{\mathbf{2},\boldsymbol{n}}\|_{2\mu},$$

we deduce that $(x, \mathbf{x^2}) \in \mathcal{C}_1^{\mu} \times \mathcal{C}_2^{2\mu}$. Finally, we have

$$\|x - x^n\|_{\mu} \longrightarrow 0 \quad \text{and} \quad \|\mathbf{x^2} - \mathbf{x}^{\mathbf{2},\boldsymbol{n}}\|_{2\mu} \longrightarrow 0,$$

owing to the fact that

$$\|x - x^n\|_{\mu} \leq \|x - x^n\|_{\gamma}\ \|x - x^n\|_{\infty}^{1-\frac{\mu}{\gamma}} \leq \left(\|x\|_{\gamma} + \|x^n\|_{\gamma}\right)\|x - x^n\|_{\infty}^{1-\frac{\mu}{\gamma}}$$

and similarly:

$$\|\mathbf{x^2} - \mathbf{x}^{\mathbf{2},\boldsymbol{n}}\|_{2\mu} \leq \left(\|\mathbf{x^2}\|_{2\gamma} + \|\mathbf{x}^{\mathbf{2},\boldsymbol{n}}\|_{2\gamma}\right)\|\mathbf{x^2} - \mathbf{x}^{\mathbf{2},\boldsymbol{n}}\|_{\infty}^{1-\frac{\mu}{\gamma}}. \quad \square$$

We will use the last result in order to get a reasonable tightness criterion for our approximation processes X^{ε} and $\mathbf{X}^{\mathbf{2},\varepsilon}$, by means of a slight elaboration of [21, Corollary 6.1]:

Proposition 4.2. *Let X^{ε} and $\mathbf{X}^{\mathbf{2},\varepsilon}$ be defined respectively by (17) and (18). If, for every $\eta > 0$, there exists $\gamma > \mu$ and $A < \infty$ such that*

$$\sup_{0<\varepsilon\leq 1} P[\|X^{\varepsilon}\|_{\gamma} > A] \leq \eta \quad \textit{and} \quad \sup_{0<\varepsilon\leq 1} P[\|\mathbf{X}^{\mathbf{2},\varepsilon}\|_{2\gamma} > A] \leq \eta, \tag{21}$$

then $(X^{\varepsilon}, \mathbf{X}^{\mathbf{2},\varepsilon})$ is tight in $\mathcal{C}_1^{\mu} \times \mathcal{C}_2^{2\mu}$.

Proof. Recall the Prokhorov theorem relating precompactness of measures *on* a space to compactness of sets *in* the space. This result states that a family M of probability measures on the Borel sets of a complete separable metric space S is weakly precompact if and only if for every $\eta > 0$ there exists a compact set $K_\eta \subset S$ such that

$$\sup_{\mu \in M} \mu\left(S \setminus K_\eta\right) \leq \eta.$$

Furthermore, it is readily checked that the couple $(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon})$ satisfies the assumption (20), which allows to apply Lemma 4.1. Hence, combining this lemma with Prokhorov's theorem, our proposition is easily proved. □

Let us turn now to the main result of this subsection:

Proposition 4.3. *The sequence $(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon})_{\varepsilon>0}$ defined in Theorem 3.1 is tight in $\mathcal{C}_1^\mu \times \mathcal{C}_2^{2\mu}$.*

Proof. Thanks to Proposition 4.2, we just have to prove that $(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon})$ verifies (21). For an arbitrary $\eta \in (0, 1)$, we will first deal with the relation

$$\sup_{0<\varepsilon\leq 1} P\left[\|X^\varepsilon\|_\gamma > A\right] \leq \eta, \tag{22}$$

for $A = A_\eta$ large enough, and $1/3 < \gamma < H$. To this purpose, let us recall some basic facts about Sobolev spaces, for which we refer to [1]: for $\alpha \in (0, 1)$ and $p \geq 1$, the Sobolev space $\mathcal{W}^{\alpha,p}([0, T];\ \mathbb{R}^n)$ is induced by the semi-norm

$$\|f\|_{\alpha,p}^p = \int_0^T \int_0^T \frac{|f(t) - f(s)|^p}{|t - s|^{1+\alpha p}} \mathrm{d}s\mathrm{d}t. \tag{23}$$

Then the Sobolev imbedding theorem states that, if $\alpha p > 1$, then $\mathcal{W}^{\alpha,p}([0, T];\ \mathbb{R}^d)$ is continuously imbedded in $\mathcal{C}_1^\gamma(\mathbb{R}^d)$ for any $\gamma < \alpha - 1/p$, where the spaces $\mathcal{C}_1^\gamma$ have been defined by relation (8), and in this case, we furthermore have that

$$\|f\|_\gamma \leq c\|f\|_{\alpha,p}, \tag{24}$$

for a positive constant $c = c_{\alpha,p}$. Notice that, in both (8) and (23), the sup part of the usual Hölder or Sobolev norm has been omitted, but can be recovered since we are dealing with fixed initial conditions. In order to prove (22), it is thus sufficient to check that, for any $p \geq 1$ sufficiently large and $\alpha < H$, the following bound holds true:

$$\sup_{0<\varepsilon\leq 1} E\left[\int_0^T \int_0^T \frac{|X^\varepsilon(t) - X^\varepsilon(s)|^p}{|t - s|^{1+\alpha p}} \mathrm{d}s\mathrm{d}t\right] \leq M_{\alpha,p} < \infty. \tag{25}$$

Invoking Lemma 6.1 and then Lemma 2.7, we easily get (see [5] page 11 for the details), for any $\varepsilon > 0$, any $t > s \geq 0$ and any integer $m \geq 1$:

$$E\left[|X^{\varepsilon,i}(t) - X^{\varepsilon,i}(s)|^{2m}\right] \leq c_{2m,H}|t - s|^{2mH}. \tag{26}$$

Note that here, and in the remainder of the proof, $c_{\{\cdot\}}$ denotes a generic constant depending only on the object(s) inside its argument, and which may take different values one formula to another one. From (26), we deduce that (25) holds for any $\alpha < H$ and p large enough, from which (22) is easily seen. Moreover, thanks to the classical Garsia–Rodemich–Rumsey lemma, see [13], for any $\varepsilon, \delta, T > 0$ and $i \in \{1, \ldots, d\}$, there exists a random variable $G^{T,\delta,\varepsilon,i}$ such that, for any $s, t \in [0, T]$:

$$|X^{\varepsilon,i}(t) - X^{\varepsilon,i}(s)| \le G^{T,\delta,\varepsilon,i}|t-s|^{H-\delta}. \tag{27}$$

Since the bound in (26) is *independent* of ε, it is easily checked that, for any integer $m \ge 1$, any $i \in \{1, \ldots, d\}$ and any $\delta, T > 0$ (δ small enough), we have

$$c_{2m,\delta} := \sup_{0<\varepsilon\le 1} E\left(|G^{T,\delta,\varepsilon,i}|^{2m}\right) < +\infty.$$

Let us turn now to the tightness of $(\mathbf{X}^{\mathbf{2},\varepsilon})_{\varepsilon>0}$. Recall first that $\mathbf{X}^{\mathbf{2},\varepsilon}_{st}(i,i) = \frac{1}{2}(X^{\varepsilon,i}_t - X^{\varepsilon,i}_s)^2$. Therefore, we deduce from (26) that

$$E\left[|\mathbf{X}^{\mathbf{2},\varepsilon}_{st}(i,i)|^{2m}\right] \le \frac{c_{4m,H}}{2^{2m}}|t-s|^{4mH}. \tag{28}$$

Assume now that $i \ne j$. We have, by applying successively (50), Lemma 6.1 and (27):

$$\begin{aligned} E[|\mathbf{X}^{\mathbf{2},\varepsilon}_{st}(i,j)|^{2m}] &\le c_m\, E\left|\int_s^t \left(X^{j,\varepsilon}_u - X^{j,\varepsilon}_s\right)^2 (t+\varepsilon-u)^{2H-1}\mathrm{d}u\right|^m \\ &+ c_m\, E\left|\int_0^s \left(X^{j,\varepsilon}_u - X^{j,\varepsilon}_s\right)^2\left((t+\varepsilon-u)^{H-\frac{1}{2}} - (s+\varepsilon-u)^{H-\frac{1}{2}}\right)^2\mathrm{d}u\right|^m \\ &+ c_{m,H}\, E\left|\int_0^t \left(\int_{s\vee v}^t |X^{j,\varepsilon}_u - X^{j,\varepsilon}_v|(u+\varepsilon-v)^{H-\frac{3}{2}}\mathrm{d}u\right)^2\mathrm{d}v\right|^m. \end{aligned}$$

This last expression can be trivially bounded by considering the case $\varepsilon = 0$, and some elementary calculations then lead to the relation

$$E[|\mathbf{X}^{\mathbf{2},\varepsilon}_{st}(i,j)|^{2m}] \le c_{m,H}|t-s|^{4mH-2m\delta}. \tag{29}$$

In order to conclude that $\mathbf{X}^2$ verifies the second inequality in (21), let us recall the following inequality from [14]: let $g \in \mathcal{C}_2(V)$ for a given Banach space V; then, for any $\kappa > 0$ and $p \ge 1$ we have

$$\|g\|_\kappa \le c\left(U_{\kappa+2/p;p}(g) + \|\delta g\|_\kappa\right) \quad \text{with } U_{\gamma;p}(g) = \left(\int_0^T\int_0^T \frac{|g_{st}|^p}{|t-s|^{\gamma p}}\mathrm{d}s\mathrm{d}t\right)^{1/p}. \tag{30}$$

By plugging inequality (28)–(29), for $\delta > 0$ small enough, into (30) and by recalling that $\delta\mathbf{X}^{2,\varepsilon} = \delta X^\varepsilon \otimes \delta X^\varepsilon$ and inequality (27), we obtain easily the second part of (21). □

5. Fdd convergence in Theorem 3.1

This section is devoted to the second part of the proof of Theorem 3.1, namely the convergence of finite-dimensional distributions. Precisely, we shall prove the following:

Proposition 5.1. *Let $(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon})$ be the approximation process defined by (17) and (18). Then*

$$\text{f.d.d.} - \lim_{\varepsilon\to 0}(X^\varepsilon, \mathbf{X}^{\mathbf{2},\varepsilon}) = (B, \mathbf{B}^{\mathbf{2}}), \tag{31}$$

where f.d.d. $-$ lim *stands for the convergence in law of the finite-dimensional distributions. Otherwise stated, for any $k \ge 1$ and any family $\{s_i, t_i;\ i \le k, 0 \le s_i < t_i \le T\}$, we have*

$$\mathcal{L} - \lim_{\varepsilon\to 0}(X^\varepsilon_{t_1}, \mathbf{X}^{\mathbf{2},\varepsilon}_{s_1t_1}, \ldots, X^\varepsilon_{t_k}, \mathbf{X}^{\mathbf{2},\varepsilon}_{s_kt_k}) = (B_{t_1}, \mathbf{B}^{\mathbf{2}}_{s_1t_1}, \ldots, B_{t_k}, \mathbf{B}^{\mathbf{2}}_{s_kt_k}). \tag{32}$$

Proof. The proof is divided into several steps. $\square$

(i) *Reduction of the problem*. For simplicity, we assume that the dimension d of B is 2 (the general case can be treated along the same lines, up to some cumbersome notations). For $i = 1, 2$, $\varepsilon > 0$ and $0 \le u \le t \le T$, let us consider

$$Y^{i,\varepsilon}(u,t) = \int_u^t (X_v^{i,\varepsilon} - X_u^{i,\varepsilon})(v-u)^{H-\frac{3}{2}}\mathrm{d}v$$

and

$$Y^{i}(u,t) = \int_u^t (B_v^{i} - B_u^{i})(v-u)^{H-\frac{3}{2}}\mathrm{d}v.$$

In this step, we shall prove that the fdd convergence (31) is a consequence of the following one:

$$\begin{aligned}
&\left(\int_0^\cdot \theta^{\varepsilon,1}(u)\mathrm{d}u, \int_0^\cdot \theta^{\varepsilon,2}(u)\mathrm{d}u, \int_0^\cdot X_u^{2,\varepsilon}\theta^{\varepsilon,1}(u)\mathrm{d}u,\right.\\
&\quad\times \left.\int_0^\cdot Y^{2,\varepsilon}(u,\cdot)\theta^{\varepsilon,1}(u)\mathrm{d}u, \int_0^\cdot X_u^{1,\varepsilon}\theta^{\varepsilon,2}(u)\mathrm{d}u, \int_0^\cdot Y^{1,\varepsilon}(u,\cdot)\theta^{\varepsilon,2}(u)\mathrm{d}u\right)\\
&\xrightarrow{\text{f.d.d.}} \left(W^1, W^2, \int_0^\cdot B_u^2\mathrm{d}W_u^1, \int_0^\cdot Y^2(u,\cdot)\mathrm{d}W_u^1, \int_0^\cdot B_u^1\mathrm{d}W_u^2, \int_0^\cdot Y^1(u,\cdot)\mathrm{d}W_u^2\right).
\end{aligned} \tag{33}$$

Indeed, assume for an instant that (33) takes place. Then, approximating the kernel $(t-\cdot)^{H-1/2}$ in L^2 by a sequence of step functions (along the same lines as in [9, Proof of Theorem 1, p. 404]), it is easily checked that we also have:

$$\begin{aligned}
&\left(X^{1,\varepsilon}, X^{2,\varepsilon}, \int_0^\cdot (\cdot+\varepsilon-u)^{H-\frac{1}{2}} X_u^{2,\varepsilon}\theta^{\varepsilon,1}(u)\mathrm{d}u,\right.\\
&\quad\times \left.\int_0^\cdot Y^{2,\varepsilon}(u,\cdot)\theta^{\varepsilon,1}(u)\mathrm{d}u, \int_0^\cdot (\cdot+\varepsilon-u)^{H-\frac{1}{2}} X_u^{1,\varepsilon}\theta^{\varepsilon,2}(u)\mathrm{d}u, \int_0^\cdot Y^{1,\varepsilon}(u,\cdot)\theta^{\varepsilon,2}(u)\mathrm{d}u\right)\\
&\xrightarrow{\text{f.d.d.}} \left(B^1, B^2, \int_0^\cdot (\cdot-u)^{H-\frac{1}{2}} B_u^2\mathrm{d}W_u^1,\right.\\
&\quad\times \left.\int_0^\cdot Y^2(u,\cdot)\mathrm{d}W_u^1, \int_0^\cdot (\cdot-u)^{H-\frac{1}{2}} B_u^1\mathrm{d}W_u^2, \int_0^\cdot Y^1(u,\cdot)\mathrm{d}W_u^2\right).
\end{aligned} \tag{34}$$

In other words, we can add the deterministic kernel $(\cdot+\varepsilon-u)^{H-\frac{1}{2}}$ in the first, second, third and fifth components of (33) without difficulty. Let us invoke now the forthcoming identity (50) in Lemma 6.3 for $s = 0$, which allows easily to go from (34) to:

$$\left(X^{1,\varepsilon}, X^{2,\varepsilon}, \mathbf{X}_{0\cdot}^{\mathbf{2},\varepsilon}(1,2), \mathbf{X}_{0\cdot}^{\mathbf{2},\varepsilon}(2,1)\right) \xrightarrow{\text{f.d.d.}} \left(B^1, B^2, \int_0^\cdot B^2\mathrm{d}B^1, \int_0^\cdot B^1\mathrm{d}B^2\right). \tag{35}$$

Finally, in order to prove our claim (32) from (35), it is enough to observe that $\mathbf{X}_{0t}^{\mathbf{2},\varepsilon}(i,i) = (X_t^{i,\varepsilon})^2/2$ and

$$\mathbf{X}_{st}^{\mathbf{2},\varepsilon}(i,j) = \mathbf{X}_{0t}^{\mathbf{2},\varepsilon}(i,j) - \mathbf{X}_{0s}^{\mathbf{2},\varepsilon}(i,j) - X_s^{i,\varepsilon}\left(X_t^{j,\varepsilon} - X_s^{j,\varepsilon}\right).$$

(ii) *Simplification of the statement* (33). For the simplicity of the exposition, we only prove (33) for a fixed t, instead of a vector $(t_1, \ldots, t_m)$. It will be clear from our proof that the general case can be elaborated easily from this particular situation, up to some additional unpleasant notations. Precisely, we shall prove that, for any $u := (u_1, \ldots, u_6) \in \mathbb{R}^6$, we have $\lim_{\varepsilon\to 0} \delta_\varepsilon = E[\exp(\mathrm{i}\langle u, U\rangle)]$, where $\delta_\varepsilon := E[\exp(\mathrm{i}\langle u, U^\varepsilon\rangle)]$, U^ε is defined by

$$
\begin{aligned}
U^\varepsilon &= u_1 \int_0^t \theta^{\varepsilon,1}(v)\mathrm{d}v + u_2 \int_0^t \theta^{\varepsilon,2}(v)\mathrm{d}v + u_3 \int_0^t X_u^{2,\varepsilon}\theta^{\varepsilon,1}(v)\mathrm{d}v \\
&\quad + u_4 \int_0^t Y^{2,\varepsilon}(v,t)\theta^{\varepsilon,1}(v)\mathrm{d}v + u_5 \int_0^t X_v^{1,\varepsilon}\theta^{\varepsilon,2}(v)\mathrm{d}v + u_6 \int_0^t Y^{1,\varepsilon}(v,t)\theta^{\varepsilon,2}(v)\mathrm{d}v,
\end{aligned}
$$

and

$$
\begin{aligned}
U &= u_1 W_t^1 + u_2 W_t^2 + u_3 \int_0^t B_v^2 \mathrm{d}W_v^1 \\
&\quad + u_4 \int_0^t Y^2(v,t)\mathrm{d}W_v^1 + u_5 \int_0^t B_v^1 \mathrm{d}W_v^2 + u_6 \int_0^t Y^1(v,t)\mathrm{d}W_v^2.
\end{aligned}
$$

In order to analyze the asymptotic behavior of δ_ε, let us first express U^ε as an integral with respect to $\theta^{\varepsilon,1}$ only. Indeed, Fubini's theorem easily yields

$$
\int_0^t X_v^{1,\varepsilon}\theta^{\varepsilon,2}(v)\mathrm{d}v = \int_0^t \mathrm{d}u\theta^{\varepsilon,1}(u) \int_u^t \mathrm{d}v\theta^{\varepsilon,2}(v)(v+\varepsilon-u)^{H-\frac{1}{2}},
$$

and the same kind of argument also gives

$$
\begin{aligned}
\int_0^t Y^{1,\varepsilon}(v,t)\theta^{\varepsilon,2}(v)\mathrm{d}v &= \int_0^t \mathrm{d}u\theta^{\varepsilon,1}(u) \int_u^t \mathrm{d}w \int_u^w \mathrm{d}v\theta^{\varepsilon,2}(v)(w-v)^{H-\frac{1}{2}} \\
&\quad \times \left((w+\varepsilon-u)^{H-\frac{1}{2}} - (v+\varepsilon-u)^{H-\frac{1}{2}}\right) \\
&\quad + \int_0^t \mathrm{d}u\theta^{\varepsilon,1}(u) \int_u^t \mathrm{d}w \int_0^u \mathrm{d}v\theta^{\varepsilon,2}(v)(w-v)^{H-\frac{3}{2}}(w+\varepsilon-u)^{H-\frac{1}{2}}.
\end{aligned}
$$

Therefore, integrating first with respect to the randomness contained in $\theta^{\varepsilon,1}$, one is allowed to write $\delta_\varepsilon = E(\Phi_\varepsilon(Z^\varepsilon)\, \mathrm{e}^{\mathrm{i}u_2 \int_0^t \theta^{\varepsilon,2}(v)\mathrm{d}v})$ where, for $f \in L^1([0,t])$, we set

$$
\Phi_\varepsilon(f) := E\left(\mathrm{e}^{\mathrm{i}\int_0^t f(u)\theta^{\varepsilon,1}(u)\mathrm{d}u}\right),
$$

and where the process Z^ε is defined by:

$$
\begin{aligned}
Z_u^\varepsilon &:= u_1 + u_3 X_u^{2,\varepsilon} + u_4 Y^{2,\varepsilon}(u,t) + u_5 \int_u^t (v+\varepsilon-u)^{H-\frac{1}{2}}\theta^{\varepsilon,2}(v)\mathrm{d}v \\
&\quad + u_6 \int_u^t \mathrm{d}w \int_u^w \mathrm{d}v\theta^{\varepsilon,2}(v)(w-v)^{H-\frac{3}{2}}\left((w+\varepsilon-u)^{H-\frac{1}{2}} - (v+\varepsilon-u)^{H-\frac{1}{2}}\right) \\
&\quad + u_6 \int_u^t \mathrm{d}w \int_0^u \mathrm{d}v\theta^{\varepsilon,2}(v)(w-v)^{H-\frac{3}{2}}(w+\varepsilon-u)^{H-\frac{1}{2}}.
\end{aligned} \tag{36}
$$

Hence setting now, for $f \in L^2([0,t])$,

$$
\Phi(f) := E\left(\mathrm{e}^{\mathrm{i}\int_0^t f(u)\mathrm{d}W_u^1}\right) = \exp\left(-\frac{1}{2}\int_0^t f^2(u)\mathrm{d}u\right),
$$

we have obtained the decomposition

$$
\delta_\varepsilon = E\left(\Phi(Z)\mathrm{e}^{\mathrm{i}u_2 W_t^2}\right) + v_\varepsilon^a + v_\varepsilon^b
$$

where the process Z is given by

$$
\begin{aligned}
Z_u &= u_1 + u_3 B_u^2 + u_4 Y^2(u,t) + u_5 \int_u^t (v-u)^{H-\frac{1}{2}} \mathrm{d}W_v^2 \\
&\quad + u_6 \int_u^t \mathrm{d}w \int_u^w \mathrm{d}W_v^2\, (w-v)^{H-\frac{3}{2}} \left((w-u)^{H-\frac{1}{2}} - (v-u)^{\frac{1}{2}}\right) \\
&\quad + u_6 \int_u^t \mathrm{d}w \int_0^u \mathrm{d}W_v^2 (w-v)^{H-\frac{3}{2}} (w-u)^{H-\frac{1}{2}}, \quad u \in [0,t],
\end{aligned}
$$

and with two remainders v_ε^a, v_ε^b defined as:

$$
\begin{aligned}
v_\varepsilon^a &:= E\left(\Phi_\varepsilon(Z^\varepsilon) \mathrm{e}^{\mathrm{i}u_2 \int_0^t \theta^{\varepsilon,2}(u)\mathrm{d}u}\right) - E\left(\Phi(Z^\varepsilon) \mathrm{e}^{\mathrm{i}u_2 \int_0^t \theta^{\varepsilon,2}(u)\mathrm{d}u}\right) \\
v_\varepsilon^b &:= E\left(\Phi(Z^\varepsilon) \mathrm{e}^{\mathrm{i}u_2 \int_0^t \theta^{\varepsilon,2}(u)\mathrm{d}u}\right) - E\left(\Phi(Z) \mathrm{e}^{\mathrm{i}u_2 W_t^2}\right).
\end{aligned}
$$

The convergence of v_ε^b above is easily established: using again the same strategy than in [9, Proof of Theorem 1] (namely reducing the problem to a convergence of the Kac–Stroock process to white noise itself via an approximation of Liouville's kernel by step functions), one has that

$$
\left(Z^\varepsilon, \int_0^t \theta^{\varepsilon,2}(u)\mathrm{d}u\right) \xrightarrow[\varepsilon\to 0]{\text{Law}} (Z, W_t^2).
$$

Note that the convergence in law in the last equation holds in the space $\mathscr{C} \times \mathbb{R}$, where $\mathscr{C} = \mathscr{C}([0,t])$ denotes the space of continuous function endowed with the uniform norm $\|\cdot\|_\infty$. In particular, it is readily checked that $\lim_{\varepsilon\to 0} v_\varepsilon^b = 0$.

Now, it remains to prove that $\lim_{\varepsilon\to 0} v_\varepsilon^a = 0$. To this aim, we notice that we can bound trivially $|\mathrm{e}^{\mathrm{i}u_2 W_t^2}|$ by 1, and then apply the forthcoming Lemma 6.2 in order to deduce that

$$
|v_\varepsilon^a| \le E\left[\left(\varepsilon^{2\alpha} c_\alpha \|Z_\varepsilon\|_\alpha \|Z_\varepsilon\|_{L^2} u^2 + \phi_{Z_\varepsilon}(\varepsilon)\frac{u^2}{2} + \psi_{Z_\varepsilon}(\varepsilon)\frac{u^4}{8} + \varphi_{Z_\varepsilon}(\varepsilon)\frac{|u|}{2}\right) \mathrm{e}^{\frac{u^2\|Z_\varepsilon\|_{L^2}^2}{2}}\right]
$$

for any $\alpha \in (0,1)$. Furthermore, it is well known that characteristic functions on a neighborhood of 0 are sufficient to identify probability laws. Consequently, using Hölder's inequality, we see that in order to get $\lim_{\varepsilon\to 0} v_\varepsilon^a = 0$, we are left to check that, for a given $u_0 > 0$,

$$
\sup_{0<\varepsilon\le 1} E\left[\|Z_\varepsilon\|_\alpha^2\right] < \infty, \tag{37}
$$

$$
\lim_{\varepsilon\to 0} E\left[\phi_{Z_\varepsilon}^2(\varepsilon)\right] = 0, \qquad \lim_{\varepsilon\to 0} E\left[\psi_{Z_\varepsilon}^2(\varepsilon)\right] = 0, \qquad \lim_{\varepsilon\to 0} E\left[\varphi_{Z_\varepsilon}^2(\varepsilon)\right] = 0, \tag{38}
$$

$$
\sup_{0<\varepsilon\le 1} E\left[\mathrm{e}^{u^2\|Z_\varepsilon\|_{L^2}^2}\right] \le M \quad \text{for all } u \le u_0. \tag{39}
$$

We are now going to see that relations (37), (38) and (39) are satisfied.

(iii) *Simplification of inequality* (39). Recall that Z^ε has been defined by (36), and decompose it as $Z_u^\varepsilon = u_1 + u_3 U_1^\varepsilon(u) + u_4 U_2^\varepsilon(u) + u_5 U_3^\varepsilon(u) + u_6 U_4^\varepsilon(u) + u_6 U_5^\varepsilon(u)$, with

$$
U_1^\varepsilon(u) = X_u^{2,\varepsilon}, \qquad U_2^\varepsilon(u) = Y^{2,\varepsilon}(u,t), \qquad U_3^\varepsilon(u) = \int_u^t (r+\varepsilon-u)^{H-\frac{1}{2}} \theta^{\varepsilon,2}(r)\mathrm{d}r
$$

$$
U_4^\varepsilon(u) = u_6 \int_u^t \mathrm{d}w \int_u^w \mathrm{d}r \theta^{\varepsilon,2}(r)(w-r)^{H-\frac{3}{2}}((w+\varepsilon-u)^{H-\frac{1}{2}} - (r+\varepsilon-u)^{H-\frac{1}{2}})
$$

$$U_5^\varepsilon(u) = \int_u^t \mathrm{d}w \int_0^u \mathrm{d}r\theta^{\varepsilon,2}(r)(w-r)^{H-\frac{3}{2}}(w+\varepsilon-u)^{H-\frac{1}{2}}.$$

In order to obtain (39), it is sufficient to check that there exists $M > 0$ such that, for $\kappa > 0$ small enough and $i = 1, \ldots, 5$, we have

$$\sup_{0<\varepsilon\le 1} E\left(\mathrm{e}^{\kappa \int_0^T U_i^\varepsilon(u)^2 \mathrm{d}u}\right) \le M. \tag{40}$$

Moreover, observe that U_i^ε can always be written under the form

$$U_i^\varepsilon(u) = \int_0^T V_i(u,r,\varepsilon)\theta^{\varepsilon,2}(r)\mathrm{d}r, \tag{41}$$

for a deterministic function $V_i(u, r, \varepsilon)$, and it is thus enough to check that

$$C_i := \sup_{u\in[0,T]} \sup_{0<\varepsilon\le 1} \int_0^T V_i^2(u,r,\varepsilon)\mathrm{d}r < \infty. \tag{42}$$

Indeed, using Lemma 6.1, we can write

$$\begin{aligned} E\left(\mathrm{e}^{\kappa \int_0^T U_i^\varepsilon(u)^2 \mathrm{d}u}\right) &= \sum_{m=0}^\infty \frac{\kappa^m}{m!} E\left[\left(\int_0^T U_i^\varepsilon(u)^2 \mathrm{d}u\right)^m\right] \\ &\le \frac{1}{T}\sum_{m=0}^\infty \frac{(T\kappa)^m}{m!}\int_0^T E\big[U_i^\varepsilon(u)^{2m}\big]\mathrm{d}u \\ &\le \frac{1}{T}\int_0^T \sum_{m=0}^\infty \frac{(2m)!(T\kappa)^m}{2^m(m!)^2}\|V_i(u,\cdot,\varepsilon)\|_{L^2}^{2m}\mathrm{d}u \le \sum_{m=0}^\infty (9T\kappa C_i)^m, \end{aligned}$$

where we have used the bound $(m/3)^m \le m! \le m^m$ in the last inequality, so that the desired conclusion follows for $\kappa > 0$ small enough.

(iv) *Proof of* (42). We shall treat separately the cases for $i = 1, \ldots, 5$. During all the computations below, $C > 0$ will denote a constant depending only on H and T, which can differ from one line to another.

(a) *Case* $i = 1$. We have $X_u^{2,\varepsilon} = \int_0^T V_1(u,r,\varepsilon)\theta^{\varepsilon,2}(r)\mathrm{d}r$ with

$$V_1(u,r,\varepsilon) = \mathbf{1}_{[0,u]}(r)(u+\varepsilon-r)^{H-\frac{1}{2}}.$$

Since

$$\int_0^T V_1^2(u,r,\varepsilon)\mathrm{d}r = \int_0^u (u+\varepsilon-r)^{2H-1}\mathrm{d}r \le \int_0^u (u-r)^{2H-1}\mathrm{d}r = \frac{u^{2H}}{2H} \le C,$$

we have that (42) takes place for $i = 1$.

(b) *Case* $i = 2$. We have $Y^{2,\varepsilon}(u,t) = \int_0^T V_2(u,r,\varepsilon)\theta^{\varepsilon,2}(r)\mathrm{d}r$, with

$$\begin{aligned} V_2(u,r,\varepsilon) = \mathbf{1}_{[0,u]}(r)\int_u^t \big((w+\varepsilon-r)^{H-\frac{1}{2}} - (u+\varepsilon-r)^{H-\frac{1}{2}}\big)(w-u)^{H-\frac{3}{2}}\mathrm{d}w \\ + \mathbf{1}_{[u,t]}(r)\int_r^t (w+\varepsilon-r)^{H-\frac{1}{2}}(w-u)^{H-\frac{3}{2}}\mathrm{d}w. \end{aligned}$$

Then $\int_0^T V_2^2(u,r,\varepsilon)\mathrm{d}r = A_{2,1}(u,\varepsilon) + A_{2,2}(u,\varepsilon)$, where

$$A_{2,1}(u,\varepsilon) = \int_0^u \left(\int_u^t \left((w+\varepsilon-r)^{H-\frac{1}{2}} - (u+\varepsilon-r)^{H-\frac{1}{2}}\right)(w-u)^{H-\frac{3}{2}}\mathrm{d}w\right)^2 \mathrm{d}r,$$

$$A_{2,2}(u,\varepsilon) = \int_u^t \left(\int_r^t (w+\varepsilon-r)^{H-\frac{1}{2}}(w-u)^{H-\frac{3}{2}}\mathrm{d}w\right)^2 \mathrm{d}r.$$

For any $\beta \in (0,1)$ and $w > u > r > 0$, we can write, for some $w^* \in (u+\varepsilon, w+\varepsilon)$:

$$\left|(w+\varepsilon-r)^{H-\frac{1}{2}} - (u+\varepsilon-r)^{H-\frac{1}{2}}\right|$$
$$\leq \frac{C|w-u|^\beta}{|w^*-r|^{(\frac{3}{2}-H)\beta}}\left(\frac{1}{|w+\varepsilon-r|^{\frac{1}{2}-H}} + \frac{1}{|u+\varepsilon-r|^{\frac{1}{2}-H}}\right)^{1-\beta} \leq \frac{C|w-u|^\beta}{|u-r|^{\frac{1}{2}+\beta-H}}.$$

Then, choosing $\beta = \frac{1}{2} - H + \delta$ (with $\delta > 0$ small enough), we can write

$$A_{2,1}(u,\varepsilon) \leq C\int_0^u \frac{\mathrm{d}r}{|u-r|^{2-4H+2\delta}} \times \left(\int_u^t \frac{\mathrm{d}w}{|w-u|^{1-\delta}}\right)^2 \leq C,$$

where we have used the fact that $2-4H < 1$ whenever $H > 1/4$. Using similar arguments, it is also possible to prove that $A_{2,2}(u,\varepsilon) \leq C$ (see [5] page 17 for the details).

(c) *Case* $i = 3$. We have

$$\int_u^t (r+\varepsilon-u)^{H-\frac{1}{2}}\theta^{\varepsilon,2}(r)\mathrm{d}r = \int_0^T V_3(u,r,\varepsilon)\theta^{\varepsilon,2}(r)\mathrm{d}r,$$

with $V_3(u,r,\varepsilon) = \mathbf{1}_{[u,t]}(r)(r+\varepsilon-u)^{H-\frac{1}{2}}$, so that the desired conclusion follows immediately since

$$\int_0^1 V_3^2(u,r,\varepsilon)\mathrm{d}r = \int_u^t (r+\varepsilon-u)^{2H-1}\mathrm{d}r \leq \int_u^t (r-u)^{2H-1}\mathrm{d}r = \frac{(t-u)^{2H}}{2H} \leq C.$$

(d) *Case* $i = 4$. We can write

$$\int_u^t \mathrm{d}w \int_u^w \mathrm{d}r(w-r)^{H-\frac{3}{2}}\left((w+\varepsilon-u)^{H-\frac{1}{2}} - (r+\varepsilon-u)^{H-\frac{1}{2}}\right)\theta^{\varepsilon,2}(r)$$

as $\int_0^T V_4(u,r,\varepsilon)\theta^{\varepsilon,2}(r)\mathrm{d}r$, with

$$V_4(u,r,\varepsilon) = \mathbf{1}_{[u,t]}(r)\int_r^t (w-r)^{H-\frac{3}{2}}\left((w+\varepsilon-u)^{H-\frac{1}{2}} - (r+\varepsilon-u)^{H-\frac{1}{2}}\right)\mathrm{d}w.$$

Then, according to the computations already performed for the analysis of $A_{2,1}$ above, we obtain, for $\delta > 0$ small enough,

$$\int_0^T V_4^2(u,r,\varepsilon)\mathrm{d}r \leq C\int_u^t \frac{1}{|r-u|^{2-4H+2\delta}}\left(\int_r^t \frac{\mathrm{d}w}{|w-r|^{1-\delta}}\right)^2 \mathrm{d}r \leq C.$$

(e) *Case* $i = 5$. We have

$$\int_u^t \mathrm{d}w \int_0^u \mathrm{d}r(w-r)^{H-\frac{3}{2}}(w+\varepsilon-u)^{H-\frac{1}{2}}\theta^{\varepsilon,2}(r) = \int_0^T V_5(u,r,\varepsilon)\theta^{\varepsilon,2}(r)\mathrm{d}r,$$

with

$$V_5(u,r,\varepsilon) = \mathbf{1}_{[0,u]}(r) \int_u^t (w-r)^{H-\frac{3}{2}} (w+\varepsilon-u)^{H-\frac{1}{2}} \mathrm{d}w.$$

Since $|w-r|^{\frac{3}{2}-H} \geq |w-u|^{1-H+\delta} |u-r|^{\frac{1}{2}-\delta}$ for $r < u < w$, we get (for $\delta > 0$ small enough) that

$$\left|(w-r)^{H-\frac{3}{2}} (w+\varepsilon-u)^{H-\frac{1}{2}}\right| \leq \frac{C}{|w-r|^{\frac{3}{2}-H} |w-u|^{\frac{1}{2}-H}} \leq \frac{C}{|u-r|^{\frac{1}{2}-\delta} |w-u|^{\frac{3}{2}-2H+\delta}}.$$

Hence, invoking again the fact that $H > 1/4$, we end up with

$$\int_0^T V_5^2(u,r,\varepsilon)\mathrm{d}r \leq C \int_0^u \frac{\mathrm{d}r}{|u-r|^{1-2\delta}} \times \left(\int_u^t \frac{\mathrm{d}w}{|w-u|^{\frac{3}{2}-2H+\delta}} \right)^2 \leq C.$$

(v) *Proof of* (38). In the previous step, we have shown in particular that, for any $i = 1, \ldots, 5$, we have $\sup_{0<\varepsilon\leq 1} \int_0^T E\big[|U_i^\varepsilon(u)|^p\big]\mathrm{d}u < \infty$ for all $p > 1$, which implies

$$\sup_{0<\varepsilon\leq 1} \int_0^T E\big[|Z_u^\varepsilon|^p\big]\mathrm{d}u < \infty, \quad \text{for all } p > 1.$$

On the other hand, a simple application of Schwarz inequality yields

$$E\big[\phi_{Z^\varepsilon}^2(\varepsilon)\big] = E\left[\left(\int_0^T (Z_u^\varepsilon)^2 \mathrm{e}^{-\frac{2u}{\varepsilon^2}} \mathrm{d}u\right)^2\right] \leq C\varepsilon^2 \int_0^T E\big[(Z_u^\varepsilon)^4\big]\mathrm{d}u,$$

and the same kind of argument also gives

$$\begin{aligned} E\big[\psi_{Z^\varepsilon}^2(\varepsilon)\big] &= E\left[\left(\int_0^T \mathrm{d}x \int_0^x \mathrm{d}y (Z_x^\varepsilon)^2 (Z_y^\varepsilon)^2 \mathrm{e}^{-\frac{2(x-y)}{\varepsilon^2}}\right)^2\right] \\ &\leq \frac{1}{2} E\left[\left(\int_0^T \mathrm{d}x \int_0^x \mathrm{d}y (Z_x^\varepsilon)^4 \mathrm{e}^{-\frac{2(x-y)}{\varepsilon^2}}\right)^2\right] + \frac{1}{2} E\left[\left(\int_0^T \mathrm{d}x \int_0^x \mathrm{d}y (Z_y^\varepsilon)^4 \mathrm{e}^{-\frac{2(x-y)}{\varepsilon^2}}\right)^2\right] \\ &\leq C\varepsilon^4 \int_0^T E\big[(Z_u^\varepsilon)^8\big]\mathrm{d}u. \end{aligned}$$

Finally, we have

$$\begin{aligned} E\big[\varphi_{Z^\varepsilon}^2(\varepsilon)\big] &= E\left[\left(\varepsilon \|Z^\varepsilon\|_{L^2} + \left(\int_0^\varepsilon (Z_u^\varepsilon)^2 \mathrm{d}u\right)^{1/2}\right)^2\right] \\ &\leq 2\varepsilon^2 E(\|Z^\varepsilon\|_{L^2}^2) + 2\int_0^\varepsilon E\big[(Z_u^\varepsilon)^2\big]\mathrm{d}u \\ &\leq 2\varepsilon^2 E(\|Z^\varepsilon\|_{L^2}^2) + 2\varepsilon^{1/2} \left(\int_0^T E\big[(Z_u^\varepsilon)^4\big]\mathrm{d}u\right)^{1/2}, \end{aligned}$$

and the proof of (38) follows immediately by putting all these facts together.

(vi) *Proof of* (37). For all $\alpha < \beta - \frac{1}{p}$, the Sobolev inequality (24) yields $\|Z^\varepsilon\|_\alpha \leq C\|Z^\varepsilon\|_{\beta,p}$, where $\|f\|_{\beta,p}$ has been defined by (23). Moreover, recall from (36) that Z^ε has the form

$$Z_t^\varepsilon - Z_s^\varepsilon = \int_0^T G(s,t,r)\theta^{\varepsilon,2}(r)\mathrm{d}r$$

for some $G(s,t,\cdot) \in L^2([0,T])$. Hence, using the definition of $\theta^{\varepsilon,2}$, we can write, for any even integer $p \geq 2$,

$$\begin{aligned}
E|Z_t^\varepsilon - Z_s^\varepsilon|^p &= \varepsilon^{-p} \int_{[0,T]^p} G(s,t,r_1)\cdots G(s,t,r_p) E\big[(-1)^{N(\frac{r_1}{\varepsilon})+\cdots+N(\frac{r_p}{\varepsilon})}\big] \mathrm{d}r_1 \cdots \mathrm{d}r_p \\
&= p!\varepsilon^{-p} \int_{[0,T]^p} G(s,t,r_1)\cdots G(s,t,r_p) \mathrm{e}^{-\frac{2(r_1-r_2)}{\varepsilon^2}} \cdots \mathrm{e}^{-\frac{2(r_{p-1}-r_p)}{\varepsilon^2}} \mathbf{1}_{\{r_1 \geq \cdots \geq r_p\}} \mathrm{d}r_1 \cdots \mathrm{d}r_p \\
&= \frac{p!}{(p/2)!} \left(\varepsilon^{-2} \int_{[0,T]^2} G(s,t,r_1) G(s,t,r_2) \mathrm{e}^{-\frac{2(r_1-r_2)}{\varepsilon^2}} \mathbf{1}_{\{r_1 \geq r_2\}} \mathrm{d}r_1 \mathrm{d}r_2 \right)^{p/2} \\
&= \frac{p!}{(p/2)!} \left(\frac{\varepsilon^{-2}}{2} \int_{[0,T]^2} G(s,t,r_1) G(s,t,r_2) E\big[(-1)^{N(\frac{r_1}{\varepsilon})+N(\frac{r_2}{\varepsilon})}\big] \mathrm{d}r_1 \mathrm{d}r_2 \right)^{p/2} \\
&= \frac{p!}{2^{\frac{p}{2}} (p/2)!} \big(E|Z_t^\varepsilon - Z_s^\varepsilon|^2\big)^{p/2}.
\end{aligned}$$

In particular, we see that, in order to achieve the proof of (37), it is enough to check that

$$E|Z_t^\varepsilon - Z_s^\varepsilon|^2 \leq C|t-s|^{H-\delta} \tag{43}$$

for some $\delta > 0$ (small enough). Actually, we shall use again the decomposition of Z^ε in terms of the U_i's, which means that it is sufficient to prove $E|U_i^\varepsilon(u) - U_i^\varepsilon(v)|^2 \leq C|u-v|^{H-\delta}$ for $i = 1, \ldots, 5$. But it is easily seen that

$$\begin{aligned}
E|U_i^\varepsilon(u) - U_i^\varepsilon(v)|^2 &= E\left| \int_0^T \big(V_i(u,r,\varepsilon) - V_i(v,r,\varepsilon)\big) \theta^{\varepsilon,2}(r) \mathrm{d}r \right|^2 \\
&\leq \int_0^T \big(V_i(u,r,\varepsilon) - V_i(v,r,\varepsilon)\big)^2 \mathrm{d}r,
\end{aligned}$$

where V_i is defined by (41), and are specified at step (iii). It is thus enough for our purposes to show that $\int_0^T (V_i(u,r,\varepsilon) - V_i(v,r,\varepsilon))^2 \mathrm{d}r \leq C|u-v|^{H-\delta}$ for $0 \leq u \leq v \leq t$, where $t \in [0,T]$, which can be done as in Step (iv) above, see [5] page 20 for the details.

The proof of Proposition 5.1 is done. □

6. Some technical lemmas

This section collect the technical results that have been used throughout the proof of Theorem 3.1. The first lemma aims at giving some estimates concerning the Kac–Stroock kernel (3), which can be seen as a elaboration of the ones contained in Delgado and Jolis [9, Lemma 2]. Notice however that these latter results are not sharp enough for our purposes, which forced us to a refinement.

Lemma 6.1. *Let $m \in \mathbb{N}$, $f, f_1, \ldots, f_{2m} \in L^2([0,T])$, $k \in \{1,2\}$ and $\varepsilon > 0$. We have:*

$$\left| E\left[\prod_{j=1}^{2m} \int_0^T f_j(r) \theta^{\varepsilon,k}(r) \mathrm{d}r \right] \right| \leq \frac{(2m)!}{2^m m!} \|f_1\|_{L^2} \ldots \|f_{2m}\|_{L^2}, \tag{44}$$

and

$$\left| E\left[\left(\int_0^T f(r) \theta^{\varepsilon,k}(r) \mathrm{d}r \right)^{2m+1} \right] \right| \leq \varphi_f(\varepsilon) \frac{(2m+1)!}{2^{m+1} m!} \|f\|_{L^2}^{2m}, \tag{45}$$

where

$$\varphi_f(\varepsilon) = \varepsilon \|f\|_{L^2} + \left(\int_0^\varepsilon |f(s)|^2 \mathrm{d}s \right)^{\frac{1}{2}}.$$

Proof. For $m \in \mathbb{N}$, $\varepsilon > 0$ and $f_1, \ldots, f_{2m} \in L^2([0,T])$, let us denote

$$\Delta_{2m}^{\varepsilon}(f_1, \ldots, f_{2m}) := E\left[\prod_{j=1}^{2m} \int_0^T f_j(r) \theta^{\varepsilon,k}(r) \mathrm{d}r \right].$$

We will need to introduce some operations on the set of permutations (in the sequel, $\mathfrak{S}_k$ stands for the set of permutations on $\{1, \ldots, k\}$): when $\tau \in \mathfrak{S}_{2m}$ and $\sigma \in \mathfrak{S}_m$, we note $\sigma \star \tau$ the element of $\mathfrak{S}_{2m}$ defined by

$$(\sigma \star \tau)(2j-1) = \tau(2\sigma(j) - 1) \quad \text{and} \quad (\sigma \star \tau)(2j) = \tau(2\sigma(j)).$$

Remark that we have $\mathrm{id} \star \tau = \tau$ and $\sigma' \star (\sigma \star \tau) = (\sigma\sigma') \star \tau$, so $\star : \mathfrak{S}_m \times \mathfrak{S}_{2m} \to \mathfrak{S}_{2m}$ defines a (right) group action of $\mathfrak{S}_m$ on $\mathfrak{S}_{2m}$. For any $\tau \in \mathfrak{S}_{2m}$, the orbit of τ has exactly $m!$ elements. Consequently, the set $\mathscr{O}$ of the orbits under the group action $\star$ has $\frac{(2m)!}{m!}$ elements and we have, by denoting τ_i one particular element of the orbit $o_i = o(\tau_i) \in \mathscr{O}$: for $r_1, \ldots, r_{2m} \in [0,1]$,

$$\begin{aligned} \mathbf{1}_{\{\forall i \neq j, r_i \neq r_j\}} &= \sum_{\tau \in \mathfrak{S}_{2m}} \mathbf{1}_{\{r_{\tau(1)} > \cdots > r_{\tau(2m)}\}} = \sum_{o_i \in \mathscr{O}} \sum_{\tau \in o_i} \mathbf{1}_{\{r_{\tau(1)} > \cdots > r_{\tau(2m)}\}} \\ &\leq \sum_{i=1}^{(2m)!/m!} \prod_{j=1}^{m} \mathbf{1}_{\{r_{2\tau_i(j)-1} > r_{2\tau_i(j)}\}}. \end{aligned} \tag{46}$$

For the reader who might not be completely convinced by this inequality, let us illustrate it by an example: when $m = 2$ and $\tau_i = \mathrm{id} \in \mathfrak{S}_4$, we have $o_i = o(\tau_i) = \{\mathrm{id}, (13)(24)\}$ and we have used

$$\begin{aligned} \sum_{\tau \in o_i} \mathbf{1}_{\{r_{\tau(1)} > r_{\tau(2)} > r_{\tau(3)} > r_{\tau(4)}\}} &= \mathbf{1}_{\{r_1 > r_2 > r_3 > r_4\}} + \mathbf{1}_{\{r_3 > r_4 > r_1 > r_2\}} \leq \mathbf{1}_{\{r_1 > r_2\}} \mathbf{1}_{\{r_3 > r_4\}} \\ &= \prod_{j=1}^{2} \mathbf{1}_{\{r_{2\tau_i(j)-1} > r_{2\tau_i(j)}\}}. \end{aligned}$$

Let us apply now inequality (46). We introduce first a notation which will prevail until the end of the article: for $\varepsilon > 0$ and $r \in \mathbb{R}_+$, we set $Q_\varepsilon(r) := \mathrm{e}^{-\frac{2r}{\varepsilon^2}} / \varepsilon^2$. Notice then that, for any $\varepsilon > 0$:

$$\begin{aligned} |\Delta_{2m}^{\varepsilon}(f_1, \ldots, f_{2m})| &\leq \frac{1}{\varepsilon^{2m}} \int_{[0,T]^{2m}} |f_1(r_1)| \ldots |f_{2m}(r_{2m})| \\ &\quad \times \left| E\left[(-1)^{\sum_{i=1}^{2m} N(\frac{r_i}{\varepsilon^2})} \right] \right| \mathrm{d}r_1 \ldots \mathrm{d}r_{2m} \\ &= \sum_{o_i \in \mathscr{O}} \sum_{\tau \in o_i} \frac{1}{\varepsilon^{2m}} \int_{[0,T]^{2m}} \mathbf{1}_{\{r_{\tau(1)} > \cdots > r_{\tau(2m)}\}} |f_1(r_1)| \cdots |f_{2m}(r_{2m})| \\ &\quad \times Q_\varepsilon\left(\sum_{i=1}^{m} (r_{\tau(2i-1)} - r_{\tau(2i)}) \right) \mathrm{d}r_1 \cdots \mathrm{d}r_{2m} \end{aligned}$$

and thus, according to (46), we obtain

$$
\begin{aligned}
|\Delta^{\varepsilon}_{2m}(f_1,\ldots,f_{2m})| &\le \sum_{i=1}^{(2m)!/m!} \prod_{j=1}^{m} \int_{[0,T]^2} \mathbf{1}_{\{r_1>r_2\}} |f_{2\tau_i(j)-1}(r_1)||f_{2\tau_i(j)}(r_2)| \\
&\quad \times Q_\varepsilon(r_1-r_2)\mathrm{d}r_1\mathrm{d}r_2 \\
&\le \frac{(2m)!}{2^m m!} \|f_1\|_{L^2}\ldots\|f_{2m}\|_{L^2},
\end{aligned}
$$

the last inequality coming from

$$
\begin{aligned}
&\int_{[0,T]^2} \mathbf{1}_{\{r_1>r_2\}} |f_k(r_1)||f_\ell(r_2)| Q_\varepsilon(r_1-r_2)\mathrm{d}r_1\mathrm{d}r_2 \\
&\le \left(\int_0^T |f_k(r_1)|^2 \left(\int_0^{r_1} Q_\varepsilon(r_1-r_2)\mathrm{d}r_2\right)\mathrm{d}r_1\right)^{\frac12} \\
&\quad \times \left(\int_0^T |f_\ell(r_2)|^2 \left(\int_{r_2}^{T} Q_\varepsilon(r_1-r_2)\mathrm{d}r_1\right)\mathrm{d}r_2\right)^{\frac12} \\
&\le \frac12 \|f_k\|_{L^2}\|f_\ell\|_{L^2}.
\end{aligned} \tag{47}
$$

This finishes the proof of (44), so let us now concentrate on (45). For $m \in \mathbb{N}$, we have

$$
\begin{aligned}
&\left| E\left[\left(\int_0^T f(r)\theta^{\varepsilon,k}(r)\mathrm{d}r\right)^{2m+1}\right]\right| \\
&\le \frac{1}{\varepsilon^{2m+1}} \int_{[0,T]^{2m+1}} \prod_{l=1}^{2m+1} |f(r_l)| \left| E\left[(-1)^{\sum_{i=1}^{2m+1} N(\frac{r_i}{\varepsilon^2})}\right]\right| \mathrm{d}r_1\ldots\mathrm{d}r_{2m+1} \\
&= \frac{2m+1}{\varepsilon^{2m+1}} \int_0^T |f(s)|\mathrm{d}s \int_{[s,T]^{2m}} \prod_{l=1}^{2m} |f(r_l)| \left| E\left[(-1)^{N(\frac{s}{\varepsilon^2})+\sum_{i=1}^{2m} N(\frac{r_i}{\varepsilon^2})}\right]\right| \mathrm{d}r_1\ldots\mathrm{d}r_{2m} \\
&\le (2m+1)\,\Delta^{\varepsilon}_{2m}(|f|,\ldots,|f|) \int_0^T |f(s)|\frac{1}{\varepsilon}\mathrm{e}^{-\frac{2s}{\varepsilon^2}}\mathrm{d}s.
\end{aligned}
$$

Since for $s > \varepsilon$, we have that $\frac{1}{\varepsilon^2}\mathrm{e}^{-\frac{2s}{\varepsilon^2}} \le \frac12$, we get that

$$
\begin{aligned}
\int_0^T |f(s)|\frac{1}{\varepsilon}\mathrm{e}^{-\frac{2s}{\varepsilon^2}}\mathrm{d}s &= \int_0^\varepsilon |f(s)|\frac{1}{\varepsilon}\mathrm{e}^{-\frac{2s}{\varepsilon^2}}\mathrm{d}s + \varepsilon\int_\varepsilon^T |f(s)|\frac{1}{\varepsilon^2}\mathrm{e}^{-\frac{2s}{\varepsilon^2}}\mathrm{d}s \\
&\le \left(\int_0^\varepsilon |f(s)|^2\mathrm{d}s\right)^{\frac12}\left(\int_0^\varepsilon Q_\varepsilon(2s)\mathrm{d}s\right)^{\frac12} + \frac{\varepsilon}{2}\int_\varepsilon^T |f(s)|\mathrm{d}s \\
&\le \frac12\left(\int_0^\varepsilon |f(s)|^2\mathrm{d}s\right)^{\frac12} + \frac{\varepsilon}{2}\|f\|_{L^2},
\end{aligned} \tag{48}
$$

and (45) follows easily. □

The following lemma aims at measuring the distance between the laws of the stochastic integrals $\int_0^T f(r)\theta^{\varepsilon,k}(r)\mathrm{d}r$ and $\int_0^T f(r)\mathrm{d}W^k_r$, whenever f is a given (deterministic) function:

Lemma 6.2. *Let $f \in \mathcal{C}^\alpha([0,T])$ for a given $\alpha \in (0,1)$, $k \in \{1,2\}$ and $\varepsilon > 0$. For any $u \in \mathbb{R}$, we have:*

$$\left| E\left[e^{iu \int_0^T f(r)\theta^{\varepsilon,k}(r)dr}\right] - E\left[e^{iu \int_0^T f(r)dW_r^k}\right] \right|$$

$$\leq \left[\varepsilon^{2\alpha} c_\alpha \|f\|_\alpha \|f\|_{L^2} u^2 + \phi_f(\varepsilon)\frac{u^2}{2} + \psi_f(\varepsilon)\frac{u^4}{8} + \varphi_f(\varepsilon)\frac{|u|}{2}\right] e^{\frac{u^2\|f\|_{L^2}^2}{2}}, \tag{49}$$

with $c_\alpha = \int_0^\infty x^\alpha e^{-2x} dx$ and

$$\phi_f(\varepsilon) = \int_0^T f^2(x) e^{-\frac{2x}{\varepsilon^2}} dx, \qquad \psi_f(\varepsilon) = \int_0^T dx f^2(x) \int_0^x dy f^2(y) e^{-\frac{2(x-y)}{\varepsilon^2}}.$$

Proof. The proof is divided into two steps.

1. *First step: control of the imaginary part.* We can write, thanks to (45):

$$\left| \mathrm{Im}\left(E\left[e^{iu \int_0^T f(r)\theta^{\varepsilon,k}(r)dr}\right] - E\left[e^{iu \int_0^T f(r)dW_r^k}\right]\right) \right| = \left| \mathrm{Im}\, E\left[e^{iu \int_0^T f(r)\theta^{\varepsilon,k}(r)dr}\right] \right|$$

$$\leq \sum_{m=0}^\infty \frac{|u|^{2m+1}}{(2m+1)!} \left| E\left[\left(\int_0^T f(r)\theta^{\varepsilon,k}(r)dr\right)^{2m+1}\right]\right| \leq \varphi_f(\varepsilon)\frac{|u|}{2} e^{\frac{u^2\|f\|_{L^2}^2}{2}}.$$

2. *Second step: control of the real part.* This step is more technical, and we will mainly get a bound on the quantity $L_{m,\varepsilon}$ defined by:

$$L_{m,\varepsilon} = \left| \frac{1}{(2m)!} \Delta_{2m}^\varepsilon(f, \ldots, f) - \frac{1}{2^m} \int_0^T f^2(s_1) ds_1 \ldots \int_0^{s_{m-1}} f^2(s_m) ds_m \right|.$$

In order to express this quantity in a suitable way for estimations, notice that $\int_0^\infty e^{-2s} ds = \frac{1}{2}$. We can thus insert this term artificially in the multiple integrals involved in the computations of $E[e^{iu \int_0^T f(r)dW_r^k}]$. This gives:

$$L_{m,\varepsilon} = \left| \frac{1}{(2m)!} \Delta_{2m}^\varepsilon(f, \ldots, f) - \int_0^T f^2(r_1) dr_1 \int_0^\infty e^{-2r_2} dr_2 \right.$$
$$\left. \ldots \int_0^{r_{2m-3}} f^2(r_{2m-1}) dr_{2m-1} \int_0^\infty e^{-2r_{2m}} dr_{2m} \right|.$$

By a telescoping sum argument, we can now write $L_{m,\varepsilon}$ as a sum of m terms, whose prototype is given by $M_{m,\varepsilon} = M_{m,\varepsilon}^1 + M_{m,\varepsilon}^2 - M_{m,\varepsilon}^3$, with

$$M_{m,\varepsilon}^1 = \int_0^T |f(r_1)| dr_1 \int_0^{r_1} |f(r_2)| Q_\varepsilon(r_1 - r_2) dr_2$$
$$\ldots \int_0^{r_{2m-2}} |f(r_{2m-1})| dr_{2m-1} \int_0^{r_{2m-1}} |f(r_{2m}) - f(r_{2m-1})| Q_\varepsilon(r_{2m-1} - r_{2m}) dr_{2m},$$

where $M_{m,\varepsilon}^2$ is defined by

$$M_{m,\varepsilon}^2 = \int_0^T f(r_1) dr_1 \int_0^{r_1} f(r_2) Q_\varepsilon(r_1 - r_2) dr_2$$
$$\ldots \int_0^{r_{2m-2}} f^2(r_{2m-1}) dr_{2m-1} \int_{r_{2m-1}}^\infty Q_\varepsilon(r_{2m}) dr_{2m},$$

and where

$$M^3_{m,\varepsilon} = \int_0^T f(r_1)\mathrm{d}r_1 \int_0^{r_1} f(r_2) Q_\varepsilon(r_1 - r_2)\mathrm{d}r_2 \\ \dots \int_{r_{2m-2}}^{r_{2m-3}} f^2(r_{2m-1})\mathrm{d}r_{2m-1} \int_0^\infty \mathrm{e}^{-2r_{2m}}\mathrm{d}r_{2m}.$$

We will now bound those three terms separately: invoking first (47), we get

$$\begin{aligned} M^1_{m,\varepsilon} &\le \frac{1}{(m-1)!}\left(\int_0^T |f(r_1)|\mathrm{d}r_1 \int_0^{r_1} |f(r_2)| Q_\varepsilon(r_1 - r_2)\mathrm{d}r_2\right)^{m-1} \\ &\quad \times \int_0^T |f(r_{2m-1})|\mathrm{d}r_{2m-1} \|f\|_\alpha \int_0^{r_{2m-1}} |r_{2m} - r_{2m-1}|^\alpha Q_\varepsilon(r_{2m-1} - r_{2m})\mathrm{d}r_{2m} \\ &\le \frac{1}{(m-1)!2^{m-1}} \|f\|^{2m-1}_{L^2} \|f\|_\alpha c_\alpha \varepsilon^{2\alpha}. \end{aligned}$$

On the other hand, (47) and (48) also yield:

$$\begin{aligned} M^2_{m,\varepsilon} &\le \frac{1}{(m-1)!}\left(\int_0^T |f(r_1)|\mathrm{d}r_1 \int_0^{r_1} |f(r_2)| Q_\varepsilon(r_1 - r_2)\mathrm{d}r_2\right)^{m-1} \\ &\quad \times \frac{1}{2}\int_0^T f^2(r_{2m-1}) Q_\varepsilon(r_{2m-1})\mathrm{d}r_{2m-1} \le \frac{1}{(m-1)!2^m} \|f\|^{2m-2}_{L^2} \phi_f(\varepsilon). \end{aligned}$$

Finally, $M^3_{m,\varepsilon}$ can be bounded in a similar way (see [5] page 28 for the details), and we get

$$M^3_{m,\varepsilon} \le \frac{1}{(m-2)!2^m} \|f\|^{2m-4}_{L^2} \psi_f(\varepsilon).$$

Our proof is now easily finished: plug our estimates on $M^1_{m,\varepsilon}$, $M^2_{m,\varepsilon}$ and $M^3_{m,\varepsilon}$ into the definition of $M_{m,\varepsilon}$, and then in the definition $L_{m,\varepsilon}$. This yields

$$\begin{aligned} &\left|\mathrm{Re}\left(E\left[\mathrm{e}^{\mathrm{i}u\int_0^T f(r)\theta^{\varepsilon,k}(r)\mathrm{d}r}\right] - E\left[\mathrm{e}^{\mathrm{i}u\int_0^T f(r)\mathrm{d}W^k_r}\right]\right)\right| \\ &\le \sum_{m=1}^\infty \frac{u^{2m}}{(2m)!}\left|\Delta^\varepsilon_{2m}(f,\dots,f) - \frac{(2m)!}{2^m m!}\left(\int_0^T f^2(r)\mathrm{d}r\right)^m\right| \\ &\le \left[\varepsilon^{2\alpha} c_\alpha \|f\|_\alpha \|f\|_{L^2} u^2 + \phi_f(\varepsilon)\frac{u^2}{2} + \psi_f(\varepsilon)\frac{u^4}{8}\right]\mathrm{e}^{\frac{u^2\|f\|^2_{L^2}}{2}}, \end{aligned}$$

which is our claim. □

The following lemma gives an alternative form for $\mathbf{X}^{\mathbf{2},\varepsilon}$ and $\mathbf{B}^{\mathbf{2}}$:

Lemma 6.3. *Fix $i, j \in \{1,\dots,d\}$, and $t > s \ge 0$. For all $\varepsilon > 0$, we have*

$$\begin{aligned} \mathbf{X}^{\mathbf{2},\varepsilon}_{st}(i,j) &= \int_0^t \left(X^{j,\varepsilon}_u - X^{j,\varepsilon}_s\right)(t+\varepsilon-u)^{H-\frac{1}{2}}\theta^{\varepsilon,i}(u)\mathrm{d}u \\ &\quad - \int_0^s \left(X^{j,\varepsilon}_u - X^{j,\varepsilon}_s\right)(s+\varepsilon-u)^{H-\frac{1}{2}}\theta^{\varepsilon,i}(u)\mathrm{d}u \\ &\quad - \alpha_H \int_0^t \mathrm{d}v\theta^{\varepsilon,i}(v)\int_{s\vee v}^t \mathrm{d}u(X^{j,\varepsilon}_u - X^{j,\varepsilon}_v)(u+\varepsilon-v)^{H-\frac{3}{2}}, \end{aligned} \tag{50}$$

where we have set $\alpha_H = 1/2 - H$. In the limit $\varepsilon \to 0$, we also have

$$\mathbf{B}^{\mathbf{2}}_{st}(i,j) = \int_s^t \big(B_u^j - B_s^j\big)(t-u)^{H-\frac{1}{2}} \mathrm{d}W_u^i - \int_0^s \big(B_u^j - B_s^j\big)\big[(t-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}\big] \mathrm{d}W_u^i - \alpha_H \int_0^t \mathrm{d}W_v^i \int_{v\vee s}^t \mathrm{d}u \big(B_u^j - B_v^j\big)(u-v)^{H-\frac{3}{2}}. \tag{51}$$

Proof. For any $\varepsilon > 0$, the process $X^{\varepsilon,i}$ is differentiable, and according to (17), we have

$$\dot{X}^{\varepsilon,i}(r) = \varepsilon^{H-1/2}\theta^{\varepsilon,i}(u) - \alpha_H \int_0^u (u+\varepsilon-v)^{H-3/2}\theta^{\varepsilon,i}(v)\mathrm{d}v.$$

Recall also that we have set $\delta X_{st}^{j,\varepsilon} = X_t^{j,\varepsilon} - X_s^{j,\varepsilon}$ for any $s,t \in [0,T]$. This allows to write:

$$\mathbf{X}^{\mathbf{2},\varepsilon}_{st}(i,j) = \int_s^t \delta X_{su}^{j,\varepsilon} \mathrm{d}X_u^{i,\varepsilon} = \varepsilon^{H-\frac{1}{2}} \int_s^t \delta X_{su}^{j,\varepsilon}\, \theta^{\varepsilon,i}(u)\mathrm{d}u - \alpha_H \int_s^t \mathrm{d}u \delta X_{su}^{j,\varepsilon} \int_0^u \mathrm{d}v (u+\varepsilon-v)^{H-\frac{3}{2}}\theta^{\varepsilon,i}(v). \tag{52}$$

Moreover, an elementary application of Fubini's theorem yields:

$$\begin{aligned}\int_s^t \mathrm{d}u\delta X_{su}^{j,\varepsilon} \int_0^u \mathrm{d}v(u+\varepsilon-v)^{H-\frac{3}{2}}\theta^{\varepsilon,i}(v) &= \int_0^t \mathrm{d}v\theta^{\varepsilon,i}(v)\int_{s\vee v}^t \mathrm{d}u\delta X_{su}^{j,\varepsilon}\,(u+\varepsilon-v)^{H-\frac{3}{2}} \\ &= \int_0^t \mathrm{d}v\theta^{\varepsilon,i}(v)\delta X_{sv}^{j,\varepsilon}\int_{s\vee v}^t \mathrm{d}r(r+\varepsilon-u)^{H-\frac{3}{2}} \\ &\quad + \int_0^t \mathrm{d}v\theta^{\varepsilon,i}(v)\int_{s\vee v}^t \mathrm{d}r\delta X_{vr}^{j,\varepsilon}(r+\varepsilon-v)^{H-\frac{3}{2}}.\end{aligned}$$

Integrating the kernel $(r+\varepsilon-u)^{H-\frac{3}{2}}$, and plugging the last identity into (52), we obtain the desired relation (50).

To get formula (51) for $\mathbf{B}^{\mathbf{2}}_{st}(i,j)$, it suffices to observe that

$$\mathbf{B}^{\mathbf{2}}_{st}(i,j) = L^2 - \lim_{\varepsilon\to 0}\int_s^t \big(B_u^j - B_s^j\big)\mathrm{d}B_u^{i,\varepsilon}$$

with $B_u^{i,\varepsilon} = \int_0^u (u+\varepsilon-v)^{H-\frac{1}{2}}\mathrm{d}W_v^i$, and then to mimic the computations allowing us to write (50) just above. Details are left to the reader (see also the proof of [3, Lemma 3]). □

Finally, the following lemma gives an estimate for the variance of $\mathbf{B}^{\mathbf{2}}_{st}(i,j)$ which is useful in the proof of Proposition 2.8:

Lemma 6.4. *There exists a constant $c > 0$, depending only on H, such that $E|\mathbf{B}^{\mathbf{2}}_{st}(i,j)|^2 \le c|t-s|^{4H}$ for all $t > s \ge 0$ and $i,j \in \{1,\ldots,d\}$.*

Proof. The case where $i = j$ is immediate by Lemma 2.7, so we only concentrate on the case where $i \neq j$. Using formula (51), we see that we have to bound the three following terms:

$$A_1 := \int_s^t E|B_u^j - B_s^j|^2 (t-u)^{2H-1}\mathrm{d}u$$

$$A_2 := \int_0^s E|B_u^j - B_s^j|^2 \left((t-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}\right)^2 \mathrm{d}u$$
$$A_3 := \int_0^t E\left|\int_{v\vee s}^t \mathrm{d}u\big(B_u^j - B_v^j\big)(u-v)^{H-\frac{3}{2}}\right|^2 \mathrm{d}v.$$

Throughout the proof, c will denote a generic constant (depending only on H, T) whose value can change from one line to another. Owing to the fact that $E|B_u^j - B_s^j|^2 \leq c|u-s|^{2H}$, see Lemma 2.7, we can write

$$A_1 \leq c\int_s^t (u-s)^{2H}(t-u)^{2H-1}\mathrm{d}u \leq c(t-s)^{2H}\int_s^t (t-u)^{2H-1}\mathrm{d}u = c(t-s)^{4H}.$$

We also get

$$\begin{aligned}
A_2 &\leq c\int_0^s (s-u)^{2H}\left((t-u)^{H-\frac{1}{2}} - (s-u)^{H-\frac{1}{2}}\right)^2 \mathrm{d}u \\
&= c\int_0^s u^{2H}\left((t-s+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 \mathrm{d}u \\
&= c(t-s)^{4H}\int_0^{\frac{s}{t-s}} u^{2H}\left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 \mathrm{d}u \\
&\leq c(t-s)^{4H}\int_0^\infty u^{2H}\left((1+u)^{H-\frac{1}{2}} - u^{H-\frac{1}{2}}\right)^2 \mathrm{d}u = c(t-s)^{4H},
\end{aligned}$$

the last integral being finite since $H < \frac{1}{2}$. Finally, we have

$$\begin{aligned}
&E\left|\int_{v\vee s}^t \mathrm{d}u\big(B_u^j - B_v^j\big)(u-v)^{H-\frac{3}{2}}\right|^2 \\
&= \int_{v\vee s}^t \mathrm{d}u\int_{v\vee s}^t \mathrm{d}w\, E\big[(B_u^j - B_v^j)(B_w^j - B_v^j)\big](u-v)^{H-\frac{3}{2}}(w-v)^{H-\frac{3}{2}} \\
&\leq c\int_{v\vee s}^t \mathrm{d}u\int_{v\vee s}^t \mathrm{d}w(u-v)^{2H-\frac{3}{2}}(w-v)^{2H-\frac{3}{2}} = c\left(\int_{v\vee s}^t (u-v)^{2H-\frac{3}{2}}\mathrm{d}u\right)^2;
\end{aligned}$$

so that

$$\begin{aligned}
A_3 &\leq c\int_0^s \big[(t-v)^{2H-\frac{1}{2}} - (s-v)^{2H-\frac{1}{2}}\big]^2\mathrm{d}v + c\int_s^t\left(\int_v^t (u-v)^{2H-\frac{3}{2}}\mathrm{d}u\right)^2\mathrm{d}v \\
&\leq c(t-s)^{4H}\int_0^\infty \big[(1+v)^{2H-\frac{1}{2}} - v^{2H-\frac{1}{2}}\big]^2\mathrm{d}v + c\int_s^t (t-v)^{4H-1}\mathrm{d}v \\
&= c(t-s)^{4H}.
\end{aligned}$$

This finishes the proof of the lemma. □

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