The stochastic heat equation on Heisenberg groups

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Outline

1. Gaussian noises
   - Noises indexed by time
   - Space-time noises
2. Parabolic Anderson model
   - Model and results
   - A basic identity
3. Heisenberg group
   - Basic geometric setting
   - Projective Fourier analysis
4. Stochastic heat equation on Heisenberg
   - A class of space-time noises
   - Existence-uniqueness for the heat equation

Outline

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Brownian motion

Definition 1.

Let

- $(\Omega, \mathcal{F}, \mathrm{P})$ probability space
- $\{B_t;\ t \geq 0\}$ stochastic process, $\mathbb{R}$-valued

We say that B is a Brownian motion if:

1. $B_0 = 0$ almost surely
2. B is Gaussian centered
3. For $0 \leq s < t$ we have

$$\mathbb{E}\left[(B_t - B_s)^2\right] = (t - s)$$

Brownian motion is non smooth ($B \in \mathcal{C}^{1/2-\varepsilon}$)

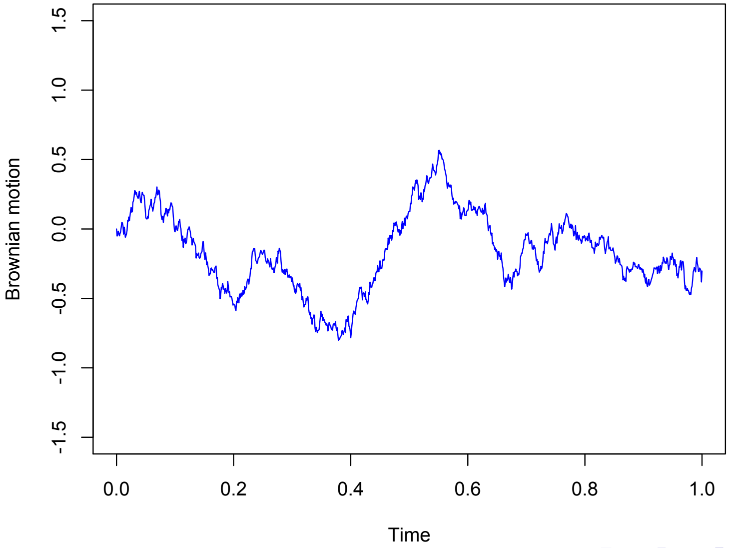
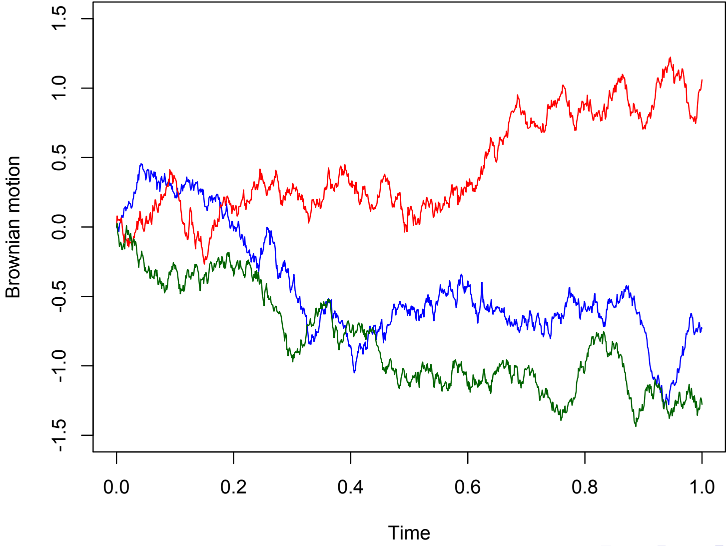


Illustration: Brownian motion is random



Fractional Brownian motion

Definition 2.

Let

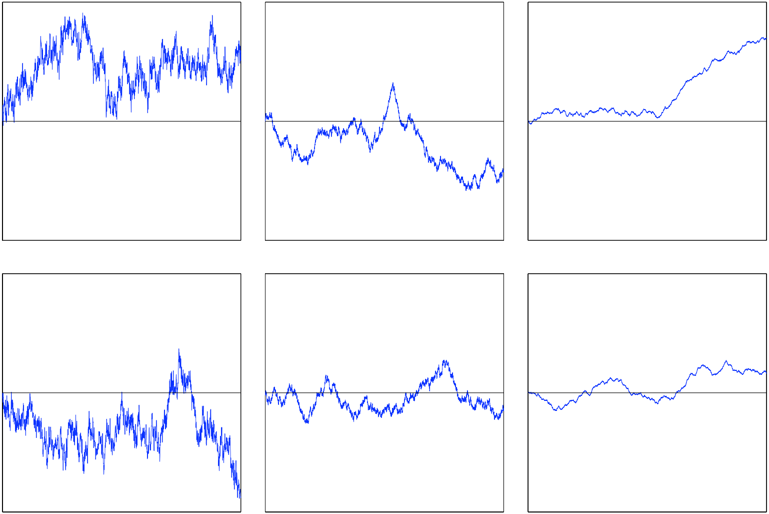
- $(\Omega, \mathcal{F}, \mathrm{P})$ probability space and $H \in (0,1)$
- $\{B_t^H;\, t \geq 0\}$ stochastic process, $\mathbb{R}$-valued

We say that B^H is a fractional Brownian motion if:

1. $B_0^H = 0$ almost surely
2. B^H is Gaussian centered
3. For $0 \leq s < t$ we have

$$\mathbb{E}\left[\left(B_t^H - B_s^H\right)^2\right] = (t-s)^{2H}$$

Examples of fBm paths ($B \in \mathcal{C}^{H-\varepsilon}$)



$H = 0.3$ $H = 0.5$ $H = 0.7$

Noises

White noise: We have

- $\dot{B}$ = distributional derivative of B
- Regularity: $\dot{B}$ element of Besov space $\mathcal{B}^{-1/2-\varepsilon}$
- Covariance:

$$\mathbb{E}\left[\dot{B}_t\,\dot{B}_s\right] = \delta(t-s)$$

Colored noise: Defined as

- $\dot{B}^H$ = distributional derivative of B^H
- Regularity: $\dot{B}^H$ element of Besov space $\mathcal{B}^{-(1-H+\varepsilon)}$
- Covariance: can also be a distribution

$$\mathbb{E}\left[\dot{B}^H_t\,\dot{B}^H_s\right] = |t-s|^{-(2-2H)}$$

Outline

Model for the noise in $\mathbb{R}^d$

Covariance function for $\dot{W}$: Gaussian noise on $\mathbb{R}_+ \times \mathbb{R}^d$, with

$$\mathbb{E}\left[\dot{W}_t(x)\,\dot{W}_s(y)\right] = |t-s|^{-\alpha_0}\,|y-x|^{-\alpha} \qquad (1)$$

Remark:

1. One can do more general than (1), with a Dalang type condition
2. Under (1), we have

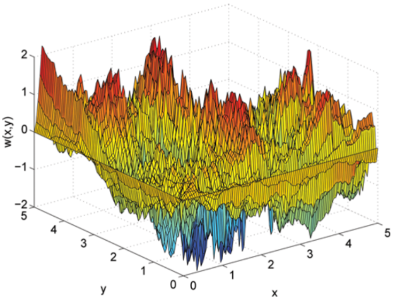
$$\dot{W}_t(\cdot) \in \mathcal{B}^{-(\alpha+\varepsilon)/2}$$

Another point of view on the noise

Noise as a derivative: One has the distributional derivative relation

$$\dot{W}_t(x) = \frac{\partial W}{\partial t \partial x}(t, x),$$

where $W \equiv$ fractional Brownian sheet



Covariance in Fourier modes

Notation for the covariance: On $\mathbb{R}_+ \times \mathbb{R}$, one can write

$$\mathbb{E}\left[\dot{W}_t(x)\,\dot{W}_s(y)\right] = \gamma_0(t-s)\,\gamma_1(y-x)$$

with the following distributional relation:

$$\gamma_j(u,v) = |u-v|^{2H_j-2}. \tag{2}$$

Covariance in Fourier modes:

- The covariance γ_j is given in Fourier mode as

$$\gamma_j(x) = \int_{\mathbb{R}} e^{\imath \xi x} |\xi_j|^{1-2H_j} d\xi$$

Outline

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Equation under consideration

Equation:

Stochastic heat equation on $\mathbb{R}^d$:

$$\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}_t(x), \tag{3}$$

with

- $t \geq 0$, $x \in \mathbb{R}^d$.
- $\dot{W}$ Gaussian noise such that
  - $\dot{W}$ white noise or fractional in time
  - $\dot{W}$ has a certain spatial covariance structure.
- $u_t(x)\,\dot{W}_t(x)$ differential: Stratonovich or Skorohod sense.

Motivation: intermittency phenomenon

Equation: $\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + \lambda\, u_t(x)\, \dot{W}_t(x)$

Phenomenon: The solution u concentrates its energy in high peaks.

Characterization: through moments
$\hookrightarrow$ Easy possible definition of intermittency: for all $k_1 > k_2 \geq 1$

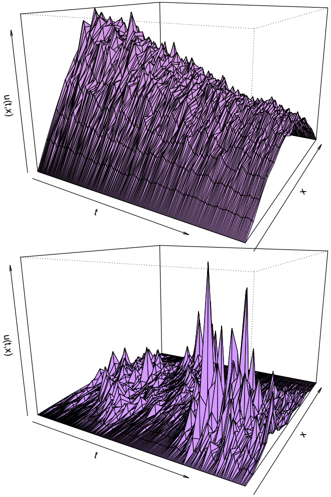
$$\lim_{t\to\infty} \frac{\mathbb{E}^{1/k_1}\left[|u_t(x)|^{k_1}\right]}{\mathbb{E}^{1/k_2}\left[|u_t(x)|^{k_2}\right]} = \infty\,.$$

Results:

- White noise in time: Khoshnevisan, Foondun, Conus, Joseph
- Fractional noise in time: Balan-Conus, Hu-Huang-Nualart-T

Intermittency: illustration (by Daniel Conus)

Simulations: for $\lambda = 0.1$, 0.5, 1 and 2.



Model for the noise on $\mathbb{R}_+ \times \mathbb{R}$ (repeated)

Covariance function for $\dot{W}$: As before,

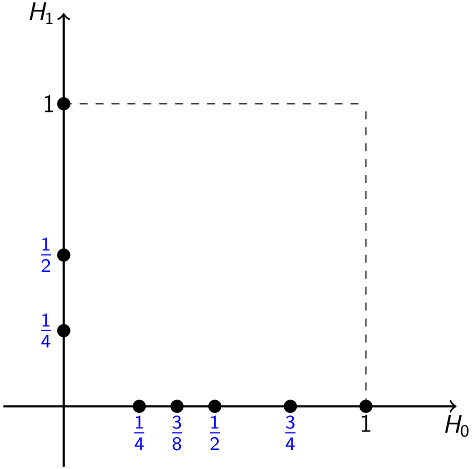
$$\mathbb{E}\left[\dot{W}_t(x)\, \dot{W}_s(y)\right] = \gamma_0(t-s)\,\gamma_1(y-x)$$

with the following distributional relation:

$$\gamma_j(u,v) = |u-v|^{2H_j-2}.$$

Existence and uniqueness results

Existence-uniqueness in the (H_0, H_1) plane: according to $2H_0 + H_1$



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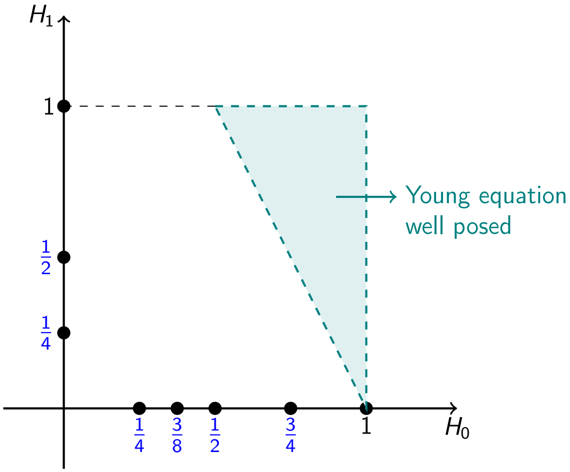

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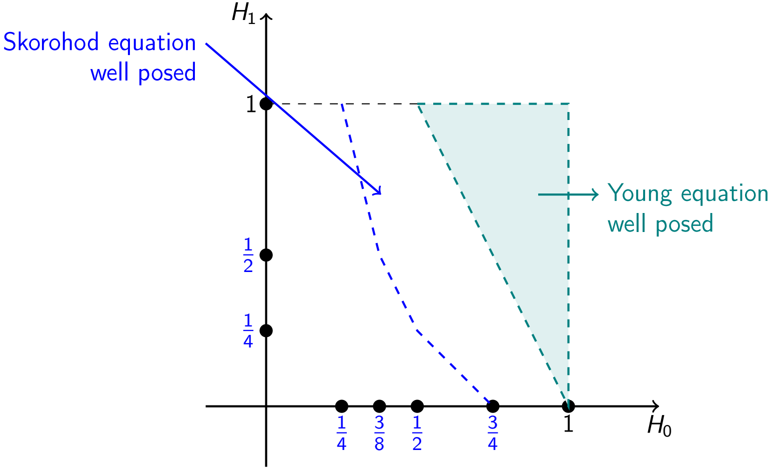

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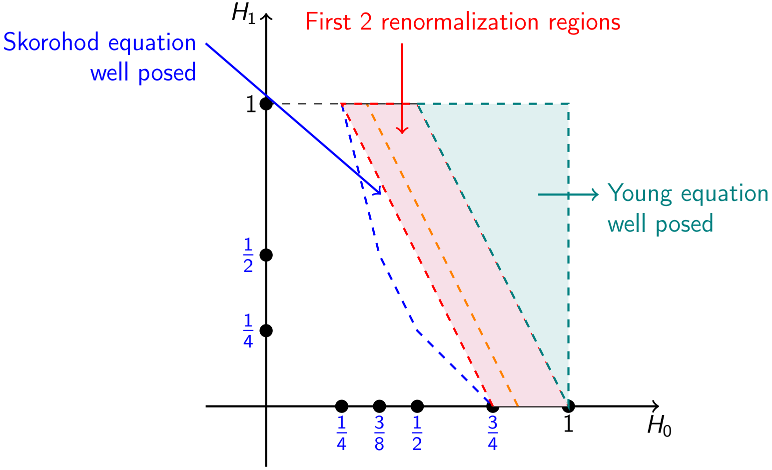

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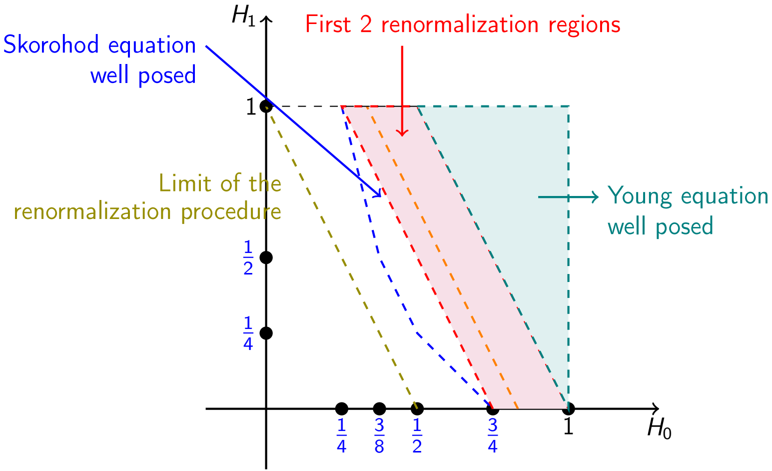

Outline

A stochastic integral

Heat kernel: Let $p_t \equiv$ heat kernel for $\frac{1}{2}\Delta$,

$$p_t(x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{x^2}{2t}\right)$$

Noise we consider: White in time, colored in space,

$$\mathbb{E}\left[\dot{W}_t(x)\,\dot{W}_s(y)\right] = \delta_0(t-s)\,|y-x|^{-\alpha}$$

A stochastic integral: Set

$$X_t(x) = \int_0^t \int_{\mathbb{R}^d} p_s(x-y)\, W(\mathrm{d}s, \mathrm{d}y)$$

Variance computation

Variance in direct mode: One can prove that

$$\mathbb{E}\left[|X_t(x)|^2\right] = \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p_s(x - y_1)|y_1 - y_2|^{-\alpha} p_s(x - y_2)\, \mathrm{d}s \mathrm{d}y_1 \mathrm{d}y_2$$

Variance in Fourier mode: We also have

$$\mathbb{E}\left[|X_t(x)|^2\right] = \int_0^t \int_{\mathbb{R}^d} |\mathcal{F} p_s(\xi)|^2 |\xi|^{d+\alpha}\, \mathrm{d}s \mathrm{d}\xi$$

$\hookrightarrow$ Much easier to handle!

Long term project

Main question:

- Do we observe a big difference in the previous exponents
  $\hookrightarrow$ under geometric settings?

Settings of interest:

- Sub-Riemannian manifolds
- On Heisenberg groups: use of Fourier
- Fractals

Related models:

- Polymers
- KPZ equation

Outline

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Group structure

Symplectic form on $\mathbb{R}^{2n}$:

$$\omega((x,y),(x',y')) = \sum_{i=1}^{n} x_i' y_i - x_i y_i'$$

Heisenberg group $\mathbf{H}^n$: Seen as $\mathbb{R}^{2n+1}$ equipped with

$$(x,y,z) \star (x',y',z') := \Big(x + x', y + y', z + z' + 2\omega\left((x,y),(x',y')\right) \Big)$$

Sub-Riemannian structure

Invariant vector fields:
At $p = (x, y, z)$ and for $i = 1, \dots, n$, given as

$$X_i(p) = \partial_{x_i} + 2y_i\partial_z, \quad Y_i(p) = \partial_{y_i} - 2x_i\partial_z, \quad Z(p) = \partial_z.$$

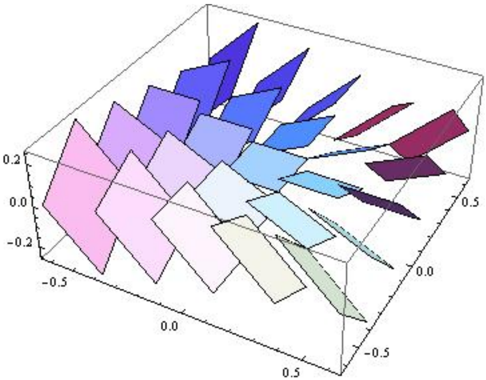
Then X_i, Y_i are the horizontal vector fields.

Horizontal sub-Laplacian: Defined by

$$\Delta = \sum_{i=1}^{n} X_i^2 + Y_i^2.$$

Horizontal tangent planes in $\mathbf{H}^n$

$$X_i(p) = \partial_{x_i} + 2y_i\partial_z, \qquad Y_i(p) = \partial_{y_i} - 2x_i\partial_z.$$



Distance

Carnot-Carathéodory distance:

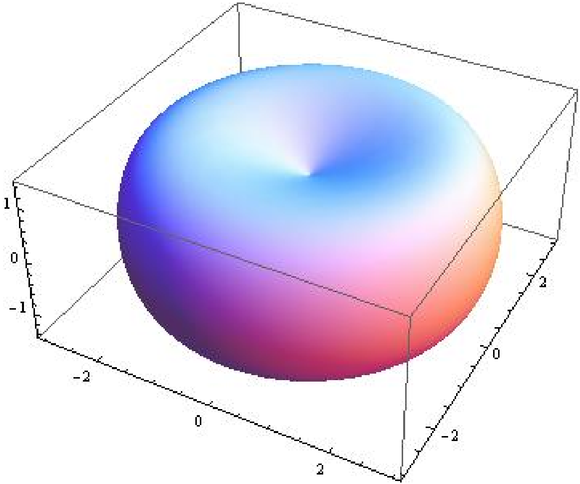
$$d_{cc}(p_1, p_2) := \inf \left\{ \int_0^1 |\dot{\gamma}(t)|_{\mathcal{H}} dt \, ; \right.$$

$$\left. \gamma : [0,1] \to \mathbf{H}^n \text{ is horizontal, } \gamma(0) = p_1, \gamma(1) = p_2 \right\}$$

Bounds on cc-distance:

$$C_1(\sqrt{|(x,y)|^2} + |z|^{\frac{1}{2}}) \leq d_{cc}(e,q) \leq C_2(\sqrt{|(x,y)|^2} + |z|^{\frac{1}{2}}))$$

Unit sphere in $\mathbf{H}^n$



Heat kernel

Definition: Solution of

$$\partial_t p_t(q) = \frac{1}{2}\Delta p_t(q), \qquad p_0(q) = \delta_e(q)$$

Gaussian type bounds: We have

$$\frac{c_1}{t^{n+1}} \exp\left(-\frac{c_2}{t}\, d_{cc}(e,q)^2\right) \le p_t(q) \le \frac{c_3}{t^{n+1}} \exp\left(-\frac{c_4}{t}\, d_{cc}(e,q)^2\right)$$

Remark: The heat kernel is more singular
$\hookrightarrow$ Than the heat kernel in $\mathbb{R}^{2n+1}$

Outline

Usual Fourier transform

A unitary group representation:

For $q = (x, y, z) \in \mathbf{H}^n$, $\lambda \in \mathbb{R}$ and $u \in L^2(\mathbb{R}^n)$ we set

$$U_q^\lambda u(\xi) = e^{-\imath\lambda(z+2x\cdot(\xi-y))} u\left(\xi - 2y\right), \quad \xi \in \mathbb{R}^n$$

Fourier transform: For $f \in L^1(\mathbf{H}^n)$, operator valued,

$$\mathcal{F}(f)(\lambda) = \int_{\mathbf{H}^n} f(q) U_q^\lambda \, d\mu(q).$$

Relation with Laplacian: We have

$$\mathcal{F}\left(\Delta f\right)(\lambda) = 4\,\mathcal{F}(f)(\lambda) \circ \Delta_{\text{osc}}^\lambda$$

with

$$\Delta_{\text{osc}}^\lambda u(x) = \sum_{j=1}^n \partial_j^2 u(x) - \lambda^2 |x|^2 u(x)$$

Projective version of Fourier

Hermite functions: Onb for $-\Delta^1_{\text{osc}}$, defined for $k \in \mathbb{N}^n$ by

$$-\Delta^1_{\text{osc}} \Phi_k = (2|k| + n)\ \Phi_k$$

Rescaled Hermite functions:

$$\Phi^\lambda_k(x) = |\lambda|^{n/4}\, \Phi_k\left(\sqrt{|\lambda|}x\right)$$

Projective Fourier: For $(m, \ell, \lambda) \in \mathbb{N}^n \times \mathbb{N}^n \times \mathbb{R}^*$, we set

$$\hat{f}(m, \ell, \lambda) = \langle \mathcal{F}(f)(\lambda)\Phi^\lambda_m, \Phi^\lambda_\ell \rangle_{L^2(\mathbb{R}^n)}$$

Properties of the projective version

Plancherel identity: We have

$$\int_{\mathbf{H}^n} |f(q)|^2 dq = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m,\ell\in\mathbb{N}^n} \int_{-\infty}^{+\infty} |\hat{f}(m,\ell,\lambda)|^2 |\lambda|^n d\lambda$$

Fourier and Laplace: For a smooth enough f,

$$\widehat{\Delta f}(m,\ell,\lambda) = -4\,|\lambda|(2|m|+n)\,\widehat{f}(m,\ell,\lambda)$$

Powers of Laplace: For $\alpha > 0$,

$$\widehat{(-\Delta)^{-\alpha} f}(m,\ell,\lambda) = 4^{-\alpha}|\lambda|^{-\alpha}(2|m|+n)^{-\alpha}\widehat{f}(m,\ell,\lambda)$$

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A class of Gaussian noises

Test function: For $\alpha \geq 0$ and φ smooth, set

$$\mathsf{W}_\alpha(\varphi) = \int_{\mathbb{R}_+} \int_{\mathbf{H}^n} \varphi(t, q) \, \mathsf{W}_\alpha(\mathrm{d}t, \mathrm{d}q)$$

Covariance: For 2 test functions φ, ψ,

$$\mathbb{E}\left[\mathsf{W}_\alpha(\varphi)\mathsf{W}_\alpha(\psi)\right] = \left\langle (-\Delta)^{-\alpha}\varphi, \, (-\Delta)^{-\alpha}\psi \right\rangle_{L^2(\mathbb{R}_+ \times \mathbf{H}^n)}$$

Properties of the Gaussian noise

Relation with white noise: One can write

$$\mathbf{W}_\alpha = (-\Delta)^{-\alpha}\mathbf{W}, \quad \text{with} \quad \mathbf{W} \equiv \text{space-time white noise on } \mathbf{H}^n$$

Inequality for the covariance: For positive test functions φ, ψ,

$$\mathbb{E}\left[\mathbf{W}_\alpha(\varphi)\mathbf{W}_\alpha(\psi)\right] \asymp \int_{\mathbb{R}_+}\int_{(\mathbf{H}^n)^2} \frac{\varphi(t,q_1)\psi(t,q_2)}{d_{cc}(q_1,q_2)^{2n+2-4\alpha}}\, dt\, d\mu(q_1)d\mu(q_2)$$

Outline

Existence-uniqueness

Theorem 3.

Consider

- Noise $\mathbf{W}_\alpha$ as in previous slide
- Stochastic heat equation on $\mathbf{H}^n$:

$$\partial_t u_t(q) = \frac{1}{2}\Delta u_t(q) + u_t(x)\,\dot{W}_t(q),$$

interpreted in the Itô sense.

Then a necessary and sufficient condition
$\hookrightarrow$ to have existence and uniqueness is

$$\alpha > \frac{n}{2}$$

Comparison with $\mathbb{R}^d$

Bessel noises: Noises on $\mathbb{R}^d$ with covariance

$$\mathbb{E}\left[\mathsf{W}_\alpha(\varphi)\mathsf{W}_\alpha(\psi)\right] = \left\langle (\mathrm{Id}-\Delta)^{-\alpha}\varphi,\, (\mathrm{Id}-\Delta)^{-\alpha}\psi \right\rangle_{L^2(\mathbb{R}_+\times\mathbb{R}^d)}$$

Condition in $\mathbb{R}^d$: In order to solve SHE,

$$\alpha > \frac{d}{4} - \frac{1}{2}$$

Condition in $\mathbf{H}^n$: The condition $\alpha > n/2$ can be read as

$$\alpha > \frac{Q}{4} - \frac{1}{2}, \quad \text{with} \quad Q = 2n + 2 = \text{Effective dimension}$$

The basic identity in $\mathbf{H}^n$

Stochastic convolution: Consider the process

$$X_t \equiv \int_0^t \int_{\mathbf{H}^n} p_{t-s}(q) \, \mathbf{W}_\alpha(\mathrm{d}s, \mathrm{d}q)$$

Variance in Fourier mode:

$$\mathbb{E}\left[X_t^2\right] = \int_{[0,t]} ds \int_{\mathbb{R}} |\lambda|^n \sum_{m \in \mathbb{N}^n} |\lambda|^{-2\alpha} (2|m| + n)^{-2\alpha} e^{-8s|\lambda|(2|m|+n)} \, \mathrm{d}\lambda$$

Lower bound: We have

$$\mathbb{E}\left[X_t^2\right] \gtrsim \int_{[0,t]} ds \int_{\mathbb{R}} |\lambda|^{n-2\alpha} e^{-8ns|\lambda|} \, \mathrm{d}\lambda$$

This is finite for $\alpha \in \left(\frac{n}{2}, \frac{n+1}{2}\right)$