Hyperbolic Anderson model in the Skorohod and rough setting

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New Interfaces of Stochastic Analysis and Rough Paths

Joint works with Xia Chen, Aurélien Deya and Jian Song

Outline

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Equation under consideration

Equation:
Stochastic heat equation on $\mathbb{R}^d$:

$$\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}_t(x), \tag{1}$$

with

- $t \geq 0$, $x \in \mathbb{R}^d$.
- $\dot{W}$ Gaussian noise such that
  - $\dot{W}$ white noise or fractional in time
  - $\dot{W}$ has a certain spatial covariance structure.
- $u_t(x)\,\dot{W}_t(x)$ differential: Stratonovich or Skorohod sense.

Motivation: intermittency phenomenon

Equation: $\partial_t u_t(x) = \frac{1}{2}\Delta u_t(x) + \lambda\, u_t(x)\, \dot{W}_t(x)$

Phenomenon: The solution u concentrates its energy in high peaks.

Characterization: through moments

$\hookrightarrow$ Easy possible definition of intermittency: for all $k_1 > k_2 \geq 1$

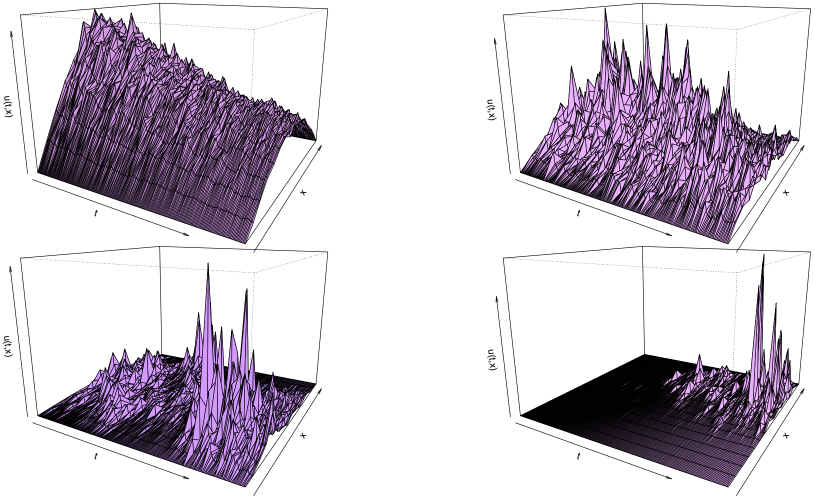
$$\lim_{t\to\infty} \frac{\mathrm{E}^{1/k_1}\left[|u_t(x)|^{k_1}\right]}{\mathrm{E}^{1/k_2}\left[|u_t(x)|^{k_2}\right]} = \infty\,.$$

Results:

- White noise in time: Khoshnevisan, Foondun, Conus, Joseph
- Fractional noise in time: Balan-Conus, Hu-Huang-Nualart-T

Intermittency: illustration (by Daniel Conus)

Simulations: for $\lambda = 0.1$, 0.5, 1 and 2.



Possible model for the noise (1)

Covariance function for $\dot{W}$: Gaussian noise on $\mathbb{R}_+ \times \mathbb{R}$, with

$$\mathrm{E}\left[\dot{W}_t(x)\,\dot{W}_s(y)\right] = \gamma_0(t-s)\,\gamma_1(y-x)$$

with the following distributional relation:

$$\gamma_j(u,v) = |u-v|^{2H_j-2}. \tag{2}$$

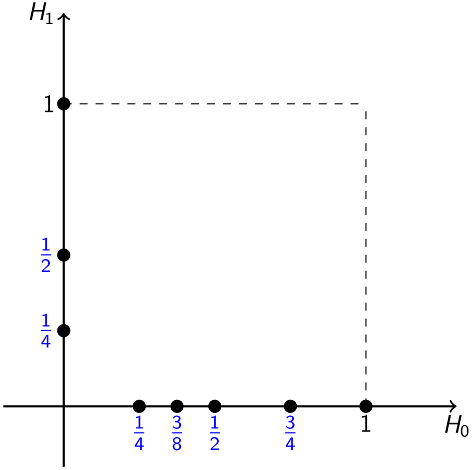
Remark:

- The covariance γ_j is given in Fourier mode as

$$\gamma_j(x) = \int_{\mathbb{R}} e^{\imath \xi x} |\xi_j|^{1-2H_j} d\xi$$

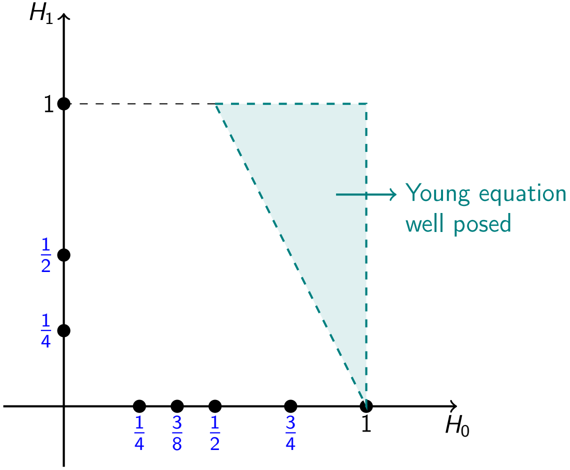
Subcritical zone: illustration

Existence-uniqueness in the (H_0, H_1) plane: according to $2H_0 + H_1$



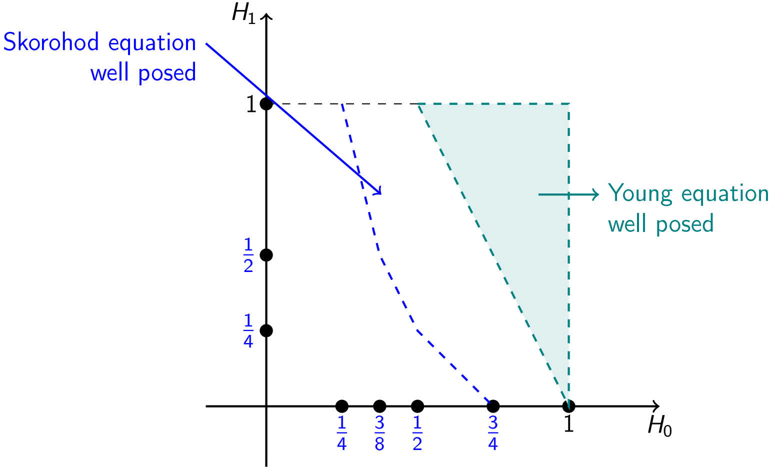
Subcritical zone: illustration

Existence-uniqueness in the (H_0, H_1) plane: according to $2H_0 + H_1$



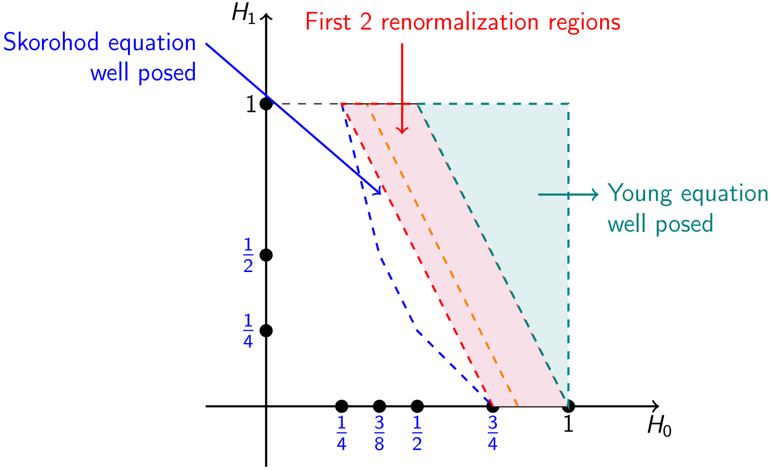
Subcritical zone: illustration

Existence-uniqueness in the (H_0, H_1) plane: according to $2H_0 + H_1$



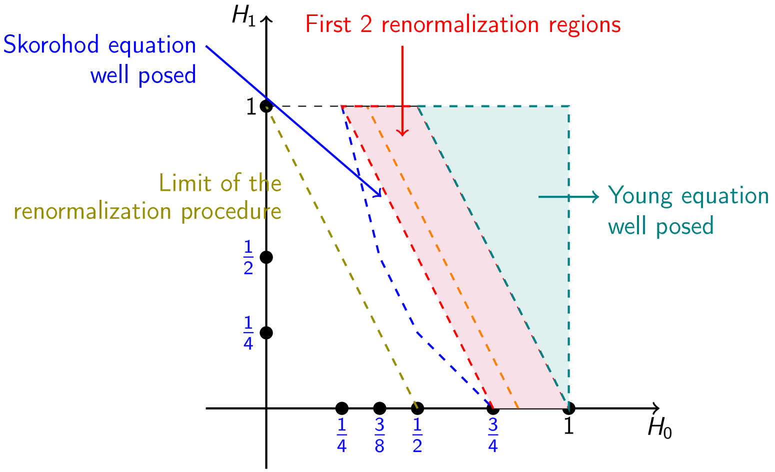
Subcritical zone: illustration

Existence-uniqueness in the (H_0, H_1) plane: according to $2H_0 + H_1$



Subcritical zone: illustration

Existence-uniqueness in the (H_0, H_1) plane: according to $2H_0 + H_1$



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Equation under consideration

Equation:
Stochastic wave equation on $\mathbb{R}^d$:

$$\partial^2_{tt} u_t(x) = \frac{1}{2} \Delta u_t(x) + u_t(x)\, \dot{W}_t(x), \tag{3}$$

with

- $t \geq 0$, $x \in \mathbb{R}^d$.
- $\dot{W}$ Gaussian noise such that
  - $\dot{W}$ has a certain space-time covariance structure.
- $u_t(x)\, \dot{W}_t(x)$ differential: Stratonovich or Skorohod sense.

Description of the noise

Covariance function for $\dot{W}$: Gaussian noise on $\mathbb{R}_+ \times \mathbb{R}^d$, with

$$\mathrm{E}\left[\dot{W}_t(x)\,\dot{W}_s(y)\right] = |t-s|^{-\alpha_0}\,\gamma(y-x)$$

with the following distributional relation:

$$\gamma(c\,x) = c^{-\alpha}\gamma(x). \tag{4}$$

Remark:

1. One can do more general than (4), with a Dalang type condition
2. Under (4), we have

$$\dot{W}_t(\cdot) \in \mathcal{B}^{-(\alpha+\varepsilon)/2}$$

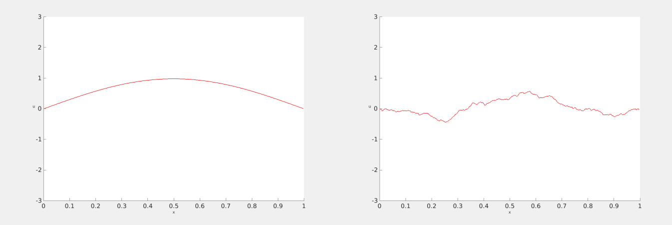
Simulation in the additive case (by David Cohen)

Equation: Stochastic wave equation on $[0,1]$:

$$\partial_{tt}^2 u_t(x) = \frac{1}{2}\Delta u_t(x) + \dot{W}_t(x), \tag{5}$$

with

- $t \geq 0$, $x \in [0,1]$.
- $\dot{W}$ space-time white noise



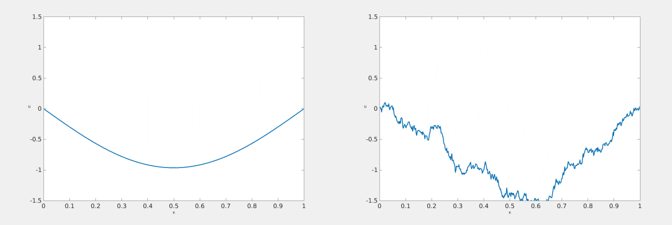
Multiplicative case (by David Cohen)

Equation: Stochastic wave equation on $[0,1]$:

$$\partial_{tt}^2 u_t(x) = \frac{1}{2}\Delta u_t(x) + u\,\dot{W}_t(x), \qquad (6)$$

with

- $t \geq 0$, $x \in [0,1]$.
- $\dot{W}$ space-time white noise



Mild formulation

Notation: We set

- $G_t(x) \equiv$ fundamental solution of the wave equation

Duhamel's principle: The solution to

$$\partial_{tt}^2 u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\,\dot{W}_t(x), \quad u(0,x) = \partial_t u(0,x) = 0$$

can be written as

$$u_t(x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\, u_s(y)\, W(\mathrm{d}s, \mathrm{d}y)$$

Fundamental solution (1)

Notation: Set

$$\rho_t = \text{ Uniform measure on sphere with radius } t.$$

Expression for the fundamental solution: We have

$$G_t(x) = \begin{cases} \frac{1}{2} 1_{[|x|<t]} & \text{if } d = 1, \\ \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{[|x|<t]} & \text{if } d = 2, , \\ \frac{1}{4\pi t} \rho_t(dx) & \text{if } d = 3, \\ \text{Derivatives of } \rho_t & \text{if } d \geq 4. \end{cases}$$

Conclusion: Ugly expressions as d gets large!

Fundamental solution (2)

Fundamental solution in Fourier modes: We have

$$\mathcal{F}G_t(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

Conclusion:
Some computations will be easier in Fourier modes!

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Existence-uniqueness, Skorohod case

Theorem 1.

We consider the Skorohod equation in $\mathbb{R}^d$ with $d \leq 3$:

$$\partial^2_{tt} u_t(x) = \frac{1}{2} \Delta u_t(x) + u_t(x) \diamond \dot{W}_t(x) \qquad (7)$$

The noise covariance is (with scaling $\gamma(c\,x) = c^{-\alpha}\gamma(x)$)

$$\mathrm{E}\left[\dot{W}_t(x)\, \dot{W}_s(y)\right] = |t-s|^{-\alpha_0}\, \gamma(y-x).$$

Then a necessary and sufficient condition
$\hookrightarrow$ to get existence-uniqueness for (7) is

$$\alpha_0 + \alpha < 3$$

Bibliography

Comparison with other contributions:

- Dalang '99:
  $\hookrightarrow$ White noise in time, Itô setting, $\alpha < 2$
- Balan '12:
  $\hookrightarrow$ Colored space-time, $\alpha < 2$
- Balan-Chen-Chen '22:
  $\hookrightarrow$ Spatial noise, $\alpha < 3$

Note:
We are improving on Balan '12 and Balan-Chen-Chen '22

Outline

Chaos expansion

Malliavin calculus notation: We denote

- $\mathcal{H} \equiv$ Cameron-Martin space related to $\dot{W}$
- $I_n \equiv$ multiple integrals with respect to $\dot{W}$

Chaos expansion: One can write

- $u(t,x) = \sum_{n=0}^{\infty} I_n(g_n(\cdot,t,x))$
- $g_n \equiv$ product of wave kernels

Reduction of the problem: We have to estimate

$$\|g_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}}^2$$

Reversed L^2 estimates

Initial expression: We have

$$\|g_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} = \int_{([0,t]^n_<)^2}\int_{(\mathbb{R}^d)^2}\left(\prod_{k=1}^n |s_k - s'_k|^{-\alpha_0}\gamma(x_k - x'_k)\right)$$

$$\times\left(\prod_{k=1}^n G_{s_k - s_{k-1}}(x_k - x_{k-1})G_{s'_k - s'_{k-1}}(x'_k - x'_{k-1})\right)dxdx'dsds'.$$

Bound for the Laplace transform: We get a less intricate expression,

$$\int_0^\infty e^{-2pt}\|g_n(t,x,\cdot)\|^2_{\mathcal{H}^{\otimes n}}dt \le \frac{p}{2}\int_{(\mathbb{R}^{d+1})^{2n}} H_p(s_1,x_1\ldots,s_n,x_n)$$

$$\times H_p(s'_1,x'_1,\ldots,s'_n,x'_n)\left(\prod_{k=1}^n |s_k - s'_k|^{-\alpha_0}\gamma(x_k - x'_k)\right)dxdx'dsds'.$$

Remainder of the strategy

In a few words:

1. Going back and forth in Fourier and direct modes
   $\hookrightarrow$ reduction to products of 1-d integrals
2. Depoissonization:
   - Take large values of p in the Laplace transform
   - Relate to one value of $t \mapsto \|g_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$

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An additive case (1)

First equation under consideration:

$$\partial_{tt}^2 u_t(x) = \frac{1}{2}\Delta u_t(x) - u_t^2(x) + \dot{W}_t(x), \tag{8}$$

Approach: Solution as perturbation of the stochastic convolution

$$\Psi_t(x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\, W(\mathrm{d}s, \mathrm{d}y)$$

Equation for $v \equiv u - \Psi$:

$$\partial_{tt}^2 v_t(x) = \frac{1}{2}\Delta v_t(x) - (v_t(x) + \Psi_t(x))^2$$

An additive case (2)

Problem: When W rough or dimension high
$\hookrightarrow$ Ψ is a distribution and Ψ^2 ill-defined

Renormalized equation: One considers

- Smooth approximation of the noise W^n
- Family $\{u^n;\ n \geq 1\}$
- $\sigma_n \sim 2^{n\gamma} t$

such that

$$\partial_{tt}^2 u_t^n(x) = \frac{1}{2}\Delta u_t^n(x) - \left[(u_t^n(x))^2 - \sigma_n(t)\right] + \dot{W}_t^n(x),$$

Then (Gubinelli-Koch-Oh, Deya) u^n converges
$\hookrightarrow$ to renormalized version of (9)

Outline

Stratonovich multiplicative setting

Equation under consideration:

$$\partial_{tt}^2 u_t(x) = \frac{1}{2}\Delta u_t(x) + u_t(x)\dot{W}_t(x), \tag{9}$$

Approach (Chen-Deya-Song-T):

- Mild form of the equation in pathwise sense:

$$u_t(x) = \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\, u_s(y) W(\mathrm{d}s, \mathrm{d}y)$$

- Smoothing effect of wave kernel G
- Young type integration

Existence-uniqueness, Stratonovich case

Theorem 2.

We consider the Stratonovich equation in $\mathbb{R}^d$ with $d \leq 3$:

$$\partial^2_{tt} u_t(x) = \frac{1}{2} \Delta u_t(x) + u_t(x)\, \dot{W}_t(x) \qquad (10)$$

The noise covariance is (with scaling $\gamma(c\,x) = c^{-\alpha}\gamma(x)$)

$$\mathrm{E}\left[\dot{W}_t(x)\, \dot{W}_s(y)\right] = |t-s|^{-\alpha_0}\, \gamma(y-x).$$

Then existence-uniqueness for (10) under condition

$$\alpha_0 + \alpha < \begin{cases} 1, & \text{if } d = 1, \\ \frac{1}{2}, & \text{if } d = 2. \end{cases}$$

Outline

Besov spaces and Strichartz estimates

Weighted Besov space: We set

- $\mathcal{B}^\alpha \equiv$ weighted Besov space with exponential weight on $\mathbb{R}^d$
- Parameters μ, p, q in $\mathcal{B}^{\alpha,\mu}_{p,q}$ not specified for simplicity

Strichartz type estimates: For all $t \in [0,1]$, it holds that

$$\left\|\mathcal{G}_t f\right\|_{\mathcal{B}^{\alpha+\rho_d}} \lesssim \|f\|_{\mathcal{B}^\alpha}, \quad \text{with } \rho_d \equiv \begin{cases} 1 & \text{if } d = 1 \\ \frac{1}{2} & \text{if } d = 2 \end{cases}$$

Remarks:

- Those Strichartz type estimates appear to be new
- They rely on Ryzhkov's version of weighted Besov spaces
- Possibility of regularity structure type expansions

Strategy for Strichartz estimates

Ingredients:

1. Consider rescaled wavelet type functions φ_ℓ
2. Ryzkhov's trick: subtle decomposition for $\varphi_\ell * \mathcal{G}_t f$
3. We are then reduce to obtain ($B_2 \equiv$ ball, radius 2)

$$\int_{\mathbb{R}^d} dy \int_{B_2} dz \left| G_t(y - 2^{-j}z) - G_t(y) \right| \lesssim 2^{-j\rho_d}$$

4. $\rho_d = \frac{1}{2}$ for $d = 2$, due to $t^{-1/2}$ singularity of G_t