Rough paths methods 2: Young integration

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Outline

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Definition of fBm

Complete probability space: $(\Omega, \mathcal{F}, \mathbf{P})$

Definition 1.

A 1-d fBm is a continuous process $B = \{B_t;\ t \geq 0\}$ such that:

- $B_0 = 0$
- B is a centered Gaussian process
- $\mathbf{E}[B_t B_s] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t-s|^{2H})$, for $H \in (0,1)$

d-dimensional fBm: $B = (B^1, \ldots, B^d)$, with B^i independent 1-d fBm

fBm: variance of the increments

Notation: If $f : [0, T] \to \mathbb{R}^d$ is a function, we shall denote:

$$\delta f_{st} = f_t - f_s, \quad \text{and} \quad \|f\|_\mu = \sup_{s,t\in[0,T]} \frac{|\delta f_{st}|}{|t-s|^\mu}$$

Variance of the increments: for a 1-d fBm,

$$\mathbf{E}[|\delta B_{st}|^2] \equiv \mathbf{E}[|B_t - B_s|^2] = |t-s|^{2H}$$

FBm regularity

Proposition 2.

FBm $B \equiv B^H$ is γ-Hölder continuous on $[0, T]$ for all $\gamma < H$, up to modification.

Proof: We have $\delta B_{st} \sim \mathcal{N}(0, |t-s|^{2H})$. Thus for $n \geq 1$,

$$\mathbf{E}\left[|\delta B_{st}|^{2n}\right] = c_n |t-s|^{2Hn} \quad \text{i.e} \quad \mathbf{E}\left[|\delta B_{st}|^{2n}\right] = c_n |t-s|^{1+(2Hn-1)}$$

Kolmogorov: B is γ-Hölder for $\gamma < (2Hn-1)/2n = H - 1/(2n)$.
Proof finished by letting $n \to \infty$.

Some properties of fBm

Proposition 3.

Let B be a fBm with parameter H. Then:

1. $\{ a^{-H} B_{at};\, t \geq 0 \}$ is a fBm (scaling)
2. $\{B_{t+h} - B_h;\, t \geq 0 \}$ is a fBm (stationarity of increments)
3. B is not a semi-martingale unless $H = 1/2$

Proof of claim 3

Semi-martingale and quadratic variation:

If B were a semi-martingale, we would get on $[0, 1]$:

$$\mathbf{P} - \lim_{n\to\infty} \sum_{i=1}^{n} (B_{i/n} - B_{(i-1)/n})^2 = \langle B \rangle_1 \,,$$

were $\langle B \rangle$ is the (non trivial) quadratic variation of B.

We will show that $\langle B \rangle$ is trivial (0 or ∞) whenever $H \neq 1/2$.

Proof of claim 3 (2)

A p-variation: Define

$$V_{n,p} = \sum_{i=1}^{n} |B_{i/n} - B_{(i-1)/n}|^p, \quad \text{and} \quad Y_{n,p} = n^{pH-1} V_{n,p}.$$

By scaling properties, we have:

$$Y_{n,p} \overset{(d)}{=} \hat{Y}_{n,p}, \quad \text{with} \quad \hat{Y}_{n,p} = n^{-1} \sum_{i=1}^{n} |B_i - B_{i-1}|^p.$$

The sequence $\{B_i - B_{i-1};\ i \geq 1\}$ is stationary and mixing
$\Rightarrow \hat{Y}_{n,p}$ converges $\mathbf{P} - a.s$ and in L^1 towards $\mathbf{E}[|B_1 - B_0|^p]$
$\Rightarrow \mathbf{P} - lim_{n\to\infty} Y_{n,p} = E[|B_1|^p]$
$\Rightarrow \mathbf{P} - lim_{n\to\infty} V_{n,p} = 0$ if $pH > 1$, ∞ if $pH < 1$

Proof of claim 3 (3)

Recall: $V_{n,p} = \sum_{i=1}^{n} |B_{i/n} - B_{(i-1)/n}|^p$

Definition: $\mathbf{P} - lim_{n\to\infty} V_{n,p}^{1/p} \equiv \mathcal{V}_p(B)$ is called p-variation of B
$\Rightarrow$ We have seen $\mathcal{V}_p(B) = 0$ if $pH > 1$, ∞ if $pH < 1$

Property: if $p_1 < p_2$, then $\mathcal{V}_{p_1}(B) \geq \mathcal{V}_{p_2}(B)$

Case $H > 1/2$: choose $p < 2$ such that $pH > 1$
$\Rightarrow \mathcal{V}_p(B) = 0 \Rightarrow \mathcal{V}_2(B) = 0$

Case $H < 1/2$: choose $p > 2$ such that $pH < 1$
$\Rightarrow \mathcal{V}_p(B) = \infty \Rightarrow \mathcal{V}_2(B) = \infty$

Conclusion: if $H \neq 1/2$, Itō's type methods do not apply in order to define stochastic integrals

Outline

Strategy for $H > 1/2$

- Generally speaking, take advantage of two aspects of fBm:
  - Gaussianity
  - Regularity

  For $H > 1/2$, regularity is almost sufficient
- Notation: $\mathcal{C}_1^\gamma = \mathcal{C}_1^\gamma(\mathbb{R}) \equiv \gamma$-Hölder functions of 1 variable
- If $H > 1/2$, $B \in \mathcal{C}_1^\gamma$ for any $1/2 < \gamma < H$ a.s
- We shall try to solve our equation in a pathwise manner

Equation under consideration

$$X_t = a + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, \quad t \in [0, T] \qquad (1)$$

- $a \in \mathbb{R}^n$ initial condition
- b, σ coefficients in C_b^1
- $B = (B^1, \ldots, B^d)$ d-dimensional Brownian motion
- B^i iid Brownian motions

Notational simplification

Simplified setting: In order to ease notations, we shall consider:

- Real-valued solution and fBm: $n = d = 1$.
  However, we shall use d-dimensional methods
- $b \equiv 0$

Simplified equation: we end up with

$$X_t = a + \int_0^t \sigma(X_s) dB_s, \quad t \in [0, T] \tag{2}$$

- $a \in \mathbb{R}$, $\sigma \in C_b^1(\mathbb{R})$
- B is a 1-d Brownian motion

Pathwise strategy

Aim: Let x be a function in $\mathcal{C}_1^\gamma$ with $\gamma > 1/2$. We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s)\, dx_s \tag{3}$$

Steps:

- Define an integral $\int z_s\, dx_s$ for $z \in \mathcal{C}_1^\kappa$, with $\kappa + \gamma > 1$
- Solve (3) through fixed point argument in $\mathcal{C}_1^\kappa$ with $1/2 < \kappa < \gamma$

Notation: We set

$$\mathcal{J}_{st}(z\, dx) = \text{"} \int_s^t z_w dx_w \text{"}$$

for reasonable extensions of Riemann's integral

Particular Riemann sums

Aim: Define $\int_0^1 z_s\, dx_s$ for $z \in \mathcal{C}_1^\kappa, x \in \mathcal{C}_1^\gamma$, with $\kappa + \gamma > 1$

Dyadic partition: set $t_i^n = i/2^n$, for $n \geq 0$, $0 \leq i \leq 2^n$

Associated Riemann sum:

$$I_n \equiv \sum_{i=0}^{2^n-1} z_{t_i^n}[x_{t_{i+1}^n} - x_{t_i^n}] = \sum_{i=0}^{2^n-1} z_{t_i^n}\, \delta x_{t_i^n t_{i+1}^n}.$$

Question: Can we define $\mathcal{J}_{01}(z\, dx) \equiv \lim_{n\to\infty} I_n$?

Possibility: Control $|I_{n+1} - I_n|$ and write (if the series is convergent):

$$\mathcal{J}_{01}(z\, dx) = I_0 + \sum_{n=0}^{\infty} (I_{n+1} - I_n).$$

Control of $I_{n+1} - I_n$

We have:

$$
\begin{aligned}
I_n &= \sum_{i=0}^{2^n-1} z_{t_i^n}\, \delta x_{t_i^n t_{i+1}^n} = \sum_{i=0}^{2^n-1} z_{t_{2i}^{n+1}} \left[\delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}}\right] \\
I_{n+1} &= \sum_{i=0}^{2^n-1} \left[z_{t_{2i}^{n+1}}\, \delta x_{t_{2i}^{n+1} t_{2i+1}^{n+1}} + z_{t_{2i+1}^{n+1}}\, \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}}\right]
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
|I_{n+1} - I_n| &= \left|\sum_{i=0}^{2^n-1} \delta z_{t_{2i}^{n+1} t_{2i+1}^{n+1}}\, \delta x_{t_{2i+1}^{n+1} t_{2i+2}^{n+1}}\right| \\
&\leq \sum_{i=0}^{2^n-1} \|z\|_\kappa |t_{2i+1}^{n+1} - t_{2i}^{n+1}|^\kappa\, \|x\|_\gamma |t_{2i+2}^{n+1} - t_{2i+1}^{n+1}|^\gamma \\
&= \frac{\|z\|_\kappa \|x\|_\gamma}{2^{\kappa+\gamma} 2^{n(\kappa+\gamma-1)}}
\end{aligned}
$$

Definition of the integral

We have seen: for $\alpha \equiv \kappa + \gamma - 1 > 0$ and $n \geq 0$:

$$|I_{n+1} - I_n| \leq \frac{c_{x,z}}{2^{\alpha n}}$$

Series convergence:
Obviously, $\sum_{n=0}^{\infty}(I_{n+1} - I_n)$ is a convergent series
$\hookrightarrow$ yields definition of $\mathcal{J}_{01}(z\, dx)$, and more generally: $\mathcal{J}_{st}(z\, dx)$

Remark:
One should consider more general partitions π, with $|\pi| \to 0$
$\hookrightarrow$ C.f Lejay (Séminaire 37)

Young integral, version 1

Proposition 4.

Let $z \in \mathcal{C}_1^{\kappa}([0,T]), x \in \mathcal{C}_1^{\gamma}([0,T])$, with $\kappa + \gamma > 1$, and $0 \leq s < t \leq T$. Let

- $(\pi_n)_{n \geq 0}$ a sequence of partitions of $[s,t]$ such that $\lim_{n \to \infty} |\pi_n| = 0$
- I_n corresponding Riemann sums

Then:

1. I_n converges to an element $\mathcal{J}_{st}(z\,dx)$
2. The limit does not depend on the sequence $(\pi^n)_{n \geq 0}$
3. Integral linear in z, and coincides with Riemann's integral for smooth z, x
4. If $0 \leq s < u < t \leq T$, we have $\mathcal{J}_{st}(z\,dx) = \mathcal{J}_{su}(z\,dx) + \mathcal{J}_{ut}(z\,dx)$

Outline

Notations: increments

Simplex: For $k \geq 2$ and $T > 0$ we set

$$\mathcal{S}_{k,T} = \{(s_1, \ldots, s_k);\, 0 \leq s_1 < \cdots < s_k \leq T\}$$

$(k-1)$-increment: Let $T > 0$, a vector space V and $k \geq 1$:

$$\mathcal{C}_k(V) \equiv \left\{ g \in C(\mathcal{S}_{k,T}; V);\ \lim_{t_i \to t_{i+1}} g_{t_1 \cdots t_k} = 0,\ i \leq k-1 \right\}$$

Remark: We mostly consider $V = \mathbb{R}$ for notational sake
$\hookrightarrow$ We write $\mathcal{C}_k = \mathcal{C}_k([0,T]; \mathbb{R})$

Notations: operator δ

Operator δ:

$$\delta : \mathcal{C}_k \to \mathcal{C}_{k+1}, \qquad \delta g_{t_1 \cdots t_{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} g_{\hat{\mathbf{t}}_i^{k+1}},$$

where

$$\begin{aligned} \mathbf{t}^{k+1} &= (t_1, \ldots, t_{k+1}) \\ \hat{\mathbf{t}}_i^{k+1} &= (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{k+1}) \end{aligned}$$

Examples: if $g \in \mathcal{C}_1$ and $h \in \mathcal{C}_2$ we have, for $s, u, t \in \mathcal{S}_{3,T}$,

$$\delta g_{st} = g_t - g_s, \qquad \text{and} \qquad \delta h_{sut} = h_{st} - h_{su} - h_{ut}.$$

First properties of δ

Proposition 5.

$\delta\delta : \mathcal{C}_k \to \mathcal{C}_{k+2}$ satisfies $\delta\delta = 0$

Notation: $\mathcal{Z}\mathcal{C}_k = [\mathcal{C}_k \cap \text{Ker}\delta]$

Proposition 6.

Let

- $k \geq 1$
- $h \in \mathcal{Z}\mathcal{C}_{k+1}$

There exists a (non unique) $f \in \mathcal{C}_k$ such that $h = \delta f$.

Proofs

Proposition 5, easy case: If $k = 1$, $g \in \mathcal{C}_1$ and $h \equiv \delta g$, then:

$$\begin{aligned}(\delta\delta g)_{sut} &= \delta h_{sut} = h_{st} - h_{su} - h_{ut} \\ &= [g_t - g_s] - [g_u - g_s] - [g_t - g_u] \\ &= 0\end{aligned}$$

Proofs (2)

Proposition 5, general case: Let $g \in \mathcal{C}_k$. Then:

$$
\begin{aligned}
(\delta\delta g)_{\mathbf{t}^{k+2}} &= \sum_{i=1}^{k+2} (-1)^{k+1-i} \delta g_{\hat{\mathbf{t}}_i^{k+2}} \\
&= \sum_{i=1}^{k+1} (-1)^{k+1-i} \delta g_{\hat{\mathbf{t}}_i^{k+2}} - \delta g_{\mathbf{t}^{k+1}} \qquad (4)
\end{aligned}
$$

Decomposition for $\hat{\mathbf{t}}_i^{k+2}$: Write $\hat{\mathbf{t}}_i^{k+2} = \mathbf{s}^{k+1}$. Then

$$
s_j = t_j \quad \text{if } j \le i-1, \quad \text{and} \quad s_j = t_{j+1} \quad \text{if } j \ge i.
$$

Proofs (3)

Computation of $\delta g_{\hat{\mathbf{t}}_i^{k+2}}$: For $i \le k+1$ we have

$$
\begin{aligned}
\delta g_{\hat{\mathbf{t}}_i^{k+2}} &= \delta g_{\mathbf{s}^{k+1}} = \sum_{j=1}^{k+1} (-1)^{k-j} g_{\hat{\mathbf{s}}_j^{k+1}} \\
&= \sum_{j=1}^{i-1} (-1)^{k-j} g_{\hat{\mathbf{t}}_{j,i}^{k+2}} + \sum_{j=i}^{k+1} (-1)^{k-j} g_{\hat{\mathbf{t}}_{i,j+1}^{k+2}} \\
&= \sum_{j=1}^{i-1} (-1)^{k-j} g_{\hat{\mathbf{t}}_{j,i}^{k+2}} + \sum_{j=i+1}^{k+2} (-1)^{k-j+1} g_{\hat{\mathbf{t}}_{i,j}^{k+2}} \\
&= \sum_{j=1}^{i-1} (-1)^{k-j} g_{\hat{\mathbf{t}}_{j,i}^{k+2}} + \sum_{j=i+1}^{k+1} (-1)^{k-j+1} g_{\hat{\mathbf{t}}_{i,j}^{k+2}} - g_{\hat{\mathbf{t}}_i^{k+1}} \qquad (5)
\end{aligned}
$$

Proofs (4)

Conclusion for Proposition 5: Plugging (5) into (4), we get

$$
\begin{aligned}
(\delta\delta g)_{\mathbf{t}^{k+2}} &= \sum_{i=1}^{k+1}(-1)^{k+1-i}\left[\sum_{j=1}^{i-1}(-1)^{k-j} g_{\hat{\mathbf{t}}_{j,i}^{k+2}} + \sum_{j=i+1}^{k+1}(-1)^{k-j+1} g_{\hat{\mathbf{t}}_{i,j}^{k+2}}\right] \\
&\qquad + \sum_{i=1}^{k+1}(-1)^{k-i} g_{\hat{\mathbf{t}}_{i}^{k+1}} - \delta g_{\mathbf{t}^{k+1}} \\
&= \sum_{1\le j<i\le k+1}(-1)^{i+j-1} g_{\hat{\mathbf{t}}_{j,i}^{k+2}} + \sum_{1\le i<j\le k+1}(-1)^{i+j} g_{\hat{\mathbf{t}}_{i,j}^{k+2}} \\
&\qquad + \delta g_{\mathbf{t}^{k+1}} - \delta g_{\mathbf{t}^{k+1}} \\
&= 0
\end{aligned}
$$

Proofs (5)

Proposition 6, strategy: We show that the following works:

$$f_{t_1\ldots t_k} = -h_{t_1\cdots t_k T}$$

Relation $\delta h = 0$: can be written as

$$\delta h_{\mathbf{t}^{k+2}} = \sum_{i=1}^{k+1} (-1)^{k+1-i} h_{\hat{\mathbf{t}}_i^{k+2}} - h_{\mathbf{t}^{k+1}} = 0 \tag{6}$$

Verification of our claim: Set $g_{t_1\ldots t_k} = h_{t_1\cdots t_k T} = -f_{t_1\ldots t_k}$. Then

$$\begin{aligned}\delta g_{\mathbf{t}^{k+1}} &= \sum_{i=1}^{k+1} (-1)^{k-i} g_{\hat{\mathbf{t}}_i^{k+1}} = \sum_{i=1}^{k+1} (-1)^{k-i} h_{\hat{\mathbf{t}}_i^{k+1} T} \\ &= -\sum_{i=1}^{k+1} (-1)^{k+1-i} h_{\hat{\mathbf{t}}_i^{k+1} T} \overset{(6)}{=} -h_{\mathbf{t}^{k+1}}\end{aligned}$$

Particular case of Proposition 6

Proposition 7.

Let

- $h \in \mathcal{ZC}_2$

Then

- There exists $f \in \mathcal{C}_1$ such that $h = \delta f$.
- f is unique up to a constant

Proof of existence:

Take $f_s = -h_{sT}$ as in the general case.

Proof of uniqueness:

A function f defined by its increments is unique up to a constant.

First relation with integrals

Proposition 8.

Let f and g two smooth functions on $[0, T]$. Define $I \in \mathcal{C}_2$ by

$$I_{st} = \int_s^t \left(\int_s^v df_w \right) dg_v, \quad \text{for} \quad s, t \in [0, T].$$

Then we have, for $s < u < t$:

$$\delta I_{sut} = [f_u - f_s][g_t - g_u] = \delta f_{su} \, \delta g_{ut}.$$

Remark: This elementary property is important:

- δ transforms integrals into products of increments.
- We have already seen that products of the type $\delta f \, \delta g$ $\hookrightarrow$ both regularities of f and g can be used.

Proof

Invoking the very definition of δ and I:

$$
\begin{aligned}
(\delta I)_{sut} &= I_{st} - I_{su} - I_{ut} \\
&= \int_s^t \left(\int_s^v df_w \right) dg_v - \int_s^u \left(\int_s^v df_w \right) dg_v - \int_u^t \left(\int_u^v df_w \right) dg_v \\
&= \int_u^t \left(\int_s^v df_w \right) dg_v - \int_u^t \left(\int_u^v df_w \right) dg_v \\
&= \int_u^t \left(\int_s^u df_w \right) dg_v \\
&= \left(\int_s^u df_w \right) \left(\int_u^t dg_v \right) = \delta f_{su}\, \delta g_{ut}
\end{aligned}
$$

Hölder spaces

Aim: take into account some regularities in $\mathcal{C}_k$.

Case k=2: if $f \in \mathcal{C}_2$, set

$$\|f\|_\mu = \sup_{(s,t)\in\mathcal{S}_{2,T}} \frac{|f_{st}|}{|t-s|^\mu}, \quad \text{and} \quad \mathcal{C}_2^\mu = \{f \in \mathcal{C}_2;\ \|f\|_\mu < \infty\}.$$

Case k=1: if $g \in \mathcal{C}_1$, set

$$\|g\|_\mu = \|\delta g\|_\mu, \quad \text{and} \quad \mathcal{C}_1^\mu = \{g \in \mathcal{C}_1;\ \|g\|_\mu < \infty\}.$$

Remark: $\|\cdot\|_\mu$ defines a semi-norm in $\mathcal{C}_1^\mu$. It is a norm on

$$\mathcal{C}_{1,a}^\mu = \{g : [0,T] \to \mathbb{R};\ g_0 = a,\ \|g\|_\mu < \infty\}$$

Hölder spaces (2)

Case k=3: if $h \in \mathcal{C}_3$, set

$$\|h\|_\mu = \sup_{(s,u,t)\in\mathcal{S}_{3,T}} \frac{|h_{sut}|}{|t-s|^\mu}$$

and

$$\mathcal{C}_3^\mu = \left\{h \in \mathcal{C}_3;\ \|h\|_\mu < \infty\right\}.$$

Operator Λ (Sewing map)

Theorem 9.

Let $\mu > 1$. There exists a unique linear application $\Lambda : \mathcal{Z}\mathcal{C}_3^\mu \to \mathcal{C}_2^\mu$ such that

$$\delta\Lambda = \mathrm{Id}_{\mathcal{Z}\mathcal{C}_3^\mu} \quad \text{and} \quad \Lambda\delta = \mathrm{Id}_{\mathcal{C}_2^\mu}.$$

Equivalent statement: for any $h \in \mathcal{C}_3^\mu$ such that $\delta h = 0$,

there exists a unique element $g = \Lambda(h) \in \mathcal{C}_2^\mu$ such that $\delta g = h$.

Furthermore, for any $\mu > 1$, the application Λ is continuous from $\mathcal{Z}\mathcal{C}_3^\mu$ to $\mathcal{C}_2^\mu$, and

$$\|\Lambda(h)\|_\mu \leq \frac{2^\mu}{2^\mu - 2}\,\|h\|_\mu, \qquad h \in \mathcal{Z}\mathcal{C}_3^\mu.$$

Second relation with integrals

Proposition 10.

Let $g \in \mathcal{C}_2$, such that $\delta g \in \mathcal{C}_3^\mu$ with $\mu > 1$. Define

$$k = (\mathrm{Id} - \Lambda\delta) g$$

Then

$$k_{st} = \lim_{|\pi_{st}| \to 0} \sum_{i=0}^{n} g_{t_i \, t_{i+1}},$$

as $|\pi_{st}| \to 0$, where π_{st} is a partition of $[s, t]$.

Interpretation: Increment k can be seen as an integral of g.

Proof of Proposition 10

An equation for g: Thanks to Proposition 7, we have

$$k = (\mathrm{Id} - \Lambda\delta)g \quad \Longrightarrow \quad \delta k = 0 \quad \Longrightarrow \quad k = \delta f,$$

for $f \in \mathcal{C}_1$ unique up to a constant. Thus:

$$g = \delta f + \Lambda\delta g \tag{7}$$

Conclusion: Thanks to (7) we have

$$S_\pi = \sum_{i=0}^{n} g_{t_i t_{i+1}} = \sum_{i=0}^{n} \delta f_{t_i t_{i+1}} + \sum_{i=0}^{n} (\Lambda\delta g)_{t_i t_{i+1}} = \delta f_{st} + \sum_{i=0}^{n} (\Lambda\delta g)_{t_i t_{i+1}}.$$

Then the last sum converges to zero, since $\Lambda\delta g \in \mathcal{C}_3^{1+}(V)$

Young integral: strategy

Smooth case: Let $f, g \in \mathcal{C}_1^1$. Define $I \in \mathcal{C}_2$ by

$$I_{st} = \int_s^t \left(\int_s^v df_w \right) dg_v, \quad \text{for} \quad s, t \in [0, T].$$

Decomposition-recomposition scheme: we have

$$I = \int df \int dg \xrightarrow{\delta} \delta f \, \delta g \xrightarrow{\Lambda} I = \int df \int dg.$$

Indeed:

- First step: already established.
- Second step: $\delta f \, \delta g \in \mathcal{Z}\mathcal{C}_3^{\mu}$ with $\mu > 1 \Longrightarrow$ Theorem 9

Important: Second step can be extended to more irregular situations $\hookrightarrow f \in \mathcal{C}_1^{\gamma}, g \in \mathcal{C}_1^{\kappa}$ with $\mu = \gamma + \kappa > 1$.

Operator Λ (repeated)

Theorem 11.

Let $\mu > 1$. There exists a unique linear application $\Lambda : \mathcal{Z}\mathcal{C}_3^{\mu} \to \mathcal{C}_2^{\mu}$ such that

$$\delta\Lambda = \mathrm{Id}_{\mathcal{Z}\mathcal{C}_3^{\mu}} \quad \text{and} \quad \Lambda\delta = \mathrm{Id}_{\mathcal{C}_2^{\mu}}.$$

Equivalent statement: for any $h \in \mathcal{C}_3^{\mu}$ such that $\delta h = 0$,

there exists a unique element $g = \Lambda(h) \in \mathcal{C}_2^{\mu}$ such that $\delta g = h$.

Furthermore, for any $\mu > 1$, the application Λ is continuous from $\mathcal{Z}\mathcal{C}_3^{\mu}$ to $\mathcal{C}_2^{\mu}$, and

$$\|\Lambda h\|_{\mu} \leq \frac{2^{\mu}}{2^{\mu} - 2} \|h\|_{\mu}, \qquad h \in \mathcal{Z}\mathcal{C}_3^{\mu}.$$

Operator Λ: uniqueness

Definition of 2 increments:

Let $M, \hat{M}$ be two elements in $\mathcal{C}_2^\mu$ such that $\delta M = \delta \hat{M} = h$.

Define $Q = M - \hat{M}$.

Then $\delta Q = 0$ and $Q \in \mathcal{C}_2^\mu$.

Contradiction:

Hence there exists an element $q \in \mathcal{C}_1$ such that $Q = \delta q$, and

$$|q_t - q_s| = |Q_{st}| \leq c|t - s|^\mu$$

Since $\mu > 1$, q is constant in $[0, T]$, and thus $Q = 0$.

Operator Λ: existence

Algebraic increment:
$\delta h = 0 \Rightarrow$ existence of $B \in \mathcal{C}_2$ such that $\delta B = h$.

Construction of a sequence:
Called M^n_{st}, defined for $s, t \in [0, T]$, with $s < t$

For $n \geq 0$, consider partition $\{r^n_i; i \leq 2^n\}$ of $[s, t]$, where

$$r^n_i = s + \frac{(t-s)i}{2^n}, \quad \text{for} \quad 0 \leq i \leq 2^n.$$

For $n \geq 0$, define

$$M^n_{st} = B_{st} - \sum_{l=0}^{2^n-1} B_{r^n_l, r^n_{l+1}}.$$

Easy step: check $M^0_{st} = 0$.

Operator Λ: existence (2)

Control of $M^n - M^{n+1}$: we have

$$M^{n+1}_{st} - M^n_{st} = \sum_{i=0}^{2^n-1} \left(B_{r^{n+1}_{2i}, r^{n+1}_{2i+2}} - B_{r^{n+1}_{2i}, r^{n+1}_{2i+1}} - B_{r^{n+1}_{2i+1}, r^{n+1}_{2i+2}} \right)$$

$$= \sum_{i=0}^{2^n-1} \delta B_{r^{n+1}_{2i}, r^{n+1}_{2i+1}, r^{n+1}_{2i+2}} = \sum_{i=0}^{2^n-1} h_{r^{n+1}_{2i}, r^{n+1}_{2i+1}, r^{n+1}_{2i+2}},$$

Since $h \in \mathcal{C}^\mu_3$ with $\mu > 1$, we get

$$\left| M^n_{st} - M^{n+1}_{st} \right| \le \frac{\|h\|_\mu (t-s)^\mu}{2^{n(\mu-1)}},$$

Taking limits: we obtain existence of $M_{st} \equiv \lim_{n\to\infty} M^n_{st}$, such that

$$|M_{st}| \le \frac{2^\mu}{2^\mu - 2} \|h\|_\mu |t-s|^\mu.$$

Operator Λ: existence (3)

More general sequences: Consider

- $\{\pi_n;\, n \geq 1\}$ sequence of partitions of $[s, t]$
- $\pi_n = \{r_0^n, r_1^n, \ldots, r_{k_n}^n,\, r_{k_n}^n\}$
- $\pi_n \subset \pi_{n+1}$, and $\lim_{n\to\infty} k_n = \infty$
- $M_{st}^{\pi_n} = B_{st} - \sum_{l=0}^{k_n} B_{r_{l+1}^n, r_l^n}$

Removing points of a partition:

For $n \geq 1$, there exists $1 \leq l \leq k_n$ such that

$$|r_{l+1}^n - r_{l-1}^n| \leq \frac{2|t-s|}{k_n} \tag{8}$$

Then

- Pick now such an index l
- Transform π_n into $\hat{\pi}$, where

$$\hat{\pi} = \left\{ r_0^n, r_1^n, \ldots, r_{l-1}^n, r_{l+1}^n, \ldots, r_{k_n}^n, r_{k_n+1}^n \right\}.$$

Operator Λ: existence (4)

Estimate for the difference: As for dyadic partitions we have

$$M_{st}^{\hat{\pi}} = M_{st}^{\pi_n} - (\delta B)_{r_{l-1}^n, r_l^n, r_{l+1}^n} = M_{st}^{\pi_n} - h_{r_{l-1}^n, r_l^n, r_{l+1}^n}.$$

and thus

$$\left| M_{st}^{\hat{\pi}} - M_{st}^{\pi_n} \right| \leq 2^{\mu} \|h\|_{\mu} \left(\frac{t-s}{k_n} \right)^{\mu}.$$

Iteration of the estimate: We repeat this operation and

- We end up with the trivial partition $\hat{\pi}_0 \equiv \{s, t\}$
- $M_{st}^{\hat{\pi}_0} = 0$
- We obtain

$$|M_{st}^{\pi_n}| \leq 2^{\mu} \|h\|_{\mu} |t-s|^{\mu} \sum_{j=1}^{k_n} j^{-\mu} \leq 2^{\mu} \|h\|_{\mu} |t-s|^{\mu} \sum_{j=1}^{\infty} j^{-\mu} \equiv c_{\mu,h} |t-s|^{\mu}.$$

Operator Λ: existence (5)

More general sequences, conclusion: By compactness arguments

- One can find a subsequence $\{\pi_m;\, m \geq 1\}$ of $\{\pi_n;\, n \geq 1\}$
- It satisfies $\lim_{m\to\infty} M_{st}^{\pi_m} = M_{st}$
- M_{st}, satisfies $M_{st} \leq c_{\mu,h}|t-s|^{\mu}$

Uniqueness of the limit: One can show
$\hookrightarrow$ That the limit does not depend on the sequence of partitions.

Operator Λ: existence (6)

Algebraic property:
We wish to show that $\delta M = h$.

Family of partitions: Consider

- $0 \le s < u < t \le T$
- π^n_{su} sequence of partitions of $[s, u]$ such that $\lim_{n\to 0} |\pi^n_{su}| = 0$
- π^n_{ut} sequence of partitions of $[u, t]$ such that $\lim_{n\to 0} |\pi^n_{ut}| = 0$
- $\pi^n_{st} = \pi^n_{su} \cup \pi^n_{ut}$

Limits along the partitions: One can construct $\pi^n_{ut}, \pi^n_{su}, \pi^n_{st}$ such that

$$\lim_{m\to\infty} M^{\pi^n_{ut}}_{ut} = M_{ut}, \quad \lim_{m\to\infty} M^{\pi^n_{su}}_{su} = M_{su}, \quad \lim_{m\to\infty} M^{\pi^n_{st}}_{st} = M_{st}.$$

Operator Λ: existence (7)

Notation: We call

- k^n_{st} the number of points of the partition π^n_{st}
- k^n_{su} the number of points of the partition π^n_{su}
- k^n_{ut} the number of points of the partition π^n_{ut}

Applying δ: We have

$$
\begin{aligned}
\delta M^{\pi^n_{st}}_{sut} &= M^{\pi^n_{st}}_{st} - M^{\pi^n_{su}}_{su} - M^{\pi^n_{ut}}_{ut} \\
&= \delta B_{sut} - \left(\sum_{l=0}^{k^n_{su}+k^n_{ut}-1} B_{r^n_l r^n_{l+1}} - \sum_{l=0}^{k^n_{su}-1} B_{r^n_l r^n_{l+1}} - \sum_{l=k^n_{su}}^{k^n_{su}+k^n_{ut}-1} B_{r^n_l r^n_{l+1}} \right) \\
&= \delta B_{sut} = h_{sut}.
\end{aligned}
$$

Taking the limit $n \to \infty$ in the latter relation, we get $\delta M_{sut} = h_{sut}$

Outline

Expression for smooth functions

Riemann integral: Let $f, g \in \mathcal{C}_1^1$
$\hookrightarrow \mathcal{J}_{st}(f\, dg)$ defined in Riemann sense and

$$\mathcal{J}_{st}(f\, dg) \equiv \int_s^t f_u\, dg_u = f_s\, \delta g_{st} + \int_s^t [f_u - f_s]\, dg_u$$
$$= f_s\, \delta g_{st} + \int_s^t \delta f_{su}\, dg_u = f_s\, \delta g_{st} + \mathcal{J}_{st}(\delta f\, dg).$$

Analysis of $\mathcal{J}(\delta f\, dg) \in \mathcal{C}_2$: for $s, u, t \in [0, T]$ we have

$$h_{sut} \equiv \left[\delta \left(\mathcal{J}(df\, dg)\right)\right]_{sut} = \delta f_{su}\, \delta g_{ut}.$$

Therefore, $f \in C_1^{\kappa}, g \in C_1^{\gamma} \Rightarrow h \in \mathcal{Z}\mathcal{C}_3^{\gamma+\kappa}$

Expression for smooth functions (2)

We have seen:
If $\kappa + \gamma > 1$ (smooth case: $\kappa = \gamma = 1$), then $h \in \mathrm{Dom}(\Lambda)$
Thus (explain convention on products),

$$\mathcal{J}(\delta f\, dg) = \Lambda(h) = \Lambda\left(\delta f\, \delta g\right),$$

and we get:

$$\mathcal{J}_{st}(f\, dg) = f_s\, \delta g_{st} + \Lambda_{st}\left(\delta f\, \delta g\right). \tag{9}$$

Generalization: RHS in (9) makes sense whenever $\kappa + \gamma > 1$
$\hookrightarrow$ natural extension of the notion of integral

Theorem 12.

Let $f \in \mathcal{C}_1^\kappa, g \in \mathcal{C}_1^\gamma$, with $\kappa + \gamma > 1$. Define

$$\mathcal{J}_{st}(f\,dg) = f_s\,\delta g_{st} + \Lambda_{st}\left(\delta f\,\delta g\right). \qquad (10)$$

Then:

1. If f, g are smooth functions
   $\hookrightarrow$ Then $\mathcal{J}_{st}(f\,dg)$ = Riemann integral
2. Generalized integral $\mathcal{J}(f\,dg)$ satisfies:
   $$|\mathcal{J}_{st}(f\,dg)| \leq \|f\|_\infty \|g\|_\gamma |t-s|^\gamma + c_{\gamma,\kappa} \|f\|_\kappa \|g\|_\gamma |t-s|^{\gamma+\kappa}.$$
3. $\mathcal{J}_{st}(f\,dg)$ coincides with usual Young integral:
   $$\mathcal{J}_{st}(f\,dg) = \lim_{|\pi_{st}| \to 0} \sum_{i=0}^{n-1} f_{t_i}\,\delta g_{t_i\,t_{i+1}}.$$

Proof

Claim 1: Already obtained at (9)

Claim 2: Recall that

$$\mathcal{J}_{st}(f\,dg) = f_s\,\delta g_{st} + \Lambda_{st}\left(\delta f\,\delta g\right).$$

Hence, setting $h = \delta f\,\delta g$:

$$\begin{aligned} |f_s\,\delta g_{st}| &\leq \|f\|_\infty \|g\|_\gamma |t-s|^\gamma \\ |\Lambda_{st}\left(\delta f\,\delta g\right)| &\leq c_{\gamma,\kappa}\|h\|_{\gamma+\kappa}|t-s|^{\gamma+\kappa} \leq c_{\gamma,\kappa}\|f\|_\kappa \|g\|_\gamma |t-s|^{\gamma+\kappa} \end{aligned}$$

Proof (2)

Claim 3:
Recall: that, if $\delta\ell \in \mathcal{C}_3^{\mu}$ with $\mu > 1$,

$$k = (\mathrm{Id} - \Lambda\delta)\ell \quad \Rightarrow \quad k_{st} = \lim_{|\pi_{st}|\to 0} \sum_{i=0}^{n} \ell_{t_i\, t_{i+1}}$$

Application: take $\ell = f\,\delta g$, namely $\ell_{st} = f_s\,\delta g_{st} \Rightarrow \delta\ell = -\delta f\,\delta g$

Conclusion: we have

$$\begin{aligned} f_s\,\delta g_{st} + \Lambda_{st}\left(\delta f\,\delta g\right) &= f_s\,\delta g_{st} - \Lambda_{st}\left(\delta(f\,\delta g)\right) = [\mathrm{Id} - \Lambda\delta](f\,\delta g) \\ &= \lim_{|\pi_{st}|\to 0} \sum_{i=0}^{n} f_{t_i}\,\delta g_{t_i t_{i+1}} \end{aligned}$$

Outline

Pathwise strategy (repeated)

Aim: Let x be a function in $\mathcal{C}_1^\gamma$ with $\gamma > 1/2$. We wish to define and solve an equation of the form:

$$y_t = a + \int_0^t \sigma(y_s)\, dx_s \tag{11}$$

Steps:

- Define an integral $\int z_s\, dx_s$ for $z \in \mathcal{C}_1^\kappa$, with $\kappa + \gamma > 1$
- Solve (11) through fixed point argument in $\mathcal{C}_1^\kappa$ with $1/2 < \kappa < \gamma$

Remark: We treat a real case and $b \equiv 0$ for notational sake.

Existence-uniqueness result

Theorem 13.

Consider

- Noise: $x \in \mathcal{C}_1^\gamma \equiv \mathcal{C}_1^\gamma([0, T])$, with $\gamma > 1/2$
- Coefficient: $\sigma : \mathbb{R} \to \mathbb{R}$ a C_b^2 function
- Equation: $\delta y = \mathcal{J}(\sigma(y)\, dx)$
- $\frac{1}{2} < \kappa < \gamma$

Then:

1. Our equation admits a unique solution y in $\mathcal{C}_1^\kappa$
2. Application $(a, x) \mapsto y$ is continuous from $\mathbb{R} \times \mathcal{C}_1^\gamma$ to $\mathcal{C}_1^\kappa$.

Fixed point: strategy

A map on a small interval:
Consider an interval $[0, \tau]$, with τ to be determined later

Consider κ such that $1/2 < \kappa < \gamma < 1$

In this interval, consider $\Gamma : \mathcal{C}_1^\kappa([0,\tau]) \to \mathcal{C}_1^\kappa([0,\tau])$ defined by:
$\Gamma(z) = \hat{z}$, with $\hat{z}_0 = a$, and for $s, t \in [0, \tau]$:

$$\delta \hat{z}_{st} = \int_s^t \sigma(z_r) dx_r = \mathcal{J}_{st}(\sigma(z)\, dx)$$

Aim: See that for a small enough τ, the map Γ is a contraction
$\hookrightarrow$ our equation admits a unique solution in $\mathcal{C}_1^\kappa([0,\tau])$

Contraction argument in $[0, \tau]$

Definition of 2 processes:

Let $z^1, z^2 \in \mathcal{C}_1^\kappa([0,\tau])$. Define $\hat{z}^i = \Gamma(z^i)$. Then

$$\delta(\hat{z}^1 - \hat{z}^2)_{st} = \int_s^t \left[\sigma(z_r^1) - \sigma(z_r^2)\right] dx_s = \mathcal{J}_{st}\left(\left[\sigma(z^1) - \sigma(z^2)\right] dx\right)$$

Evaluation of the difference:

$$\left|\mathcal{J}_{st}(\left[\sigma(z^1) - \sigma(z^2)\right] dx)\right| \leq \|\sigma(z^2) - \sigma(z^1)\|_\infty \|x\|_\gamma |t-s|^\gamma + c_{\gamma,\kappa} \|\sigma(z^2) - \sigma(z^1)\|_\kappa \|x\|_\gamma |t-s|^{\gamma+\kappa}$$

Important step: Control of

$$\|\sigma(z^2) - \sigma(z^1)\|_\kappa$$

Control of $\|\sigma(z^2) - \sigma(z^1)\|_\kappa$

Lemma 14.

Let $\sigma \in C_b^2$. We have

$$\|\sigma(z^2) - \sigma(z^1)\|_\kappa \leq c_{\sigma,\tau} \left(1 + \|z^1\|_\kappa + \|z^2\|_\kappa\right) \|z^2 - z^1\|_\kappa$$

Problem: Application $\sigma : \mathcal{C}_1^\kappa([0,\tau]) \to \mathcal{C}_1^\kappa([0,\tau])$ is only locally Lipschitz

Solution: Decomposition of the fixed point argument:

1. If τ small enough and M large enough: existence of an invariant ball $B(0,M)$ by map Γ in $\mathcal{C}_{1,a}^\kappa([0,\tau])$
2. Within the invariant ball, usual contraction argument

Invariant ball

Lemma 15.

Let

- $c = c_{\sigma,x,\gamma,\kappa}$ be a constant
- $\tau \leq \inf\left\{ \left(\frac{1}{2c}\right)^{\frac{1}{\gamma-\kappa}}, \left(\frac{1}{2c}\right)^{\frac{1}{\gamma}} \right\}$

Then ball $B(0,1)$ in $\mathcal{C}^{\kappa}_{1,a}([0,\tau])$ is invariant by Γ.

Invariant ball: proof

Bound on Γ:

$$
\begin{aligned}
|\mathcal{J}_{st}(\sigma(z)dx)| &\le \|\sigma(z)\|_\infty \|x\|_\gamma |t-s|^\gamma + c_{\gamma,\kappa} \|\sigma(z)\|_\kappa \|x\|_\gamma |t-s|^{\gamma+\kappa} \\
&\le \|\sigma\|_\infty \|x\|_\gamma |t-s|^\kappa \tau^{\gamma-\kappa} + c_{\gamma,\kappa} \|\sigma'\|_\infty \|z\|_\kappa \|x\|_\gamma |t-s|^\kappa \tau^\gamma \\
&\le c_{\gamma,\kappa,\sigma} \|x\|_\gamma \left[\tau^{\gamma-\kappa} + \|z\|_\kappa \tau^\gamma\right] |t-s|^\kappa \\
&\le c \left[\tau^{\gamma-\kappa} + \|z\|_\kappa \tau^\gamma\right] |t-s|^\kappa
\end{aligned}
$$

Invariant ball: proof (2)

Inequality for M: We have seen

$$\|\Gamma(z)\|_{\kappa} \leq c \left[\tau^{\gamma-\kappa} + \|z\|_{\kappa}\tau^{\gamma}\right].$$

Hence, if M satisfies:

$$c \left[\tau^{\gamma-\kappa} + M\tau^{\gamma}\right] \leq M, \tag{12}$$

ball $B(0, M)$ invariant by Γ.

Remark:
We have used $\gamma > \kappa$ in order to gain a contraction factor $\tau^{\gamma-\kappa}$

Invariant ball: proof (3)

Solving (12): Write

$$(12) \iff c\left[\tau^{\gamma-\kappa} + M\tau^{\gamma}\right] \le M \iff M\left(1 - c\tau^{\gamma}\right) \ge c\,\tau^{\gamma-\kappa}$$

First condition on τ: $c\tau^{\gamma} \le \frac{1}{2}$. Then a sufficient condition for (12) is

$$M \ge 2c\,\tau^{\gamma-\kappa}$$

Second condition on τ: We take $M = 1$ and $\tau \le \left(\frac{1}{2c}\right)^{\frac{1}{\gamma-\kappa}}$

Conclusion: Relation (12) satisfied and $B(0,1)$ invariant if

$$\tau \le \left(\frac{1}{2c}\right)^{\frac{1}{\gamma}} \wedge \left(\frac{1}{2c}\right)^{\frac{1}{\gamma-\kappa}} \equiv \tau_1(\gamma, \kappa, \sigma, \|x\|_{\gamma})$$

Contraction argument in $[0, \tau_1]$

Recall: Setting $K_{st} \equiv \mathcal{J}_{st}([\sigma(z^1) - \sigma(z^2)]\, dx)$, we have seen

$$|K_{st}| \leq \|\sigma(z^2) - \sigma(z^1)\|_\infty \|x\|_\gamma |t-s|^\gamma + c_{\gamma,\kappa} \|\sigma(z^2) - \sigma(z^1)\|_\kappa \|x\|_\gamma |t-s|^{\gamma+\kappa}$$

Bounds on Hölder norms:

On $[0, \tau_2]$ with $\tau_2 \leq \tau_1$ we have (cf Lemma 14)

$$\|\sigma(z^2) - \sigma(z^1)\|_\kappa \leq 3 c_{\sigma,\tau_2} \|z^2 - z^1\|_\kappa$$

and

$$\begin{aligned} \|\sigma(z^2) - \sigma(z^1)\|_\infty &\leq c_\sigma \|z^2 - z^1\|_\infty \\ &\leq c_\sigma \tau_2^\kappa \|z^2 - z^1\|_\kappa \\ &= c_{\sigma,\tau_2} \|z^2 - z^1\|_\kappa \end{aligned}$$

Contraction argument in $[0, \tau_1]$ (2)

Bound on K: Owing to previous computations we get

$$|K_{st}| \leq c_{\sigma,T} \|x\|_\gamma \tau_2^{\gamma-\kappa} \|z^2 - z^1\|_\gamma |t-s|^\kappa$$

Recall: Let $z^1, z^2 \in \mathcal{C}_1^\kappa([0, \tau_2])$. Define $\hat{z}^i = \Gamma(z^i)$. Then

$$\delta(\hat{z}^1 - \hat{z}^2)_{st} = K_{st}$$

Contraction: We have obtained

$$\|\Gamma(z^2) - \Gamma(z^1)\|_\kappa \leq c_{\sigma,T} \|x\|_\gamma \tau_2^{\gamma-\kappa} \|z^2 - z^1\|_\kappa$$

Considering $\tau_2 \leq \inf\{\tau_1, \, (2c_{\sigma,T}\|x\|_\gamma)^{-1/(\gamma-\kappa)}\}$ this yields

$$\|\Gamma(z^2) - \Gamma(z^1)\|_\kappa \leq \frac{1}{2} \|z^2 - z^1\|_\kappa$$

Contraction argument in $[0, \tau_1]$ (3)

Existence-uniqueness on a small interval:

Thanks to Banach's fixed point theorem, for

$$\tau_2 \leq \inf\left\{\tau_1, \frac{1}{(2c_{\sigma,T}\|x\|_\gamma)^{1/\kappa}}\right\},$$

we get unique solution of (11) in $\mathcal{C}^\kappa_{1,a}([0, \tau_2])$

From $[0, \tau]$ to $[\tau, 2\tau]$

New map Γ: In $[\tau, 2\tau]$, consider the map

$$\Gamma : \mathcal{C}_1^\kappa([\tau, 2\tau]) \to \mathcal{C}_1^\kappa([\tau, 2\tau])$$

defined by: $\Gamma(z) = \hat{z}$, with $\hat{z}_\tau = a_\tau$, where

- $a_\tau \equiv$ final value of the solution in $[0, \tau]$
- For $s, t \in [\tau, 2\tau]$, $\delta\hat{z}_{st} = \mathcal{J}_{st}(z\, dx)$

New fixed point argument: the same fixed point arguments yield a unique solution y of $y_t = a_\tau + \int_\tau^t f(y_s)\, dx_s$ in $\mathcal{C}_1^\kappa([\tau, 2\tau])$.

Remark:
In order to use the very same arguments, need a bound on $\sigma, \sigma', \sigma''$

Continuity with respect to initial condition (1)

Notation: We set

- y^a solution of equation (11) with initial condition a
- a_1, a_2 two initial conditions
- $z = y^{a_2} - y^{a_1}$

Equation for z:

$$\delta z_{st} = [\sigma(y_u^{a_1}) - \sigma(y_u^{a_2})]\, \delta x_{st} + \Lambda\left(\delta[\sigma(y_u^{a_1}) - \sigma(y_u^{a_2})]\delta x\right)$$

Continuity with respect to initial condition (2)

Notation: Set, for $\tau_1 > 0$,

- $\|w\|_\gamma = \|w\|_{\gamma,[0,\tau_1]}$ for a path w
- $Z_s = \sup_{r \le s} |z_s|$
- $c_1 = c_\sigma \|x\|_\gamma$
- $c_2 = c_{\kappa,\gamma,\sigma} \left(1 + \|y^{a_1}\|_\kappa + \|y^{a_2}\|_\kappa\right) \|x\|_\gamma$

Bound for z: We get

$$\|\delta z_{st}\| \le c_1 Z_s |t-s|^\gamma + c_2 \|z\|_\kappa |t-s|^{\gamma+\kappa}$$

Bound for Z: We trivially have

$$Z_s \le |a^1 - a^2| + \|z\|_\kappa \, \tau_1^\kappa$$

Continuity with respect to initial condition (3)

Bound for the Hölder norm of z: We have

$$\begin{aligned}\|z\|_\kappa &\leq c_1 \tau_1^{\gamma-\kappa}\left(|a^1 - a^2| + \|z\|_\kappa \, \tau_1^\kappa\right) + c_2 \|z\|_\kappa \tau_1^\gamma \\ &\leq c_1 \tau_1^{\gamma-\kappa} |a^1 - a^2| + (c_1 + c_2)\tau_1^\gamma \|z\|_\kappa\end{aligned}$$

Choosing τ_1: such that $\tau_1 \leq 1$ and

$$\tau_1 = \left(\frac{c_3}{1 + \|x\|_\gamma}\right)^{\frac{1}{\gamma}} \quad \Longrightarrow \quad (c_1 + c_2)\tau_1^\gamma = \frac{1}{2}$$

Conclusion on a small interval: On $[0, \tau_1]$ we have

$$\begin{aligned}\|z\|_{\kappa;\,[0,\tau_1]} &\leq 2c_1 \tau_1^{\gamma-\kappa} |a^1 - a^2| \\ |z_{\tau_1}| &\leq |z_0| + \tau_1^\gamma \|z\|_{\gamma;[0,\tau_1]} \leq \left(1 + c_4 \tau_1^\gamma\right) |a^1 - a^2|\end{aligned}$$

Continuity with respect to initial condition (4)

Iteration of the estimate: For $j \geq 0$ and setting $d_x = 1 + c_4 \tau_1^\gamma$ we get

$$\|z\|_{\kappa;\,[j\tau_1,(j+1)\tau_1]} \leq d_x^j |a^1 - a^2|$$

Patching small interval estimates: Consider

$$j\tau_1 \leq s < (j+1)\tau_1 < k\tau_1 \leq t < (k+1)\tau_1$$

Then

$$|\delta z_{st}| \leq |\delta z_{s,(j+1)\tau_1}| + \sum_{l=j+1}^{k-1} |\delta z_{l\tau_1,(l+1)\tau_1}| + |\delta z_{k\tau_1,t}|$$

Continuity with respect to initial condition (5)

Patching small interval estimates, ctd:

We get (recall $d_x - 1 = c_4 \tau_1^\gamma$)

$$
\begin{aligned}
\frac{|\delta z_{st}|}{|a^1 - a^2|} &\leq d_x^j |(j+1)\tau_1 - s|^\gamma + \sum_{l=j+1}^{k-1} d_x^l \tau_1^\gamma + d_x^k |t - k\tau_1|^\gamma \\
&\leq d_x^j |(j+1)\tau_1 - s|^\gamma + d_x^{j+1} \frac{d_x^{k-j-1} - 1}{d_x - 1} \tau_1^\gamma + d_x^k |t - k\tau_1|^\gamma \\
&\leq c_5 \, d_x^k \, \left(|(j+1)\tau_1 - s|^\gamma + \tau_1^\gamma + |t - k\tau_1|^\gamma \right) \\
&\leq c_6 \, d_x^k \, |t - s|^\gamma
\end{aligned}
$$

Continuity with respect to initial condition (6)

Bound on k: In previous computations,

$$k\tau_1 \leq T \quad \Longrightarrow \quad k \leq \frac{T}{\tau_1}$$

Conclusion for Hölder's norm: We have obtained

$$\begin{aligned} \|z\|_{\gamma;[0,T]} &\leq c_6\, d_x^{T/\tau_1} |a^1 - a^2| = c_6 \exp\left(\frac{T}{\tau_1} \ln(d_x)\right) |a^1 - a^2| \\ &\leq c_6 \exp\left(c_7 \left(1 + \|x\|_\gamma\right)^{1/\gamma}\right) |a^1 - a^2| \end{aligned}$$

Continuity proved!

Control of $\|\sigma(z^2) - \sigma(z^1)\|_\kappa$ (repeated)

Lemma 16.

Let $\sigma \in C_b^2$. We have

$$\|\sigma(z^2) - \sigma(z^1)\|_\kappa \le c_{\sigma,\tau} \left(1 + \|z^1\|_\kappa + \|z^2\|_\kappa\right) \|z^2 - z^1\|_\kappa$$

Proof: For $\lambda, \mu \in [0,1]$, define the path

$$a(\lambda, \mu) = z_s^1 + \lambda \left(z_t^1 - z_s^1\right) + \mu \left(z_s^2 - z_s^1\right) + \lambda\mu \left(z_t^2 - z_s^2 - z_t^1 + z_s^1\right)$$

Then

$$a(0,0) = z_s^1, \quad a(0,1) = z_s^2, \quad a(1,0) = z_t^1, \quad a(1,1) = z_t^2$$

Proof

Let $G(\lambda, \mu) \equiv \sigma(a(\lambda, \mu))$, and

$$\Delta_{st}^{12} \equiv \left[\sigma(z_t^2) - \sigma(z_t^1)\right] - \left[\sigma(z_s^2) - \sigma(z_s^1)\right]$$

We have:

$$\Delta_{st}^{12} = G(1,1) - G(1,0) - G(0,1) + G(0,0) = \int_0^1 \int_0^1 \partial^2_{\lambda,\mu} G \, d\lambda d\mu$$

Set $\hat{z} \equiv z^2 - z^1$ and compute:

$$\begin{aligned}
\partial^2_{\lambda,\mu} G &= \partial^2_{\lambda,\mu} a \, \sigma'(a) + \partial_\lambda a \, \partial_\mu a \, \sigma''(a) \\
\partial_\lambda a &= \mu \, \delta z^2_{st} + [1-\mu] \, \delta z^1_{st} \\
\partial_\mu a &= \lambda \hat{z}_t + [1-\lambda] \hat{z}_s \\
\partial^2_{\lambda,\mu} a &= \delta \hat{z}_{st}
\end{aligned}$$

Proof (2)

Thus:

$$\left|\partial_\lambda a\right| = \left|\mu\,\delta z^2_{st} + [1-\mu]\,\delta z^1_{st}\right| \le \left[\|z^1\|_\kappa + \|z^2\|_\kappa\right] |t-s|^\kappa$$
$$|\partial_\mu a| = |\lambda \hat{z}_t + [1-\lambda]\hat{z}_s| \le \|z^1 - z^2\|_\kappa\ \tau^\kappa$$
$$|\partial^2_{\lambda,\mu} a| = |\delta \hat{z}_{st}| \le \|z^1 - z^2\|_\kappa |t-s|^\kappa,$$

and

$$\begin{aligned} \partial^2_{\lambda,\mu} G &= \left|\partial^2_{\lambda,\mu} a\,\sigma'(a) + \partial_\lambda a\,\partial_\mu a\,\sigma''(a)\right| \\ &\le \left|\partial^2_{\lambda,\mu} a\right|\ \|\sigma'\|_\infty + |\partial_\lambda a|\ \ |\partial_\mu a|\ \ \|\sigma''\|_\infty \end{aligned}$$

The result is now easily deduced.