Convergence to equilibrium for rough differential equations

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Outline

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Definition of fBm

Definition 1.

A 1-d fBm is a continuous process $X = \{X_t;\ t \in \mathbb{R}\}$ such that $X_0 = 0$ and for $H \in (0, 1)$:

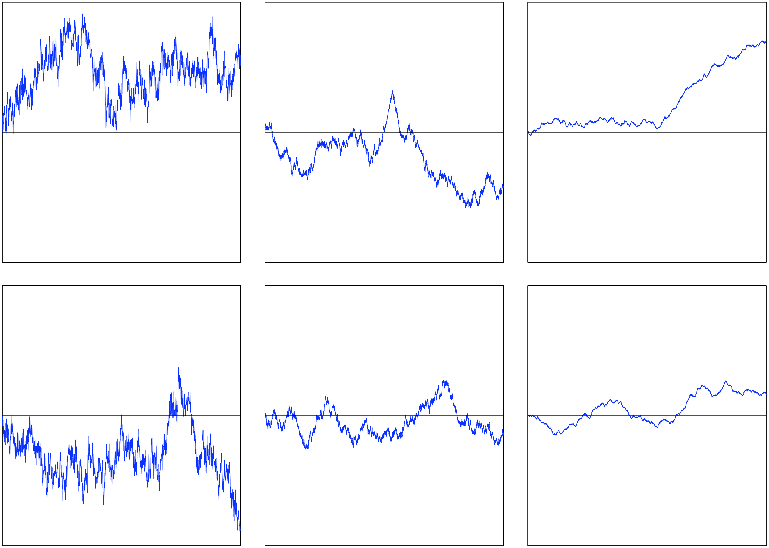
- X is a centered Gaussian process
- $\mathbf{E}[X_t X_s] = \frac{1}{2}(|s|^{2H} + |t|^{2H} - |t - s|^{2H})$

d-dimensional fBm: $X = (X^1, \ldots, X^d)$, with X^i independent 1-d fBm

Variance of increments:

$$\mathbf{E}[|\delta X^j_{st}|^2] \equiv \mathbf{E}[|X^j_t - X^j_s|^2] = |t - s|^{2H}$$

Examples of fBm paths



$H = 0.35$ $\qquad$ $H = 0.5$ $\qquad$ $H = 0.7$

System under consideration

Equation:

$$dY_t = b(Y_t)dt + \sigma(Y_t)\, dX_t, \qquad t \geq 0 \tag{1}$$

Coefficients:

- $x \in \mathbb{R}^d \mapsto \sigma(x) \in \mathbb{R}^{d\times d}$ smooth enough
- $\sigma = (\sigma_1, \ldots, \sigma_d) \in \mathbb{R}^{d\times d}$ invertible
- $\sigma^{-1}(x)$ bounded uniformly in x
- $X = (X^1, \ldots, X^d)$ is a d-dimensional fBm, with $H > \frac{1}{3}$

Resolution of the equation:

- Thanks to rough paths methods
  $\hookrightarrow$ Limit of Wong-Zakai approximations

Illustration of ergodic behavior

Equation with damping: $dY_t = -\lambda Y_t \, dt + dX_t$

Simulation: For 2 values of the parameter λ

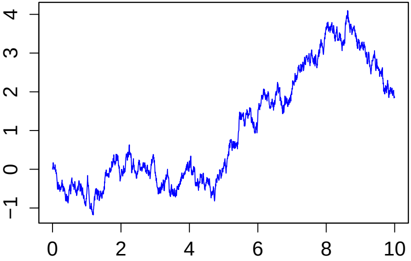


Figure: $H = 0.7$, $d = 1$, $\lambda = 0.1$

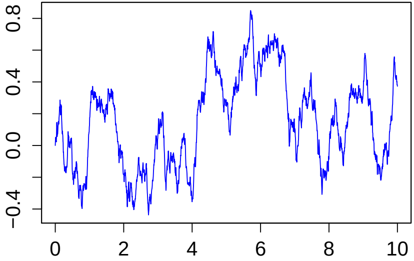
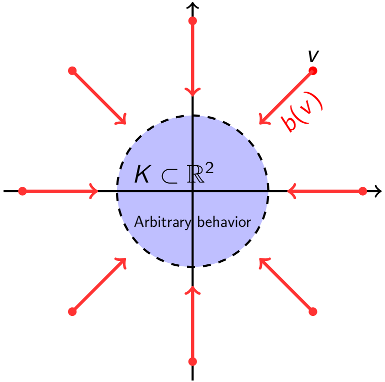


Figure: $H = 0.7$, $d = 1$, $\lambda = 3$

Coercivity assumption for b

Hypothesis: for every $v \in \mathbb{R}^d$, one has

$$\langle v, b(v)\rangle \leq C_1 - C_2\|v\|^2$$



Interpretation of the hypothesis:
Outside of a compact $K \subset \mathbb{R}^d$,
$b(v) \simeq -\lambda v$ with $\lambda > 0$

Ergodic results for equation (1)

Brownian case: If X is a Brownian motion and b coercive

- Exponential convergence of $\mathcal{L}(X_t)$ to invariant measure μ
- Markov methods are crucial
- See e.g Khashminskii, Bakry-Gentil-Ledoux

Fractional Brownian case: If X is a fBm and b coercive

- Markov methods not available
- Existence and uniqueness of invariant measure μ, when $H > \frac{1}{3}$
  $\hookrightarrow$ Series of papers by Hairer et al.
- Rate of convergence to μ:
  - When $\sigma \equiv \mathrm{Id}$: Hairer
  - When $H > \frac{1}{2}$ and further restrictions on σ: Fontbona–Panloup

Main result (loose formulation)

Theorem 2.

Let

- $H > \frac{1}{3}$, equation $dY_t = b(Y_t)dt + \sigma(Y_t)\, dX_t$
- Y unique solution with initial condition μ_0
- μ unique invariant measure

Then for all $\varepsilon > 0$ we have:

$$\|\mathcal{L}(Y_t^{\mu_0}) - \mu\|_{\mathrm{tv}} \leq c_\varepsilon t^{-(\frac{1}{8}-\varepsilon)}$$

Remark:

- Subexponential (non optimal) rate of convergence
- This might be due to the correlation of increments for X

Outline

1. Setting and main result
2. Convergence to equilibrium for diffusion processes
   - Poincaré inequality
   - Coupling method
3. Elements of proof

Outline

Poincaré and convergence to equilibrium

Theorem 3.

Let X be a diffusion process. We assume:

- μ is a symmetrizing measure, with Dirichlet form $\mathcal{E}$
- Poincaré inequality: $\mathrm{Var}_\mu(f) \leq \alpha\, \mathcal{E}(f)$

Then the following inequality is satisfied:

$$\mathrm{Var}_\mu(P_t f) \leq \exp\left(-\frac{2t}{\alpha}\right) \mathrm{Var}_\mu(f)$$

Comments on the Poincaré approach

Remarks:

1. Theorem 3 asserts that

$$X_t \xrightarrow{(d)} \mu, \qquad \text{exponentially fast}$$

2. The proof relies on identity $\partial_t P_t = L P_t$
   $\hookrightarrow$ Hard to generalize to a non Markovian context

3. One proves Poincaré with Lyapunov type techniques
   $\hookrightarrow$ Coercivity enters into the picture

Outline

A general coupling result

Proposition 4.

Consider:

- Two processes $\{Z_t;\, t \geq 0\}$ and $\{Z'_t;\, t \geq 0\}$
- A coupling $(\hat{Z}, \hat{Z}')$ of (Z, Z')

We set

$$\tau = \inf\left\{t \geq 0;\, \hat{Z}_u = \hat{Z}'_u \text{ for all } u \geq t\right\}$$

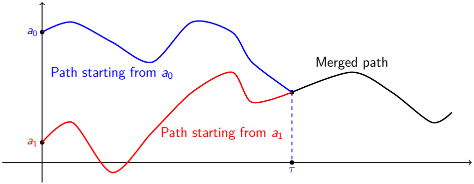
Then we have:

$$\|\mathcal{L}(Z_t) - \mathcal{L}(Z'_t)\|_{\mathrm{tv}} \leq 2\,\mathbf{P}\,(\tau > t)$$

Comments on the coupling method

1. Proposition 4 is general, does not assume a Markov setting
   $\hookrightarrow$ can be generalized (unlike Poincaré)

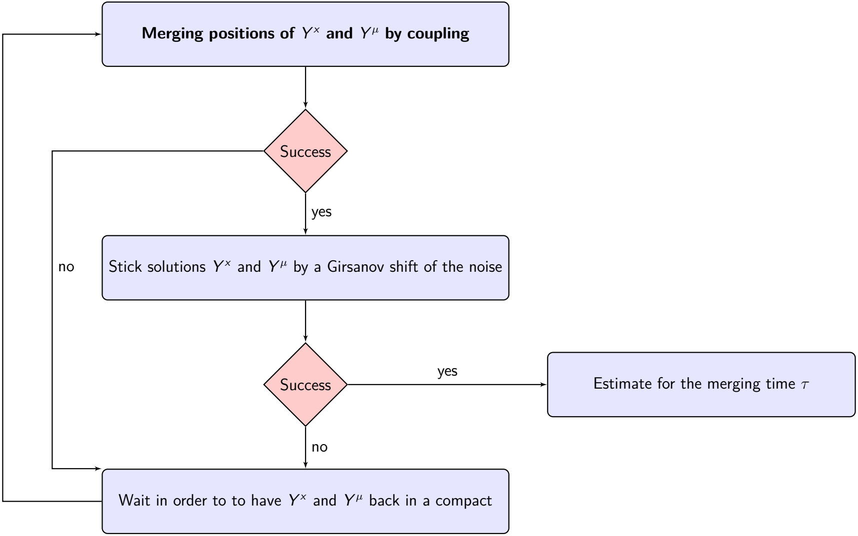
2. In a Markovian setting
   $\hookrightarrow$ Merging of paths a soon as they touch



3. In our case
   $\hookrightarrow$ We have to merge both Y, Y' and the noise

Outline

Algorithmic view of the coupling



Merging positions (1)

Simplified setting:
We start at $t = 0$, and consider $d = 1$

Effective coupling: We wish to consider y^0, y^1 and h such that

- We have

$$\begin{cases} dy_t^0 = b(y_t^0)\, dt + \sigma(y_t^0)\, dX_t \\ dy_t^1 = b(y_t^1)\, dt + \sigma(y_t^1)\, dX_t + h_t\, dt \end{cases}$$

- Merging condition: $y_0^0 = a_0$, $y_0^1 = a_1$ and $y_1^0 = y_1^1$

Computation of the merging probability:
Through Girsanov's transform involving the shift h

Merging positions (2)

Generalization of the problem:

We wish to consider a family $\{y^\xi, h^\xi; \xi \in [0,1]\}$ such that

- We have
$$dy_t^\xi = b(y_t^\xi)\,dt + \sigma(y_t^\xi)\,dX_t + h_t^\xi\,dt$$
- Merging condition:
$$y_0^\xi = a_0 + \xi(a_1 - a_0), \qquad y_1^0 = y_1^1, \qquad h^0 \equiv 0$$

Remark:

Here y has to be considered as a function of 2 variables t and ξ

Merging positions (3)

Solution of the problem: Consider a system with tangent process

$$\begin{cases} dy_t^\xi = \left[b(y_t^\xi) - \int_0^\xi d\eta \, \jmath_t^\eta \right] dt + \sigma(y_t^\xi) \, dX_t \\ d\jmath_t^\xi = b'(y_t^\xi)\jmath_t^\xi \, dt + \sigma'(y_t^\xi)\jmath_t^\xi \, dX_t \end{cases}$$

and initial condition $y_0^\xi = a_0 + \xi(a_1 - a_0)$, $\jmath_0^\xi = a_1 - a_0$

Heuristics: A simple integrating factor argument shows that

$$\partial_\xi y_t^\xi = \jmath_t^\xi(1 - t), \quad \text{and thus} \quad \partial_\xi y_1^\xi = 0$$

Hence y^ξ solves the merging problem

Merging positions (4)

Rough system under consideration: for $t, \xi \in [0,1]$

$$\begin{cases} dy_t^\xi = \left[b(y_t^\xi) - \int_0^\xi d\eta \, \jmath_t^\eta \right] dt + \sigma(y_t^\xi) \, dX_t \\ d\jmath_t^\xi = b'(y_t^\xi) \jmath_t^\xi \, dt + \sigma'(y_t^\xi) \jmath_t^\xi \, dX_t \end{cases}$$

Then y_1^ξ does not depend on ξ!

Difficulties related to the system:

1. $t \mapsto y_t$ is function-valued
2. Unbounded coefficients, thus local solution only
3. Conditioning $\Longrightarrow$ additional drift term with singularities
4. Evaluation of probability related to Girsanov's transform