A coupling between Sinai's random walk and Brox's diffusion

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Ongoing & immature joint work with Xi Geng, Mihai Gradinaru

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1. Introduction
   - Sinai's random walk
   - Brox diffusion
   - From Sinai to Brox
2. Main result and strategy of proof
   - Aim and main result
   - Strategy

Environment for Sinai's random walk

Random environment:

- Sequence of i.i.d random variables $\{\omega_x;\ x \in \mathbb{Z}\}$
- Defined on a probability space $(\Omega, \mathcal{G}, \mathbf{P})$

Elliptic assumption: For $\kappa \in (0, 1/2)$, **P**-almost surely we have

$$\omega_x \in [\kappa, 1-\kappa]$$

Recurrence assumption: For all $x \in \mathbb{Z}$ we have

$$\mathbf{E}[\xi_x] = 0, \quad \text{where} \quad \xi_x = \ln\left(\frac{1-\omega_x}{\omega_x}\right). \tag{1}$$

Definition of Sinai's random walk

Quenched probability: Conditioned on the environment ω,

$$(\hat{\Omega}, \mathcal{F}, \mathbb{P}^{\omega})$$

Discrete walk: process $\{X_n^d;\, n \geq 0\}$ with $X_0 = 0$ and

$$\begin{aligned} \mathbb{P}^{\omega}\left(X_{n+1}^d = x+1 \middle|\, X_n^d = x\right) &= \omega_x \\ \mathbb{P}^{\omega}\left(X_{n+1}^d = x-1 \middle|\, X_n^d = x\right) &= 1-\omega_x. \end{aligned}$$

Note:
The d in X_n^d stands for discrete

Large time behavior for Sinai's walk

Variance of ξ: We set

$$\sigma^2 = \mathbf{E}\left[(\xi_x)^2\right] = \mathbf{E}\left[\left(\ln\left(\frac{1-\omega_x}{\omega_x}\right)\right)^2\right]$$

Annealed limit theorem (Sinai):
There exists a random variable L such that

$$\frac{\sigma^2 X_n^d}{(\ln(n))^2} \xrightarrow{\mathbf{P}-(d)} L$$

Description of L (Kesten):
Complicated functional of a Brownian path

Simulations (courtesy Jon Peterson)

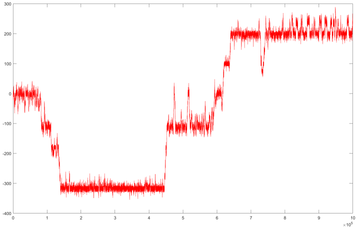


Figure: With a $\beta(10, 10)$ environment

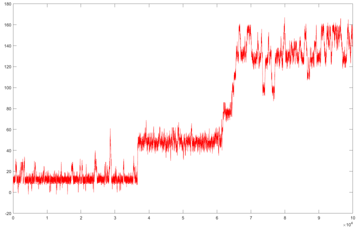


Figure: With a $\beta(5, 5)$ environment

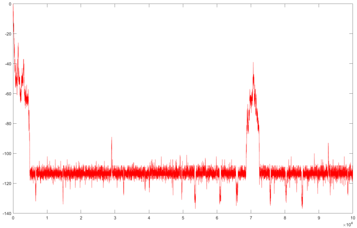


Figure: With a $\beta(5, 5)$ environment

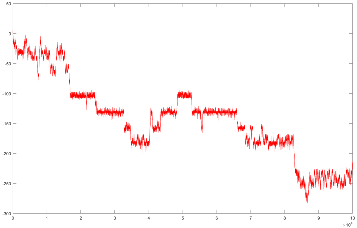


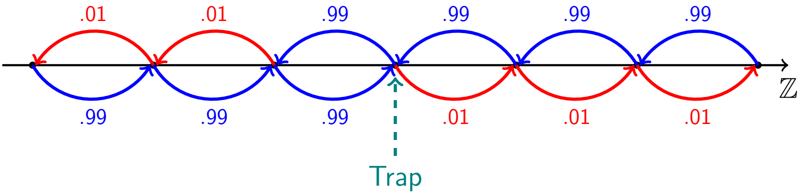
Figure: With a $\beta(5, 5)$ environment

Example of trap

Specific environment: Assume

$$\mathbf{P}(\omega_x = .99) = \mathbf{P}(\omega_x = .01) = \frac{1}{2}$$

Possible realization with a trap:



Characterization of traps

Potential for the random walk: For $x \in \mathbb{Z}$, set

$$W(x) = \sum_{j \in [\![0,x]\!]} \xi_j, \quad \text{where} \quad \xi_j = \ln\left(\frac{1-\omega_j}{\omega_j}\right)$$

Then $x \mapsto W(x)$ is a simple random walk

Role of W:

The potential W shows up in the analysis of hitting times for X^d

Characterization of traps: x_0 such that

$x \mapsto W(x)$ has a valley at x_0

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Brox diffusion

Random environment: Double sided Brownian motion

$$\{W(x);\, x \in \mathbb{R}\}$$

Formal definition: For another Brownian motion B, X^c solves

$$dX_t^c = -\frac{1}{2}\dot{W}(X_t^c)\, dt + dB_t \tag{2}$$

This equation is ill-defined (drift term in $\mathcal{C}^{-1/2-\varepsilon}$)!

Remark:
X^c is the continuous-time & continuous-space equivalent of X^d

Definition of Brox diffusion

A very weak definition:

X^c can be constructed as a Markov process with generator

$$\begin{aligned} \mathcal{L}^c f(x) &= \frac{1}{2}\Delta^c f(x) - \frac{1}{2}\dot{W}(x)\nabla f(x) \\ &= \frac{1}{2}e^{W(x)}\partial_x \left[e^{-W(x)}\partial_x f\right](x) \end{aligned}$$

Related Dirichlet form:

$$\mathcal{E}^c(f) = -\langle \mathcal{L}^c f,\, f\rangle_{L^2(\mathbb{R};\, e^{-W}dx)} = \frac{1}{2}\int_{\mathbb{R}} e^{-W(x)}|\partial_x f(x)|^2\, dx. \qquad (3)$$

Partial conclusion:

X^c exists as a Markov process on a certain probability space

Semi-pathwise constructions of Brox diffusion

Ohashi-Russo-Teixera (2020):

- Review of martingale methods for SDEs with distributional drifts
- Case of path dependent SDEs with distributional drifts

Hu-Le-Mytnik (2017): Explicit weak solution to (2) thanks to

- Mc-Kean representation of the 1-d diffusion
- Considerations on Brownian local time for W

Delarue-Diel (2016): Explicit weak solution to (2) thanks to

- Pathwise solution of some PDEs related to (3)
  $\hookrightarrow$ Rough paths method
- Related martingale problem
- Easy to generalize to higher dimensions

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Scalings

Space-time: For $\varepsilon_n \to 0$, we consider (parabolic scaling)

$$t_j = j/\varepsilon_n^2, \qquad x \in \varepsilon_n \mathbb{Z}$$

Initial environment: Recall that

$$\xi_x = \ln\left(\frac{1-\omega_x}{\omega_x}\right) \quad \Longleftrightarrow \quad \omega_x = \frac{1}{1+e^{\xi_x}}$$

Rescaled environment: We take

$$\xi_x^n = \sqrt{\varepsilon_n}\,\xi_x \quad \text{and} \quad \omega_x^n = \frac{1}{1+e^{\xi_x^n}} \quad \left(\simeq \frac{1}{2}\right)$$

Convergence in law

Rescaled random walk: We set

- $\hat{X}_n \equiv$ random walk on $\varepsilon_n\mathbb{Z}$ with environment $\omega^n_{x/\varepsilon_n}$
- $X^n_t := \hat{X}^n_{\lfloor t/\varepsilon_n^2 \rfloor}$

A result by Seignourel (2001):
We have (in $D([0,\infty))$ and for the annealed probability)

$$\lim_{n\to\infty} \{X^n_t;\ t \geq 0\} \stackrel{(d)}{=} \{X^c_t;\ t \geq 0\}$$

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Aim

Objective in a few words:

- Find a concrete coupling between X^n and X^c where
  - X^n is the rescaled Sinai walk
  - X^c is Brox diffusion
- For this coupling, find a rate of convergence
- This will give an upper bound on

$$\|X^n_{[0,T]} - X^c_{[0,T]}\|_{\mathrm{TV}}$$

Main result

Theorem 1.

There exists $(\Omega, \mathcal{F}, \mathbf{P})$ on which we can define

- Two Brownian motions W and B
- A rescaled environment ω^n
- A family of Sinai walks X^n on $\varepsilon_n \mathbb{Z}$ based on ω^n,

and such that the following holds true:

1. There exists a weak solution X^c to

$$X_t^c = -\frac{1}{2} \int_0^t \dot{W}(X_s^c)\, ds + B_t$$

2. Given a time horizon T and $\kappa < \frac{1}{6}$ we have

$$\sup_{t \le T} |X_t^c - X_t^n| \le c_{T,W,B}\, n^{-\kappa}$$

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Rough paths formulation

A rough operator: Recall that

$$\mathcal{L}^c f(x) = \frac{1}{2}\Delta^c f(x) - \frac{1}{2}\dot{W}(x)\nabla f(x)$$

A rough PDE: For $g \in C_b^2$ one solves a mild form of

$$\partial_t f_t(x) - \mathcal{L}^c f_t(x) = g_t(x), \quad t \in [0, \tau], \, x \in \mathbb{R}. \tag{4}$$

This uses heavy rough paths machinery

Martingale problem

Related martingale problem:

For f as in (4), the following process is a martingale

$$M_t = f_t(X_t^c) - f_0(X_0^c) - \int_0^t g(X_s^c)\, ds$$

This gives raise to a weak solution of Brox equation

More explicit version of the weak solution:

There exists $(\Omega, \mathcal{F}, \mathbf{P})$ on which we can define

- Two Brownian motions W, B, and X^c continuous process

such that X^c solves the equation

$$X_t^c = -\frac{1}{2} \int_0^t \dot{W}(X_s^c)\, ds + B_t$$

Approximation strategy (1)

Main point: Pathwise approximation of the PDE

$$\partial_t f_t(x) - \mathcal{L}^c f_t(x) = g_t(x), \quad \mathcal{L}^c f(x) = \frac{1}{2}\left(\Delta^c f(x) - \dot{W}(x)\nabla f(x)\right)$$

Strong approximation:
One can construct a rescaled random walk W^n such that

$$\|W^n - W\|_{\mathcal{C}^\alpha} = O\left(n^{-(1/2-\alpha-\varepsilon)}\right), \qquad \text{a.s}$$

This is a result by Komlos-Major-Tusnady (1976)
It is applied for $\alpha = \frac{1}{3} + \varepsilon \Longrightarrow$ exponent $\kappa = \frac{1}{6}$

Approximation strategy (2)

Discretized operator: Consider

$$\mathcal{L}^n f(x) = \frac{1}{2}\left(\Delta^n f(x) - \dot{W}^n(x)\nabla^n f(x)\right)$$

Discretized PDE: Of the form

$$\partial_t^n f_t^n(x) - \mathcal{L}^n f_t^n(x) = g_t^n(x)$$

Convergence: If f is the solution to (4), we get

$$\|f - f^n\|_{1/4-\varepsilon,1/2-\varepsilon} \leq \frac{C}{n^\kappa}$$

Approximation strategy (3)

A closer look a the rough mild PDE: Of the form

$$\partial_x f_t(x) = \partial_x P_t f_0(x) + \int_0^t \int_{\mathbb{R}} \partial_x p_{t-s}(x-y) g_s(y) dyds - \frac{1}{2} \int_0^t \int_{\mathbb{R}} \partial_{xx}^2 p_{t-s}(x-y) \left(\int_a^y \partial_z f_s(z) dW(z) \right) dyds. \quad (5)$$

Approximation of the rough integral: By sums of the form

$$\sum_i \frac{1}{2} \left(\partial_z f_s(z_i) + \partial_z f_s(z_{i+1}) \right) W_{t_i, t_{i+1}}$$

This is related to convergence of trapezoid rules for rough paths
$\hookrightarrow$ Cf Liu-Selk-T (2020)