Application of functions of 2 variables (1)

favorite virus

Situation:

- Fraction of students infected by `FV` is r on 9/12
- We have n random encounters with students on 9/12

Function:

The probability of meeting at least one student with `FV` is

$$p(n, r) = 1 - (1 - r)^n$$

This requires probability theory and is admitted

Question:

Draw level curves

Function $\quad p(n,r) = 1-(1-r)^n \quad r \in [0,1]$

Level curve: Fix p_0. We wish to have

Note: p_0 should be in the range of $p(n,r)$. This range is $[0,1]$

$$p(n,r) = p_0$$

$$\Leftrightarrow \quad 1-(1-r)^n = p_0$$

$$\Leftrightarrow \quad (1-r)^n = 1-p_0$$

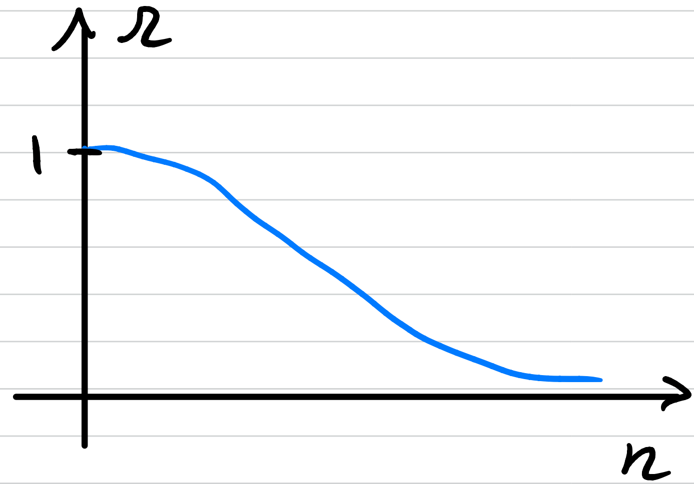
$\wedge \frac{1}{n}$

$$\Leftrightarrow \quad 1-r = (1-p_0)^{\frac{1}{n}}$$

$$\Leftrightarrow \quad \boxed{r = r(n) = 1-(1-p_0)^{\frac{1}{n}}}$$

Graphing the level curves

$$r = r(n) = 1 - (1-p_0)^{1/n}$$



As $n \to 0^+$, $\quad \frac{1}{n} \longrightarrow +\infty$

$$\underbrace{(1-p_0)}_{\in (0,1)}{}^{1/n} \xrightarrow{n \to \infty} 0$$

$$r(n) = 1 - (1-p_0)^{1/n} \xrightarrow{n \to \infty} \boxed{1}$$

As $n \to \infty$, $\quad \frac{1}{n} \longrightarrow 0$

$$(1-p_0)^{1/n} \longrightarrow 1$$

$$r(n) = 1 - (1-p_0)^{1/n} \longrightarrow \boxed{0}$$

Application of functions of 2 variables (2)

Function:

$$p(n, r) = 1 - (1 - r)^n$$

Useful values of z:

For $p_0 \in [0, 1]$, the curve $p(n, r) = p_0$ is non empty

Level curves for $p_0 \in [0, 1]$:

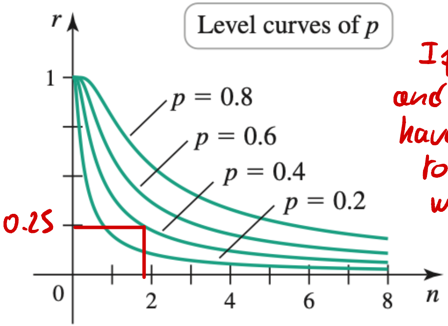
$$r = 1 - (1 - p)^{1/n}$$

Application of functions of 2 variables (3)

Function:

$$p(n, r) = 1 - (1 - r)^n$$

Depiction of level curves:



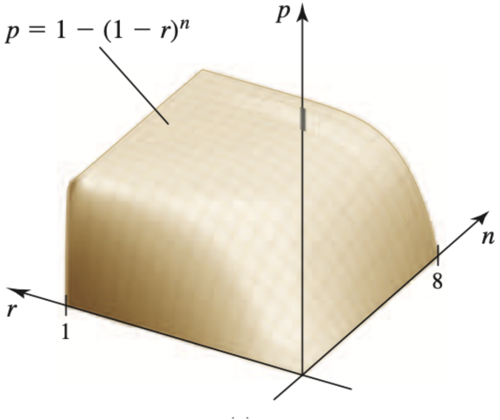
If rate is 25% and we wish to have $\leq$ 40% chances to meet a student with FV, one cannot meet more than 1 student!

Application of functions of 2 variables (4)

Function:

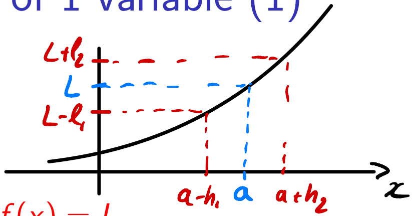
$$p(n, r) = 1 - (1 - r)^n$$

Depiction of function:



Outline

Continuity for functions of 1 variable (1)



Limit: The assertion

$$\lim_{x \to a} f(x) = L$$

means that $f(x)$ can be made as close to L as we wish
$\hookrightarrow$ by making x close to a

If $x \in [a-h_1, a+h_2]$
then $f(x) \in [L-\ell_1, L+\ell_2]$

Remark: If $\lim_{x \to a} f(x) = L$, then
the limit should not depend on the way $x \to a$

Continuity for functions of 1 variable (2)

Continuity: The function f is continuous at point a if

$$\lim_{x \to a} f(x) = f(a)$$

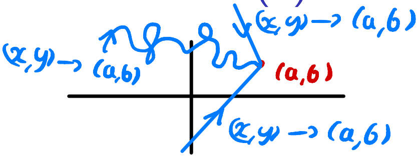
Examples of continuous functions:

- Polynomials
- sin, cos, exponential
- Rational functions (on their domain)
- Log functions (on their domain)

Ex of rational function:

$$\frac{x^2+5}{7x^5-3x^2+4}$$

Continuity for functions of 2 variables (1)



Limit: The assertion

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

means that $f(x,y)$ can be made as close to L as we wish
$\hookrightarrow$ by making (x,y) close to (a,b)

Remark: If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then
the limit should not depend on the way $(x,y) \to (a,b)$

Rmk: for $x \in \mathbb{R}$, there is just 1 way to have $x \to a$. In $\mathbb{R}^2$ there are plenty of ways to have $(x,y) \to (a,b)$

Continuity for functions of 2 variables (2)

Continuity: The function f is continuous at point a if

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

Examples of continuous functions:

- Polynomials $x^2y^5 - 4xy^2$
- sin, cos, exponential $\sin(x^2+y)$ e^{-x^2+y}
- Rational functions (on their domain) $\frac{x^5-y}{y^3+4x}$
- Log functions (on their domain) $\log(\sin(x-y))$

Logarithmic example (1)

Function:

$$\ln\left(\frac{1+y^2}{x^2}\right)$$

Problem: Continuity at point

$$(1,0)$$

Function : $f(x,y) = \ln\left(\frac{1+y^2}{x^2}\right)$

Continuity : f is continuous at any point (x,y) such that

$$\frac{1+y^2}{x^2} > 0 \quad \text{(and well-defined, i.e } x \neq 0 \text{)}$$

At point $(1,0)$

$$\frac{1+y^2}{x^2} = \frac{1+0^2}{1^2} = 1 > 0$$

Thus $\boxed{f \text{ continuous at } (1,0)}$

Logarithmic example (2)

Continuity: f is the log of a rational function
$\hookrightarrow$ Continuous wherever it is defined

Definition at point $(1,0)$: We have

$$f(1,0) = 0$$

This is well defined

Conclusion: f is continuous at $(1,0)$, that is

$$\lim_{(x,y)\to(1,0)} f(x,y) = f(1,0) = 0$$

Rational function example (1)

Function:

$$f(x, y) = \frac{y^2 - 4x^2}{2x^2 + y^2}$$

Problem: Continuity at point

$$(0, 0)$$

Function $f(x,y) = \dfrac{y^2 - 4x^2}{2x^2 + y^2}$

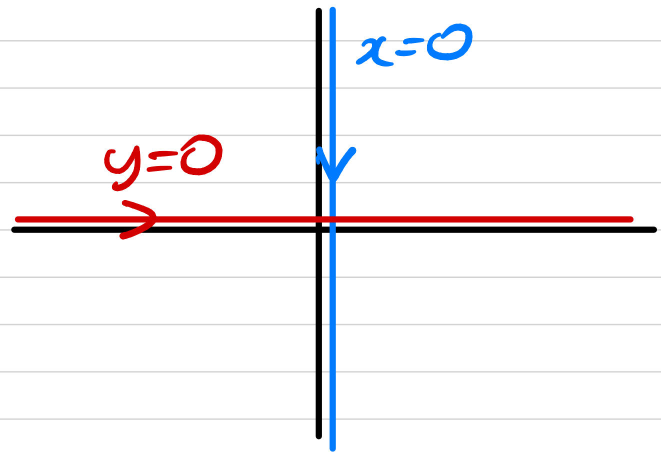
Continuity f is continuous at any point (x,y) s.t. $2x^2 + y^2 \neq 0$

Problem At $(0,0)$, we do have $2x^2 + y^2 = 0$. We cannot conclude

moreover at $(0,0)$, $f(x,y) = \frac{0}{0} \to$ undetermined

We are going to look at limits along different paths

Function $f(x,y) = \dfrac{y^2 - 4x^2}{2x^2 + y^2}$



Limit along line $x=0$

$$f(0,y) = \frac{y^2 - 0}{0 + y^2} = 1$$

Limit along line $y=0$

$$f(x,0) = \frac{0 - 4x^2}{2x^2 + 0} = -2$$

We get 2 different limits for 2 different paths $\Rightarrow$ $\boxed{f \text{ is not continuous at } (0,0)}$

Rational function example (2)

Continuity: f is a rational function
$\hookrightarrow$ Continuous wherever it is defined

Definition at point $(0,0)$: We have

$$f(0,0) = \frac{0}{0}$$

This is not well defined, therefore general result cannot be applied

Rational function example (3)

Two paths: We have

$$\begin{aligned} \text{Along } x = 0, \quad \lim_{(x,y)\to(0,0),\, x=0} \frac{y^2 - 4x^2}{2x^2 + y^2} &= 1 \\ \text{Along } y = 0, \quad \lim_{(x,y)\to(0,0),\, y=0} \frac{y^2 - 4x^2}{2x^2 + y^2} &= -2 \end{aligned}$$

We get 2 different limits

Conclusion:
f is not continuous at point $(0,0)$

Another rational function example (1)

Function:

$$f(x,y) = \frac{x^2 - y^2}{x + y}$$

Problem: Continuity at point

$$(0,0)$$

We will see :
$\frac{0}{0}$, but f is
continuous at $(0,0)$