Gradient

(i) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$ in $\mathbb{R}^2$

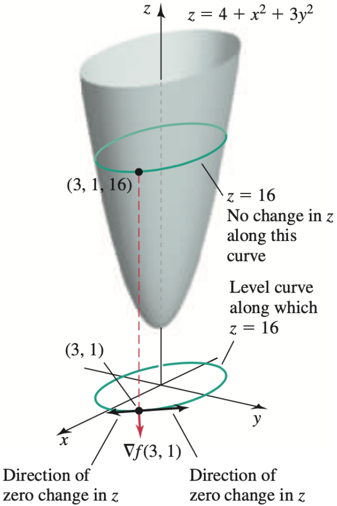
$\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z) \rangle$ in $\mathbb{R}^3$

Steepest ascent / descent At (x,y)

In the direction $\begin{cases} \nabla f(x,y) \in \mathbb{R}^2, \\ -\nabla f(x,y), \end{cases}$ rate $\begin{cases} |\nabla f(x,y)| \\ -|\nabla f(x,y)| \end{cases}$

Level curve In a direction $\perp \nabla f(x,y)$, f is locally "constant"

Example of steepest descent (6)



Gradient and level curves

Theorem 8.

Let

- f differentiable function at (x, y)
- Hypothesis: $\nabla f(a, b) \neq 0$

Then:

The line tangent to the level curve of f at (a, b)
is
orthogonal to $\nabla f(a, b)$

Hyperboloid example (1)

Function:

$$z = f(x,y) = \sqrt{1 + 2x^2 + y^2}$$

We know that from Thm 8.
we just want to verify this fact on this example

Questions:

1. Verify that the gradient at $(1,1)$ is orthogonal to the corresponding level curve at that point.
2. Find an equation of the line tangent to the level curve at $(1,1)$

Function $f(x,y) = z = \sqrt{1 + 2x^2 + y^2}$

① Gradient at $(x,y) = (1,1)$

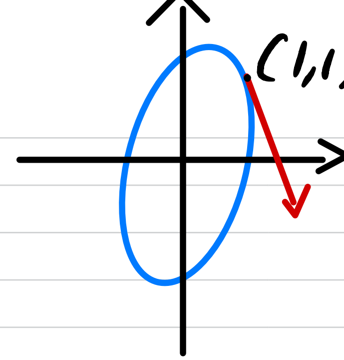
$\nabla f(x,y) = \langle f_x , f_y \rangle$

$= \left\langle \dfrac{2x}{(1+2x^2+y^2)^{1/2}} , \dfrac{y}{(1+2x^2+y^2)^{1/2}} \right\rangle$

At $(1,1)$, we get

$$\boxed{\nabla f(1,1) = \left\langle 1 , \frac{1}{2} \right\rangle}$$

Function $f(x,y) = z = \sqrt{1+2x^2+y^2}$



② Level curve

If $(x,y) = (1,1)$, then $z = \sqrt{4} = 2$. Thus the level curve going through $(x,y,z) = (1,1,2)$ is given by

$$(1+2x^2+y^2)^{\frac{1}{2}} = 2$$

$$\overset{^2}{\Leftrightarrow} \quad 1+2x^2+y^2 = 4 \quad \Leftrightarrow \quad \underbrace{2x^2+y^2-3}_{F(x,y)} = 0$$

ellipse

$F(x,y) = 0$

Tangent at (1,1)

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{4x}{2y}$$

At (1,1) we get $\boxed{\frac{dy}{dx} = -\frac{4}{2} = -2}$

Tangent vector We have seen $\frac{dy}{dx} = -2$

Thus a tangent vector is

$$\boxed{\vec{t} = \langle 1, -2 \rangle}$$

If $f'(t) = a$, then a tangent vector at $(t, f(t))$ is $\langle 1, a \rangle = \vec{t}$

③ Verifying Thm 8

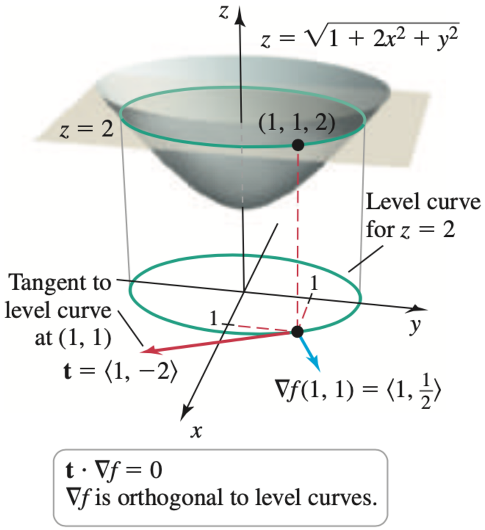
Thm 8 asserts that $\nabla f \perp \vec{t}$

Here $\nabla f(1,1) \cdot \vec{t} = \langle 1, \frac{1}{2} \rangle \cdot \langle 1, -2 \rangle$

$= 1 - 1 = 0$

Thus $\boxed{\nabla f(1,1) \perp \vec{t}}$

Hyperboloid example (2)



Hyperboloid example (3)

Point on surface:

Given by $(1, 1, 2) \Longrightarrow$ On level curve $z = 2$

Equation for level curve: Ellipse of the form

$$1 + 2x^2 + y^2 = 4 \quad \Longleftrightarrow \quad 2x^2 + y^2 = 3$$

Implicit derivative:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{F_x}{F_y} = -\frac{2x}{y}$$

Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x}(1) = -2$$

Hyperboloid example (4)

Tangent vector: Proportional to

$$\mathbf{t} = \langle 1, -2 \rangle$$

Gradient of f:

$$\nabla f(x, y) = \left\langle \frac{2x}{\sqrt{1 + 2x^2 + y^2}}, \frac{y}{\sqrt{1 + 2x^2 + y^2}} \right\rangle$$

Thus

$$\nabla f(1, 1) = \left\langle 1, \frac{1}{2} \right\rangle$$

Orthogonality: We have

$$\mathbf{t} \cdot \nabla f(1, 1) = 0$$

Hyperboloid example (5)

Tangent line to level curve: At point $(1,1)$ we get

$$f_x(1,1)(x-1) + f_y(1,1)(y-1) = 0,$$

that is

$$y = -2x + 3$$

Generalization to 3 variables

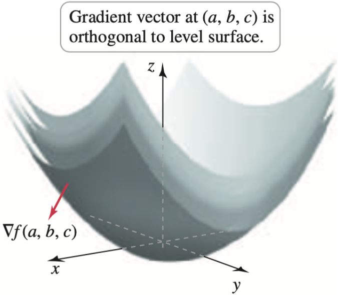
Situation:

- We have a function $w = f(x, y, z)$
- Each w_0 results in a level surface

$$f(x, y, z) = w_0$$

Gradient on level surface:

Will be $\perp$ to level surface



Example of tangent plane (1)

Function:

$$f(x, y, z) = xyz$$

Gradient:

$$\nabla f(x, y, z) = \langle yz, xz, xy \rangle$$

Thus

$$\nabla f(1, 2, 3) = \langle 6, 3, 2 \rangle$$

Example of tangent plane (2)

Plane tangent to level surface:

$$\langle 6, 3, 2\rangle \cdot \langle x - 1, y - 2, z - 3\rangle = 0$$

We get

$$6x + 3y + 2z = 18$$

Outline

Linear approximation for functions of 1 variable

Situation: We have

- $y = f(x)$

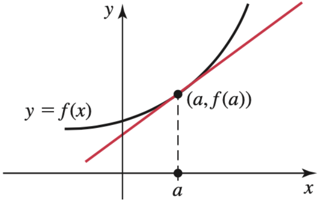
Tangent vector at a:

$$\boxed{\mathbf{t} = (1, f'(a))}$$

Linear approximation: Near a we have

$$f(x) \simeq f(a) + f'(a)(x - a)$$

Taylor's approximation (1st order)



Tangent plane for $F(x, y, z) = 0$

Definition 9.

Let $F(x, y, z)$ be such that

- F differentiable at $P(a, b, c)$
- $\nabla F \neq 0$
- S is the surface $F(x, y, z) = 0$

Then the tangent plane at (a, b, c) is given by

$$F_x(a, b, c)(x - a) + F_y(a, b, c)(y - b) + F_z(a, b, c)(z - c) = 0$$

Rmk We can also write the eq. as

$$\nabla F(a, b, c) \cdot \langle x-a, y-b, z-c \rangle = 0$$

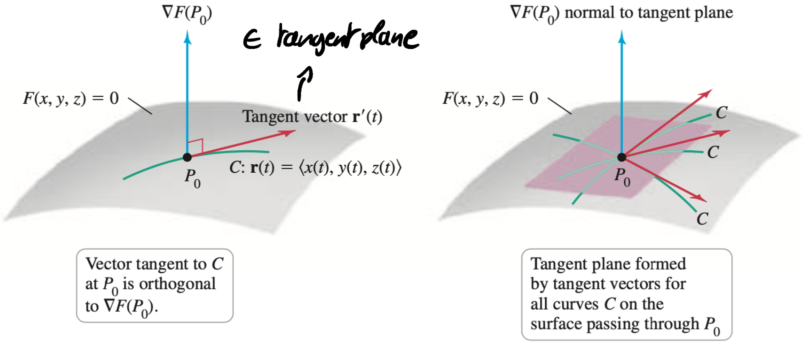
∇F is normal to the tangent plane

Interpretation of tangent plane

Tangent plane as collection of tangent vectors: If

- S is the surface $F(x, y, z) = 0$
- $\mathbf{r}$ is a curve passing through (a, b, c) at time t

Then $\mathbf{r}'(t) \in$ tangent plane



Example of tangent plane (1)

Surface: Ellipsoid of the form

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$$

check: $F(0, 4, 3/5) = 0$

Questions:

1. Tangent plane at $(0, 4, \frac{3}{5})$
2. What tangent planes to S are horizontal?

Function $F(x,y,z) = \frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1$

Surface $F(x,y,z) = 0$

Point $P(0, 4, 3/5)$

Q: When is the tangent plane horizontal?

↳ should be $(0, 0, \pm 1)$

Gradient

$$\nabla F(x,y,z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle$$

At point $P(0, 4, 3/5)$

$$\nabla F(0, 4, 3/5) = \left\langle 0, \frac{8}{25}, \frac{6}{5} \right\rangle$$

Tangent plane

$$0 \times (x-0) + \frac{8}{25}(y-4) + \frac{6}{5}(z - 3/5)$$

$$\Leftrightarrow \cdots \Leftrightarrow \quad \boxed{4y + 15z = 25}$$

Gradient

$$\nabla F(x,y,z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle$$

Horizontal planes when

$\nabla F(x,y,z)$ is in the direction of $\langle 0,0,1 \rangle$

i.e

$$\frac{2x}{9} = 0 \quad \text{and} \quad \frac{2y}{25} = 0 \iff x = 0, \; y = 0$$

On the surface $\frac{x^2}{9} + \frac{y^2}{25} + z^2 - 1 = 0$, this gives

$$z^2 = 1 \iff z = \pm 1$$

Points with horizontal planes: $\boxed{(0, 0, \pm 1)}$

Example of tangent plane (2)

Gradient: We have

$$\nabla F(x,y,z) = \left\langle \frac{2x}{9}, \frac{2y}{25}, 2z \right\rangle$$

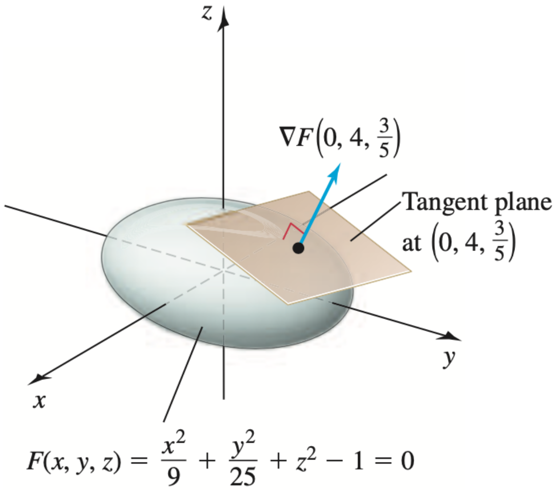
Thus

$$\nabla F(0,4,\frac{3}{5}) = \left\langle 0, \frac{8}{25}, \frac{6}{5} \right\rangle$$

Tangent plane:

$$4y + 15z = 25$$

Example of tangent plane (3)



Example of tangent plane (4)

Horizontal plane: When the normal vector is of the form

$$\mathbf{n} = (0, 0, c), \quad \text{with} \quad c \neq 0$$

Horizontal tangent plane: When the normal vector ∇F is of the form

$$\nabla F(x, y, z) = (0, 0, c) \quad \Longleftrightarrow \quad F_x = 0, \; F_y = 0, \; F_z \neq 0$$

Solutions: Horizontal tangent plane for

$$(0, 0, 1) \quad \text{and} \quad (0, 0, -1)$$