Outline

Recap We have seen how to compute

(i) max $f(x,y)$ for $(x,y) \in \mathbb{R}^2$

- critical points
- Second derivative test (max, min, saddle)

(ii) max $f(x,y)$ for $(x,y) \in R$, closed region

- critical points in R
- Study the function on the boundary of R (↑ simple shape)

(iii) Today: max $f(x,y)$ under a constraint $g(x,y) = 0$

Global aim

Objective function:

$$f = f(x, y)$$

Constraint: We are moving on a curve of the form

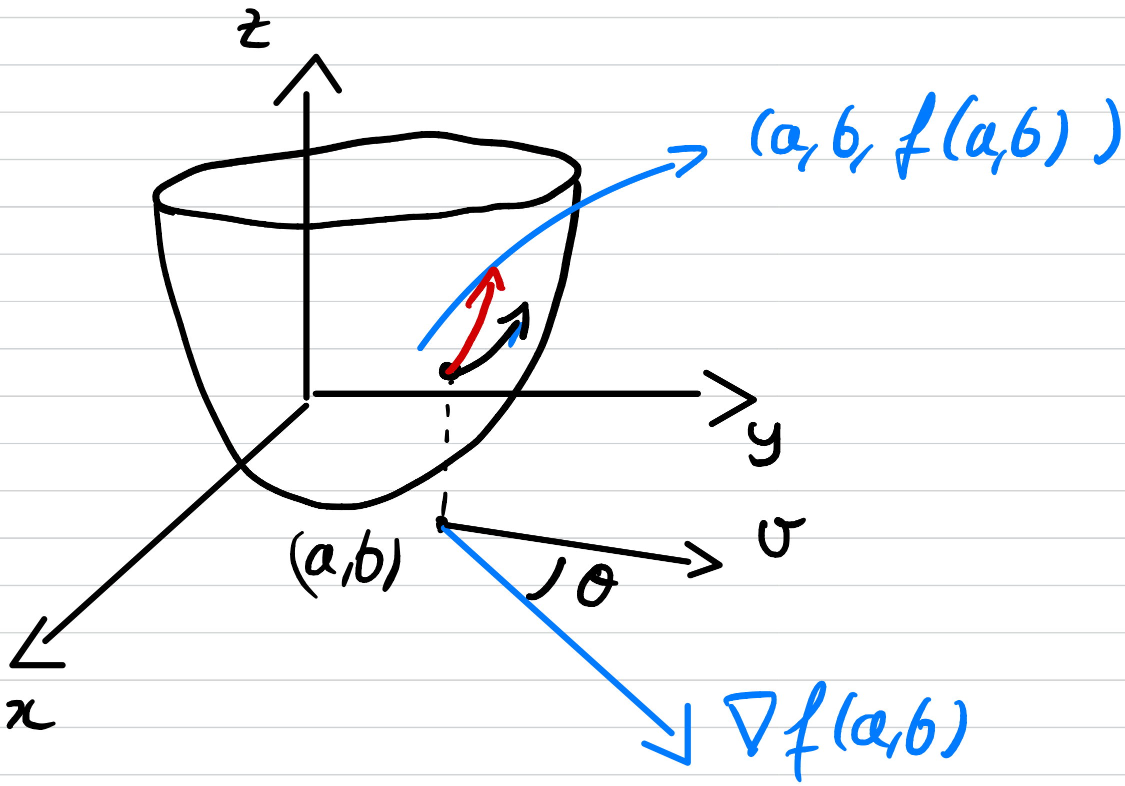
$$g(x, y) = 0$$

Optimization problem: Find

$$\max f(x, y), \quad \text{subject to} \quad g(x, y) = 0$$

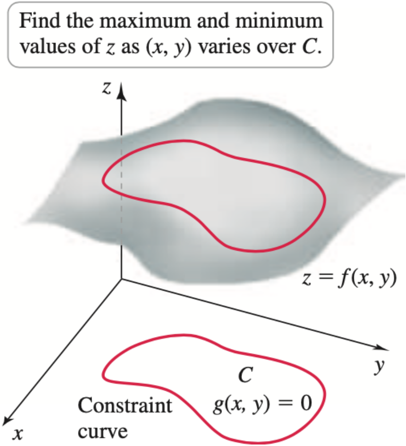
Gradients and ascents

We have seen that $\nabla f(x,y)$ is direction of max ascent.



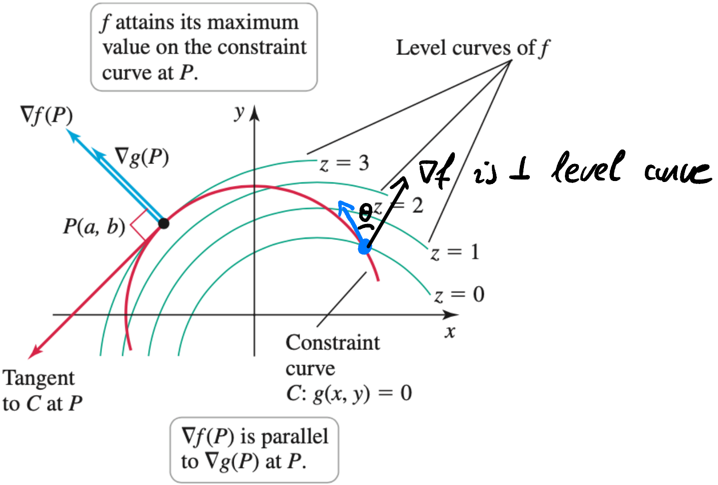
Rmk If we move along a direction $\vec{v}$ s.t. if θ = angle between $\vec{v}$ and $\nabla f(a,b)$ satisfies $\cos(\theta) > 0$, we will still move up (but not at an optimal rate)

Optimization problem: illustration



Lagrange multipliers intuition (1)

Rmk Tangent to red curve is ⊥ ∇g



Lagrange multipliers intuition (2)

Some observations from the picture:

1. $P(a,b)$ on the level curve of f
   $\Longrightarrow$ Tangent to level curve $\perp \nabla f(a,b)$
2. $P(a,b)$ gives a maximum of f on curve C
   $\Longrightarrow$ Tangent to level curve $\parallel$ Tangent to constraint curve
3. Constraint is $g(x,y) = 0$
   $\Longrightarrow$ Tangent to constraint curve $\perp \nabla g(a,b)$

Conclusion (Lagrange's idea):
At the maximum under constraint we have

$$\nabla f(a,b) \parallel \nabla g(a,b)$$

Lagrange multipliers procedure

Optimization problem: Find

$$\max f(x,y), \quad \text{subject to} \quad g(x,y)=0$$

Recipe:

1. Find the values of x, y and λ such that ($v_1 \parallel v_2$ iff $v_1 = \lambda v_2$ for $\lambda \in \mathbb{R}$)

$$\nabla f(x,y) = \lambda \nabla g(x,y), \quad \text{and} \quad g(x,y)=0$$

2. Select the largest and smallest corresponding function values.
   $\hookrightarrow$ We get absolute max and min values of f s.t constraint.

Example of Lagrange multipliers (1)

Optimization problem: Find

$$\max f(x,y), \quad \text{with} \quad f(x,y) = x^2 + y^2 + 2,$$

subject to the constraint

$$g(x,y) = x^2 + xy + y^2 - 4 = 0$$

Function $f(x,y) = x^2 + y^2 + 2$

Constraint $g(x,y) = 0$, $g(x,y) = x^2 + xy + y^2 - 4$

① $\nabla f(x,y) = \langle 2x, 2y \rangle$

$\nabla g(x,y) = \langle 2x + y, x + 2y \rangle$

We wish to find λ, x, y s.t.

$\nabla f(x,y) = \lambda \nabla g(x)$

$$\Leftrightarrow \begin{cases} 2x = \lambda(2x + y) \\ 2y = \lambda(x + 2y) \end{cases} \Leftrightarrow \boxed{\begin{cases} (2\lambda - 2)x + \lambda y = 0 \\ \lambda x + (2\lambda - 2)y = 0 \end{cases}}$$

Linear system of 2 equations

System

$$\begin{cases} (2\lambda-2)x + \lambda y = 0 \\ \lambda x + (2\lambda-2)y = 0 \end{cases}$$

not a valid point, since $g(0,0) = -4 \neq 0$

Generally speaking this system will have $(0,0)$ as the unique solution, unless

"det" $= 0 \iff (2\lambda-2)(2\lambda-2) - \lambda^2 = 0$

$\iff 4\lambda^2 - 8\lambda + 4 - \lambda^2 = 0$

$\iff 3\lambda^2 - 8\lambda + 4 = 0$

Two solutions: $\boxed{\lambda = 2 \quad \text{or} \quad \lambda = \frac{2}{3}}$

Case $\alpha = 2$

System

$$\begin{cases} (2\alpha-2)x + \alpha y = 0 \\ \alpha x + (2\alpha-2)y = 0 \end{cases}$$

System becomes

$$2x + 2y = 0 \iff y = -x$$

We should now check that $g(x,-x) = 0$

We have

$$g(x,-x) = x^2 + x \cdot (-x) + (-x)^2 - 4$$

$$= x^2 - 4$$

$f(2,-2) = f(-2,2) = 10$

Thus $g(x,-x) \overset{?}{=} 0 \iff x = \pm 2$

We get 2 points of interest: $(2,-2)$, $(-2,2)$

Case $d = 2/2$ Same kind of computations

$$f\left(\pm \frac{2}{\sqrt{3}}, \pm \frac{2}{\sqrt{3}}\right) = \frac{14}{3} < 10$$

Example of Lagrange multipliers (2)

Computing the gradients: We get

$$\nabla f(x,y) = \langle 2x, 2y \rangle, \qquad \nabla g(x,y) = \langle 2x+y, x+2y \rangle$$

Lagrange constraint 1:

$$f_x = \lambda g_x \quad \Longleftrightarrow \quad 2x = \lambda\,(2x+y) \tag{1}$$

Lagrange constraint 2:

$$f_y = \lambda g_y \quad \Longleftrightarrow \quad 2y = \lambda\,(x+2y) \tag{2}$$

Example of Lagrange multipliers (3)

System for x, y: Gathering (1) and (2), we get

$$2(\lambda - 1)x + \lambda y = 0, \qquad \lambda x + 2(\lambda - 1)y = 0$$

This has solution $(0, 0)$ unless

$$\lambda = 2, \quad \text{or} \quad \lambda = \frac{2}{3}$$

Example of Lagrange multipliers (4)

Case $\lambda = 2$: We get $x = -y$. The constraint

$$x^2 + xy + y^2 - 4 = 0$$

becomes

$$x^2 - 4 = 0$$

Solutions:

$$x = 2, \quad \text{and} \quad x = -2$$

Corresponding values of f: We have

$$f(2, -2) = f(-2, 2) = 10$$

Example of Lagrange multipliers (5)

Case $\lambda = \frac{2}{3}$: We get $x = y$. The constraint

$$x^2 + xy + y^2 - 4 = 0$$

becomes

$$3x^2 - 4 = 0$$

Solutions:

$$x = \frac{2}{\sqrt{3}}, \quad \text{and} \quad x = -\frac{2}{\sqrt{3}}$$

Corresponding values of f: We have

$$f\left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right) = f\left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right) = \frac{14}{3}$$

Example of Lagrange multipliers (6)

Absolute maximum:

For function f on the curve C defined by $g = 0$,

$$\text{Maximum} = 10, \quad \text{obtained for} \quad (2, -2)\,,\ (-2, 2)$$

Absolute minimum:

For function f on the curve C defined by $g = 0$,

$$\text{Minimum} = \frac{14}{3}, \quad \text{obtained for} \quad \left(\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right),\ \left(-\frac{2}{\sqrt{3}}, -\frac{2}{\sqrt{3}}\right)$$

Example of Lagrange multipliers (7)