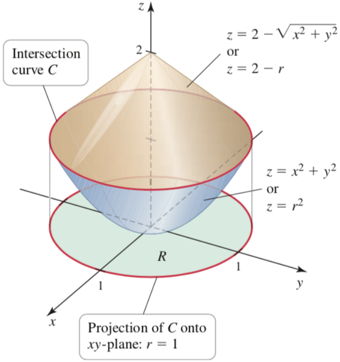
Region bounded by 2 surfaces (1)

Problem: Find the volume bounded by

- Paraboloid $z = x^2 + y^2$
- Cone $z = 2 - \sqrt{x^2 + y^2}$



Intersection of cone & paraboloid. when

$$z = \overbrace{x^2+y^2}^{r^2} = 2 - \overbrace{\sqrt{x^2+y^2}}^{r}$$

In polar coordinates, we get

$$r^2 = 2 - r \iff r^2 + r - 2 = 0$$

product of roots

Roots: 1 (trivial) and -2 not a physical solution

Thus intersection is

$$x^2+y^2 = 1$$

Integration region :

$$R = \{ x^2+y^2 \le 1 \}$$

Volume

$\rightarrow$ awful expression

$$V = \iint_{\{x^2+y^2 \leq 1\}} \left(2 - \sqrt{x^2+y^2}\right) - (x^2+y^2) \, dx \, dy$$

In polar

$= \{x, y \text{ s.t. } x^2+y^2 \leq 1\}$

$$R = \{0 \leq r \leq 1 \quad , \quad 0 \leq \theta \leq 2\pi\}$$

$$V = \int_0^{2\pi} \int_0^1 (2 - r - r^2) \, r \, dr \, d\theta$$

$$= 2\pi \int_0^1 (2r - r^2 - r^3) \, dr$$

$$= 2\pi \quad r^2 - \frac{1}{3} r^3 - \frac{1}{4} r^4 \Big|_0^1$$

$$= 2\pi \left(1 - \frac{1}{3} - \frac{1}{4}\right)$$

$$\boxed{V = \frac{5\pi}{6}}$$

Region bounded by 2 surfaces (2)

Expression as an integral: We have

$$V = \int\int_R \left(2 - \sqrt{x^2+y^2} - \left(x^2+y^2\right)\right) \mathrm{d}A,$$

Integration region: The region R is defined as

$R \equiv$ region with boundary C
given as intersection of paraboloid and cone

Region bounded by 2 surfaces (3)

Definition of C: Write

$$x^2 + y^2 = 2 - \sqrt{x^2 + y^2}$$

In polar coordinates in the plane, this gives

$$r^2 + r - 2 = 0$$

Physical solution to the equation: Circle in the xy-plane,

$$x^2 + y^2 = 1$$

Region bounded by 2 surfaces (4)

Volume in polar coordinates: We have

$$\begin{aligned} V &= \int\int_R \left(2 - \sqrt{x^2+y^2} - \left(x^2+y^2\right)\right) \mathrm{d}A \\ &= \int_0^{2\pi}\int_0^1 \left(2 - r - r^2\right) r \, \mathrm{d}r\mathrm{d}\theta \\ &= \int_0^{2\pi} \left(r^2 - \frac{1}{3}r^3 - \frac{1}{4}r^4\right)\Big|_0^1 \end{aligned}$$

Thus

$$V = \frac{5\pi}{6}$$

Outline

Triple integral approximation (1)

Aim: For $w = f(x, y, z)$, compute
$\hookrightarrow$ The integral of f on a domain $D \subset \mathbb{R}^3$

Approximation:

- Divide D into boxes centered at (x_k^*, y_k^*, z_k^*)
- Area of each box: $\Delta V_k = \Delta x_k \Delta y_k \Delta z_k$

Then we obtain the integral as a limit

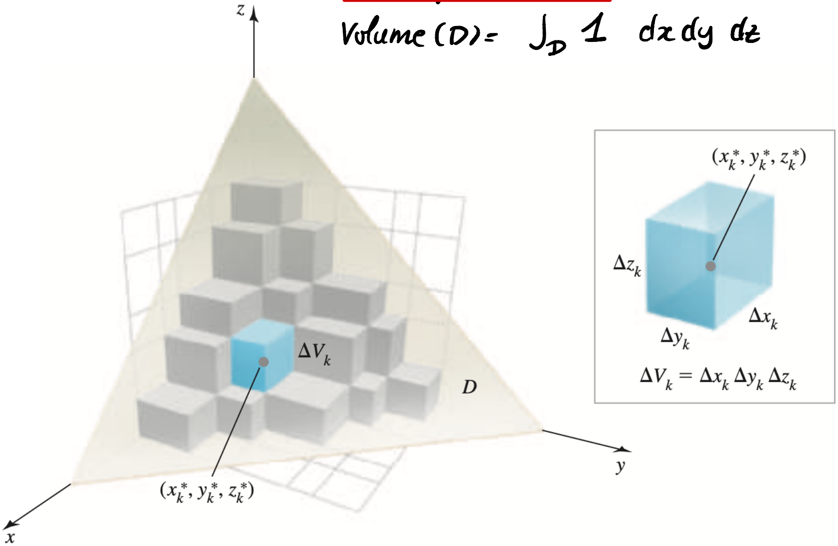
$$\int\int\int_D f(x, y, z)\, \mathrm{d}V = \sum_{k=1}^{n} f(x_k^*, y_k^*, z_k^*)\, \Delta V_k$$

$\lim_{n \to \infty}$

Triple integral approximation (2)

To compute volumes:

Volume $(D) = \int_D 1 \, dx\, dy\, dz$



Choosing the order of integration

Theorem 3.

Let

- f continuous function on $\mathbb{R}^3$
- D domain of the form

fixed bounds

$$D = \Big\{(x, y, z);\, a \le x \le b,\, g(x) \le y \le h(x),\, G(x, y) \le z \le H(x, y)\Big\}$$

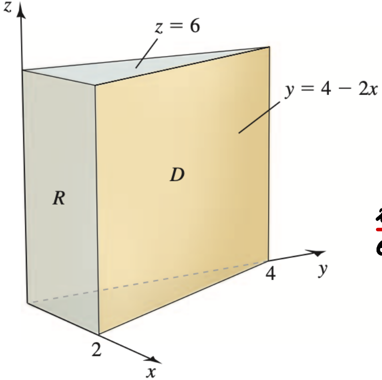
Then we have

$$\int\int\int_D f(x, y, z)\,\mathrm{d}V = \int_a^b \int_{g(x)}^{h(x)} \int_{G(x,y)}^{H(x,y)} f(x, y, z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

Volume of a prism (1)

Problem: Compute the volume of a prism D

- In the first octant $x \geq 0, y \geq 0, z \geq 0$
- Bounded by planes $y = 4 - 2x$ and $z = 6$



Rmk If x, z are fixed, we have

$0 \leq y \leq 4 - 2x$

Rmk

xz-trace given by

$R = \{0 \leq x \leq 2, \; 0 \leq z \leq 6\}$

$x = 2$ is given by eq $\begin{cases} y = 4 - 2x \\ y = 0 \end{cases}$

$\Rightarrow 4 - 2x = 0$

$\Rightarrow x = 2$

Integration : Due to the shape of D, will do

$$V = \int_0^6 \int_0^2 \int_0^{4-2x} 1 \; dy\, dx\, dz$$

$$= \int_0^6 \int_0^2 \overbrace{(4-2x)}^{\text{does not depend on } z} \, dx \; dz$$

$$= 6 \int_0^2 (4-2x)\,dx$$

$$= 6 \left(4x - x^2 \Big|_0^2 \right)$$

$$V = 24$$

Volume of a prism (2)

Strategy of integration:

1. Upper surface: $y = 4 - 2x$
2. Base: $y = 0$, $0 \leq x \leq 2$, $0 \leq z \leq 6$
   $\hookrightarrow$ We get a rectangle (easy surface)

Conclusion: an easy way to integrate is in this order,

$$\mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z$$

Volume of a prism (3)

Integral computation: We get

$$
\begin{aligned}
V &= \int_0^6 \int_0^2 \int_0^{4-2x} \mathrm{d}y \, \mathrm{d}x \, \mathrm{d}z \\
&= \int_0^6 \int_0^2 (4-2x) \, \mathrm{d}x \, \mathrm{d}z
\end{aligned}
$$

Thus we get

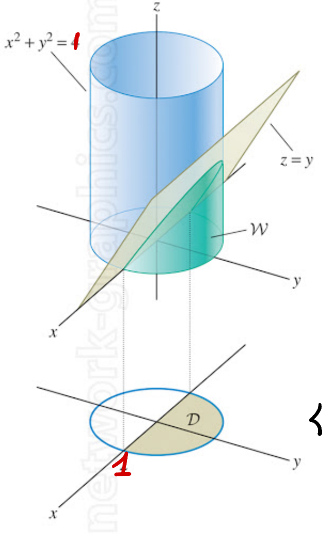
$$V = 24$$

Volume of a wedge (1)

Problem:

Compute the volume of the cylinder $C : x^2 + y^2 = 1$ delimited by

- xy-plane $z = 0$
- Plane $z = y$
- $z \geq 0$



Rmk If x, y are fixed, then we consider $0 \leq z \leq y$. In particular, $y \geq 0$

Order: $dz\, dy\, dx$

xy-trace: $x^2 + y^2 \leq 1$ and $y \geq 0$

half disk: $\{-1 \leq x \leq 1,\ 0 \leq y \leq \sqrt{1-x^2}\}$

Integral

$$V = \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{y} 1 \, dz \, dy \, dx$$

should be cylindrical coordinates in $\mathbb{R}^3$

$$= \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} y \, dy \, dx$$

polar coordinates is an option

$$= \int_{-1}^{1} \left(\frac{1}{2} y^2 \Big|_{0}^{\sqrt{1-x^2}} \right) dx$$

$$= \frac{1}{2} \int_{-1}^{1} (1-x^2) \, dx$$

$$= \frac{1}{2} \times x - \frac{1}{3} x^3 \Big|_{-1}^{1}$$

$$\boxed{V = \frac{2}{3}}$$