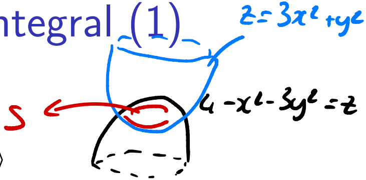
Stokes theorem for a surface integral (1)



Vector field:

$$\mathbf{F} = \langle -y, x, z \rangle$$

Surface: Part of a paraboloid within another paraboloid

$$S: \quad z = 4 - x^2 - 3y^2 \quad \bigcap \quad \left\{ z \geq 3x^2 + y^2 \right\},$$

with $\mathbf{n}$ pointing upward

Corresponding curve:
Intersection of the 2 paraboloids

Problem: In order to avoid a parametrization of S
$\hookrightarrow$ Evaluate $\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S$ as a line integral

Curve $(z = 4 - x^2 - 3y^2) \cap (z = 3x^2 + y^2)$

Thus $4 - x^2 - 3y^2 = 3x^2 + y^2$

$\Leftrightarrow \quad x^2 + y^2 = 1$ (circle in xy-plane)

Parametric form in xy-plane

$C_{xy} : \{ \langle \cos(t), \sin(t) \rangle \; ; \; 0 \leq t \leq 2\pi \}$

Corresponding curve in $\mathbb{R}^3$ $\vec{r}\,(t)$

$C : \{ \langle \cos(t), \sin(t), 3\cos^2(t) + \sin^2(t) \rangle ; \; 0 \leq t \leq 2\pi \}$

$r'(t)$

$C : \{ \langle \cos(t), \sin(t), 3\cos^2(t) + \sin^2(t) \rangle ; \quad 0 \le t \le 2\pi \}$

Line integral $\quad \vec{F} = \langle -y, x, z \rangle$

$\vec{r}'(t) = \langle -\sin(t), \cos(t), -6\cos(t)\sin(t) + 2\sin(t)\cos(t) \rangle$

$\vec{r}'(t) = \langle -\sin(t), \cos(t), -4\cos(t)\sin(t) \rangle$

Thus

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -\sin(t), \cos(t), 3\cos^2(t) + \sin^2(t) \rangle \cdot \langle -\sin(t), \cos(t), -4\cos(t)\sin(t) \rangle \, dt$$

$$= \int_0^{2\pi} \left(\sin^2(t) + \cos^2(t) - 12\cos^3(t)\sin(t) - 4\sin^3(t)\cos(t) \right) dt$$

$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \Big(\overbrace{\sin^2(t) + \cos^2(t)}^{1} - 12\cos^3(t)\sin(t) - 4\sin^3(t)\cos(t) \Big)\, dt$$

$$= 2\pi - 12 \int_0^{2\pi} \cos^3(t)\sin(t)\, dt - 4 \int_0^{2\pi} \sin^3(t)\cos(t)\, dt$$

$u^3 u'$ with $u = \sin(t)$

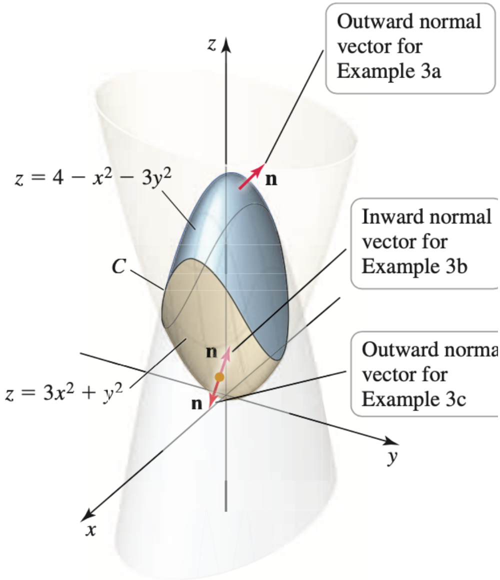
$$= 2\pi - \underbrace{\sin^4(t)\Big|_0^{2\pi}}_{0} + 3 \underbrace{\cos^4(t)\Big|_0^{2\pi}}_{0}$$

Thus $\int_C \vec{F} \cdot d\vec{r} = 2\pi$

and by Stokes,

$$\boxed{\iint_S \operatorname{Curl}(\vec{F}) \cdot \vec{n}\, dS = \int_C \vec{F} \cdot d\vec{r} = 2\pi}$$

Stokes theorem for a surface integral (2)



Stokes theorem for a surface integral (3)

Equation for C: For the intersection of the paraboloids we get

$$4 - x^2 - 3y^2 = 3x^2 + y^2 \quad \Longleftrightarrow \quad x^2 + y^2 = 1$$

Parametric equation for x, y: We choose

$$x = \cos(t), \quad y = \sin(t), \qquad 0 \le t \le 2\pi,$$

which is compatible with the orientation of S

Parametric equation for C: Writing $z = 3x^2 + y^2$ we get

$$\mathbf{r}(t) = \left\langle \cos(t), \sin(t), 3\cos^2(t) + \sin^2(t) \right\rangle$$

Stokes theorem for a surface integral (4)

Parametric equation for $\mathbf{F}$: Along C we have

$$\mathbf{F} = \langle -y, x, z \rangle = \left\langle -\sin(t), \cos(t), 3\cos^2(t) + \sin^2(t) \right\rangle$$

Dot product: We have

$$\mathbf{F}(t) \cdot \mathbf{r}'(t) = \left\langle -\sin(t), \cos(t), 3\cos^2(t) + \sin^2(t) \right\rangle \cdot \langle -\sin(t), \cos(t), -4\cos(t)\sin(t) \rangle$$

We get

$$\mathbf{F}(t) \cdot \mathbf{r}'(t) = 1 - 12\cos^3(t)\sin(t) - 4\sin^3(t)\cos(t)$$

Stokes theorem for a surface integral (5)

Line integral:

$$\oint_C \mathbf{F} \cdot \mathbf{T} \, \mathrm{d}s = \oint_C \mathbf{F}(t) \cdot \mathbf{r}'(t) \, \mathrm{d}t$$
$$= \int_0^{2\pi} \mathrm{d}t$$

Thus we get

$$\oint_C \mathbf{F}(t) \cdot \mathbf{r}'(t) \, \mathrm{d}t = 2\pi$$

Stokes theorem for a surface integral (6)

Computation of the surface integral: We have

$$\int\int_S \mathrm{Curl}(\mathbf{F}) \cdot \mathbf{n} \, \mathrm{d}S = \oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi$$

Remark:
We get a positive flux (normal is oriented like $\mathrm{Curl}(\mathbf{F})$)

Outline

The main theorem

Theorem 24.

Consider

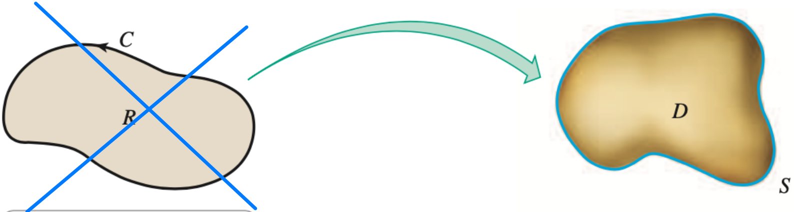
- A simply connected region D in $\mathbb{R}^3$
- D is enclosed by an oriented surface S
- $\mathbf{F} = \langle f, g, h \rangle$ vector field in $\mathbb{R}^3$
- $\mathrm{Div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$ $= f_x + g_y + h_z$

Then we have

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int\int\int_D \mathrm{Div}(\mathbf{F}) \, \mathrm{d}V$$

From Green to divergence

From 2-d to 3-d:



Flux form of Green's Theorem:

$$\oint_C \mathbf{F} \cdot \mathbf{n}\, ds = \iint_R \operatorname{div} \mathbf{F}\, dA$$

Divergence Theorem:

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \operatorname{div} \mathbf{F}\, dV$$

Verifying divergence theorem (1)

Vector field:

$$\mathbf{F} = \langle x, y, z \rangle$$

Surface: Sphere S of the form

$$S : \quad x^2 + y^2 + z^2 = a^2$$

Corresponding domain: Ball of the form

$$B = \left\{ x^2 + y^2 + z^2 \leq a^2 \right\}$$

Strategy: (i) Compute $\iiint_B \operatorname{div}(\vec{F})\, dV$

(ii) Compute $\iint_S \vec{F} \cdot \vec{n}\, dS$

(iii) Check (i) = (ii)

Problem:
Verify divergence theorem in this context

Divergence

$$\vec{F} = \langle \overset{f}{x}, \overset{g}{y}, \overset{h}{z} \rangle$$

$$\text{Div}(\vec{F}) = f_x + g_y + h_z = 1 + 1 + 1 = 3$$

(we have seen: $\text{Div}(\vec{F}) > 0 \Rightarrow \vec{F}$ induces a flux outward)

Divergence integral

$$\iiint_B \underbrace{\text{Div}(\vec{F})}_{=3}\, dV = 3 \iiint_B dV$$

→ Ball, radius $= a$

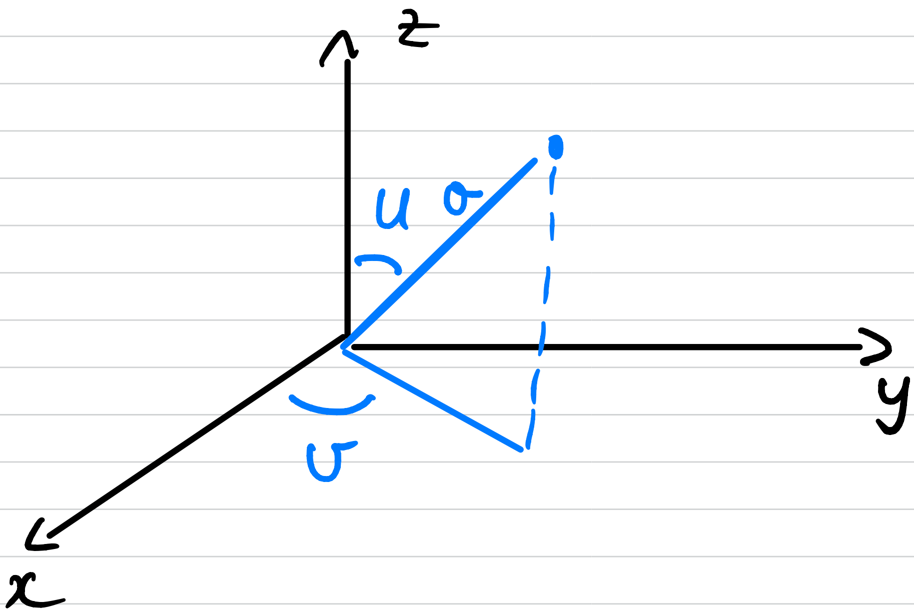
$$= 3 \, \text{Vol}(B)$$

$$= 3 \times \frac{4}{3} \pi a^3$$

$$\boxed{\iiint_B \text{Div}(\vec{F})\, dV = 4\pi a^3}$$

Parameterization of S   In spherical, $u = \varphi$, $v = \theta$

$$S = \{ \langle a \sin(u)\cos(v),\ a \sin(u)\sin(v),\ a\cos(u) \rangle ;$$
$$0 \leq u \leq \pi, \quad v \leq \theta \leq 2\pi \}$$



$$\vec{r} = \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v),\ a\cos(u)\rangle,$$

$$\vec{r}'_u = \langle a\cos(u)\cos(v),\ a\cos(u)\sin(v),\ -a\sin(u)\rangle$$

$$\vec{r}'_v = \langle -a\sin(u)\sin(v),\ a\sin(u)\cos(v),\ 0\rangle$$

$$\vec{n} = \vec{r}'_u \times \vec{r}'_v$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos(u)\cos(v) & a\cos(u)\sin(v) & -a\sin(u) \\ -a\sin(u)\sin(v) & a\sin(u)\cos(v) & 0 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ a\cos(u)\cos(v) & a\cos(u)\sin(v) \\ -a\sin(u)\sin(v) & a\sin(u)\cos(v) \end{matrix}$$

$$\vec{i}\,(0 + a^2\sin^2(u)\cos(v))$$

$$\vec{j}\,(a^2\sin^2(u)\sin(v) - 0)$$

$$\vec{k}\,(a^2\sin(u)\cos(u)\cos^2(v) + a^2\sin(u)\cos(u)\sin^2(v))$$

$$\boxed{\vec{n} = a^2\langle \sin^2(u)\cos(v),\ \sin^2(u)\sin(v),\ \sin(u)\cos(u)\rangle}$$

$\vec{F} = \langle x, y, z \rangle$

$\vec{n} = a^2 \langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u) \rangle$

Surface integral

$$\iint_S \vec{F} \cdot \vec{n} \, dS$$

$$= a^2 \int_0^{\pi} \int_0^{2\pi} \langle \overbrace{a\sin(u)\cos(v)}^{x}, \overbrace{a\sin(u)\sin(v)}^{y}, \overbrace{a\cos(u)}^{z} \rangle \cdot \langle \sin^2(u)\cos(v), \sin^2(u)\sin(v), \sin(u)\cos(u) \rangle \, dv \, du$$

$$= a^3 \int_0^{\pi} \int_0^{2\pi} \left(\sin^3(u)\cos^2(v) + \sin^3(u)\sin^2(v) + \sin(u)\cos^2(u) \right) dv \, du$$

$$= a^3 \int_0^{\pi} \int_0^{2\pi} \left(\sin^3(u) + \sin(u)\cos^2(u) \right) dv \, du$$

$$= 2\pi a^3 \int_0^{\pi} \sin(u) \left(\sin^2(u) + \cos^2(u) \right) du$$

$$= 2\pi a^3 \int_0^{\pi} \sin(u) \, du = -2\pi a^3 \cos(u) \Big|_0^{\pi}$$

$$= 4\pi a^3$$

Here $\boxed{\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \operatorname{div}(\vec{F}) \, dV}$

Verifying divergence theorem (2)

Expression for $\mathrm{Div}(\mathbf{F})$: We have

$$\mathrm{Div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$$

Computation: We find

$$\mathrm{Div}(\mathbf{F}) = 3$$

Verifying divergence theorem (3)

Volume integral: We have

$$\begin{aligned}\int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V &= 3\int\int\int_D \mathrm{d}V \\ &= 3\,\mathrm{Vol}(D)\end{aligned}$$

Thus

$$\int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V = 4\pi a^3$$

Verifying divergence theorem (4)

Parametrization of S: We take

$$\mathbf{r}(u,v) = \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v), a\cos(u)\rangle\,, \quad (u,v)\in R,$$

with

$$R = \{0 \le u \le \pi,\ 0 \le v \le 2\pi\}$$

Verifying divergence theorem (5)

Normal vector: We have

$$\mathbf{t}_u = \langle a\cos(u)\cos(v),\ a\cos(u)\sin(v), -a\sin(u)\rangle\,,$$
$$\mathbf{t}_v = \langle -a\sin(u)\sin(v),\ a\sin(u)\cos(v), 0\rangle\,,$$

Thus

$$\mathbf{t}_u \times \mathbf{t}_v = \left\langle a^2\sin^2(u)\cos(v),\ a^2\sin^2(u)\sin(v), a^2\cos(u)\sin(u)\right\rangle$$

Verifying divergence theorem (6)

Surface integral: We get

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = \int\int_R \mathbf{F}\cdot(\mathbf{t}_u \times \mathbf{t}_v)\,\mathrm{d}A$$

$$= \int_0^{2\pi}\int_0^{\pi} \langle a\sin(u)\cos(v),\ a\sin(u)\sin(v), a\cos(u)\rangle$$

$$\cdot\left\langle a^2\sin^2(u)\cos(v),\ a^2\sin^2(u)\sin(v), a^2\cos(u)\sin(u)\right\rangle\,\mathrm{d}u\mathrm{d}v$$

$$= a^3\int_0^{2\pi}\int_0^{\pi}\sin(u)\,\mathrm{d}u\mathrm{d}v$$

We get

$$\int\int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S = 4\pi a^3$$

Verifying divergence theorem (7)

Verification: We have found

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int\int\int_D \mathrm{Div}(\mathbf{F}) \, \mathrm{d}V = 4\pi a^3$$