Divergence theorem

$$\iint_S \vec{F} \cdot \vec{n} \, dS = \iiint_V \operatorname{div}(\vec{F}) \, dV$$

Computing a flux with the divergence (1)

Vector field:

$$\mathbf{F} = xyz \langle 1, 1, 1 \rangle = \langle xyz, xyz, xyz \rangle$$

Domain: Cube of the form

$$D: \quad \{0 \leq x \leq 1,\, 0 \leq y \leq 1,\, 0 \leq z \leq 1\}$$

Corresponding surface S:
6 faces of the cube

Rmk The 6 faces are long to parametrize $\Rightarrow$ it might be easier to compute $\iint_S \vec{F} \cdot \vec{n}\, dS$ as $\iiint_V \mathrm{div}(\vec{F})\, dV$

Problem: In order to avoid a parametrization of S
$\hookrightarrow$ Evaluate $\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S$ as a volume integral

Vector field

$$\vec{F} = \langle \overbrace{xyz}^{f}, \overbrace{xyz}^{g}, \overbrace{xyz}^{h} \rangle$$

Divergence

$$\operatorname{Div}(\vec{F}) = f_x + g_y + h_z$$

$$= yz + xz + xy$$

Volume integral

$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_D \operatorname{div}(\vec{F})\, dV$$

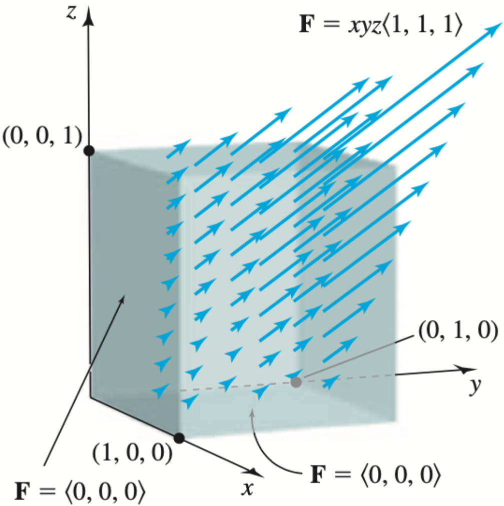
$$= \int_0^1 \int_0^1 \int_0^1 (yz + xz + xy)\, dz\, dy\, dx$$

$$= 3 \int_0^1 \int_0^1 \int_0^1 xy\, dz\, dy\, dx \quad \text{(symmetry in } x, y, z\text{)}$$

$$= 3 \overbrace{\int_0^1 x\, dx}^{1/2} \times \overbrace{\int_0^1 y\, dy}^{1/2} \times \overbrace{\int_0^1 dz}^{1}$$

$$= \frac{3}{4}$$

Computing a flux with the divergence (2)



Computing a flux with the divergence (3)

Expression for $\mathrm{Div}(\mathbf{F})$: We have

$$\mathrm{Div}(\mathbf{F}) = \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(xyz)$$

Computation: We find

$$\mathrm{Div}(\mathbf{F}) = yz + xz + xy$$

Computing a flux with the divergence (4)

Volume integral: We get

$$\int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V = \int_0^1\int_0^1\int_0^1 (yz + xz + xy)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$$

We obtain:

$$\int\int\int_D \mathrm{Div}(\mathbf{F})\,\mathrm{d}V = \frac{3}{4}$$

Computing a flux with the divergence (5)

Computation of the surface integral: The flux of **F** through S is

$$\int\int_S \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}S = \int\int\int_D \mathrm{Div}(\mathbf{F}) \, \mathrm{d}V = \frac{3}{4}$$

Remark:
We get a positive outward flux