Trace

Definition 10.

Let

- S a surface in $\mathbb{R}^3$

Then

1. A trace of S is the set of points at which S intersects a plane that is parallel to one of the coordinate planes.
2. The traces in the coordinate planes are called the xy-trace, the xz-trace, and the yz-trace

Elliptic paraboloid (1)

Problem: Graph the surface

$$z = \frac{x^2}{16} + \frac{y^2}{4}$$

Traces:

- xy-trace: ellipse, whenever $z_0 \geq 0$
- xz-trace: parabola
- yz-trace: parabola

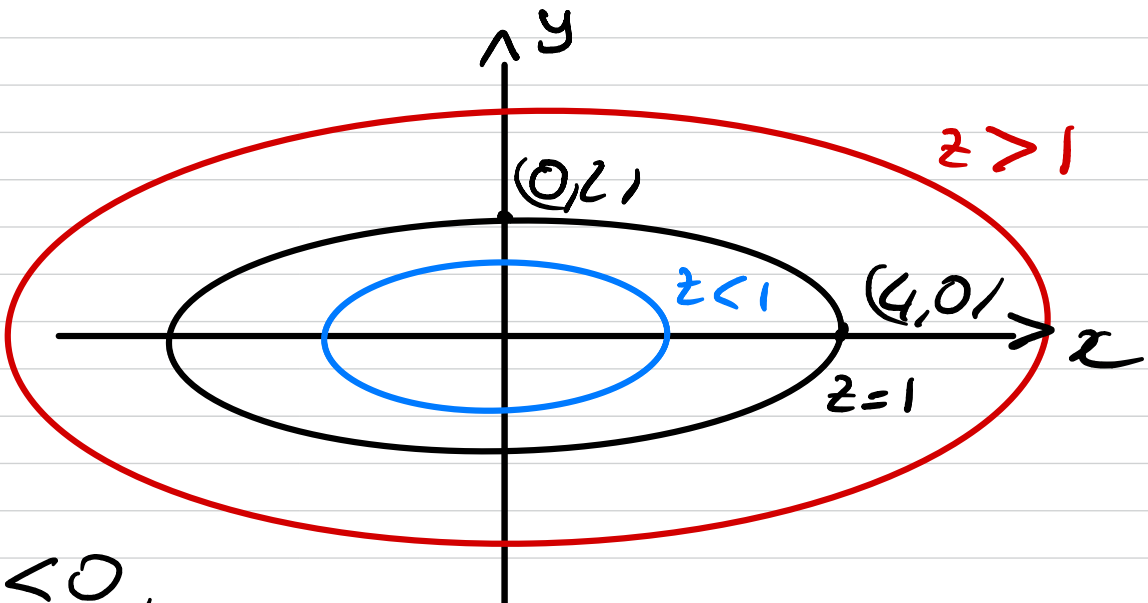
Surface

$$S: \frac{x^2}{16} + \frac{y^2}{4} = z$$

xy-traces Fix $z = z_0$. Then for $z_0 = 1$, the eq becomes

$$\frac{x^2}{16} + \frac{y^2}{4} = 1 \quad \Rightarrow a = 4, \quad b = 2$$

($16 = a^2$, $4 = b^2$)



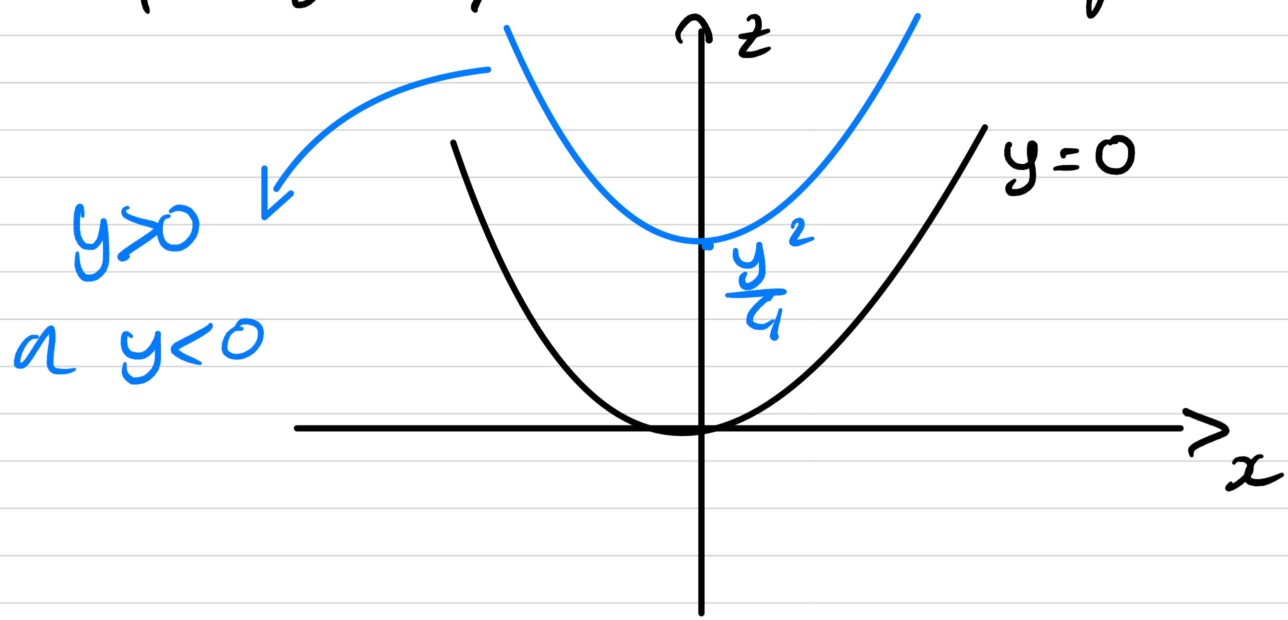
Rmk For $z < 0$, the trace is the empty set ($\emptyset$)

S: $\boxed{\dfrac{x^2}{16} + \dfrac{y^2}{4} = z}$

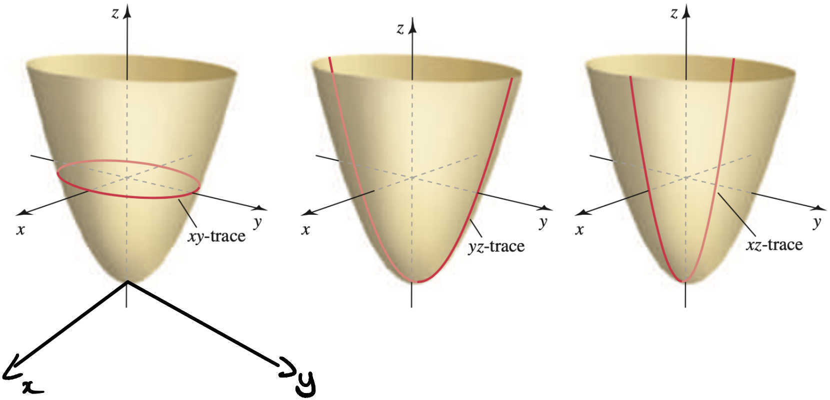
xz-traces If $y = y_0$, we get an eq

$$z = \frac{x^2}{16} + \frac{y_0^2}{4} \longrightarrow \text{parabola}$$

If $y_0 = 0$, then the parabola hits $(0,0)$



Elliptic paraboloid (2)



Graphing a cylinder (1)

Problem: Graph the cylinder ↗ z free variable ⇒ cylinder

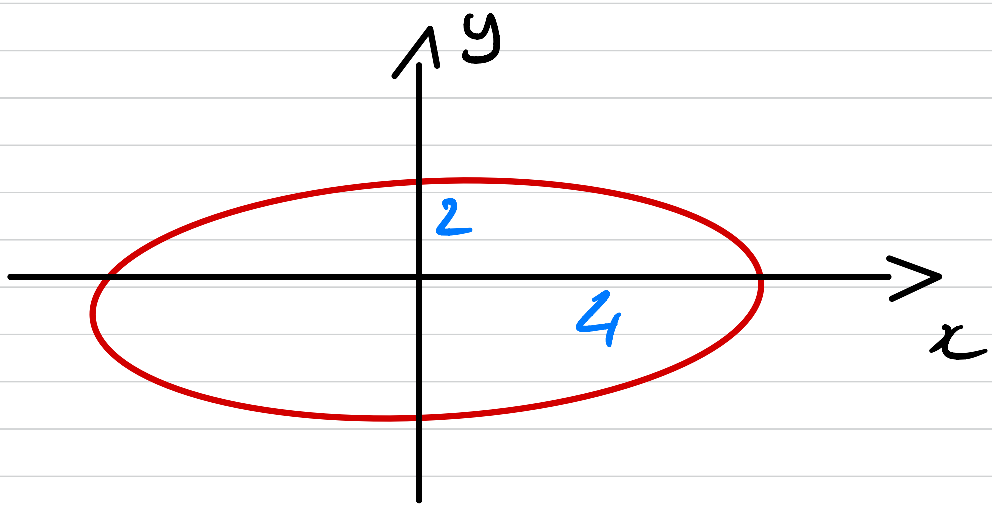
$$S: \; x^2 + 4y^2 = 16$$

Cylinder $\boxed{x^2 + 4y^2 = 16}$

xy-trace $x^2 + 4y^2 = 16$

$$\Leftrightarrow \quad \frac{x^2}{16} + \frac{y^2}{4} = 1$$

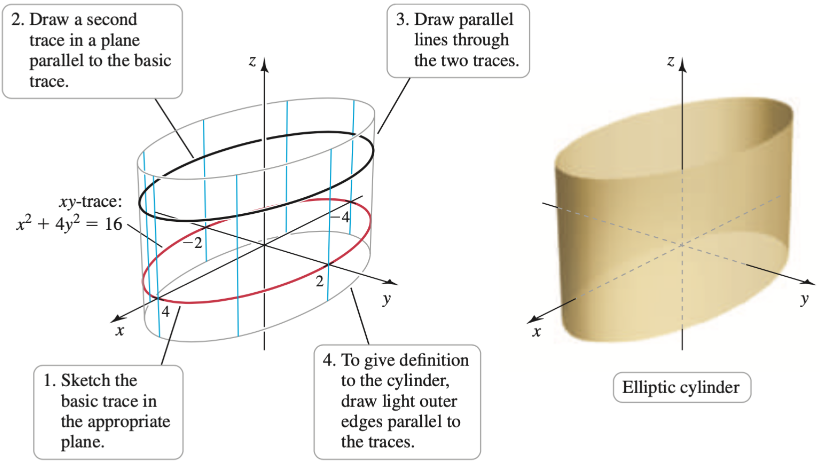
Ellipse with $a = 4$, $b = 2$



Graphing a cylinder (2)

1. Cylinder feature: Since z absent from equation
   $\hookrightarrow$ S is a cylinder with lines $\parallel$ to z axis
2. xy-trace: Ellipse of the form
$$\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1$$
3. Draw:
   - 1 trace in xy-plane
   - Another trace in e.g plane $z = 1$
   - Lines between those 2 traces

Graphing a cylinder (3)



Quadric surfaces

Analytic definition: Given by an equation of the form

$$S: \; Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0$$

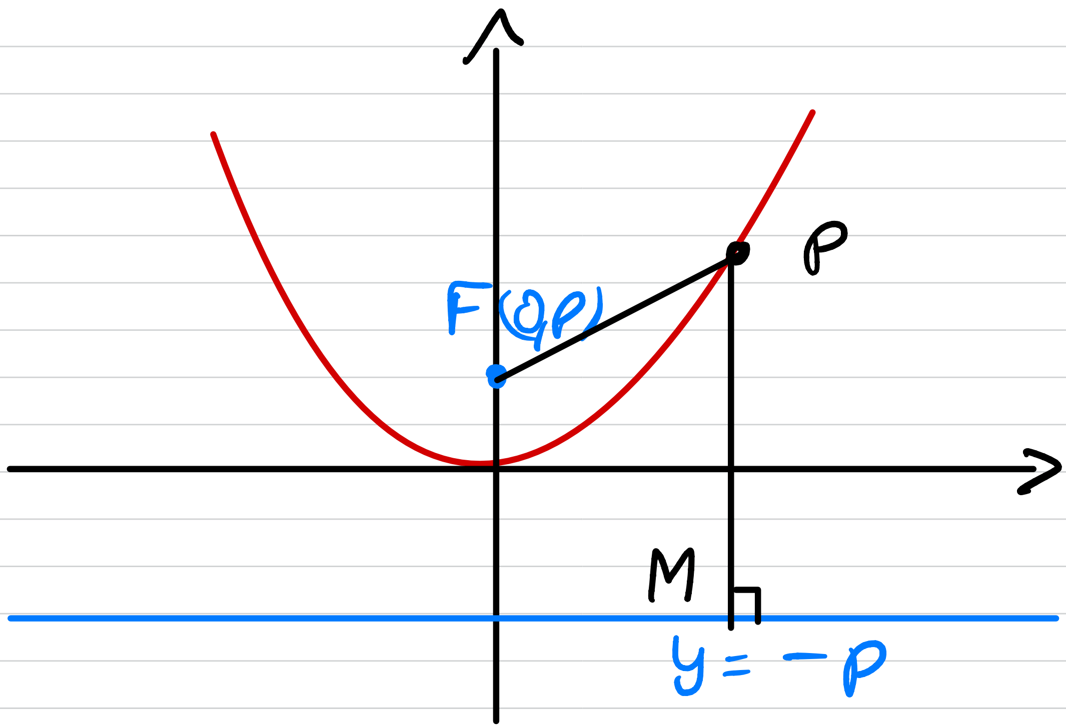
Strategy for graphing:

1. Intercepts. Determine the points, if any, where the surface intersects the coordinate axes.
2. Traces. Finding traces of the surface helps visualize the surface.
3. Completing the figure. Draw smooth curves that pass through the traces to fill out the surface.

Curves in $\mathbb{R}^2$ defined by a quad expression

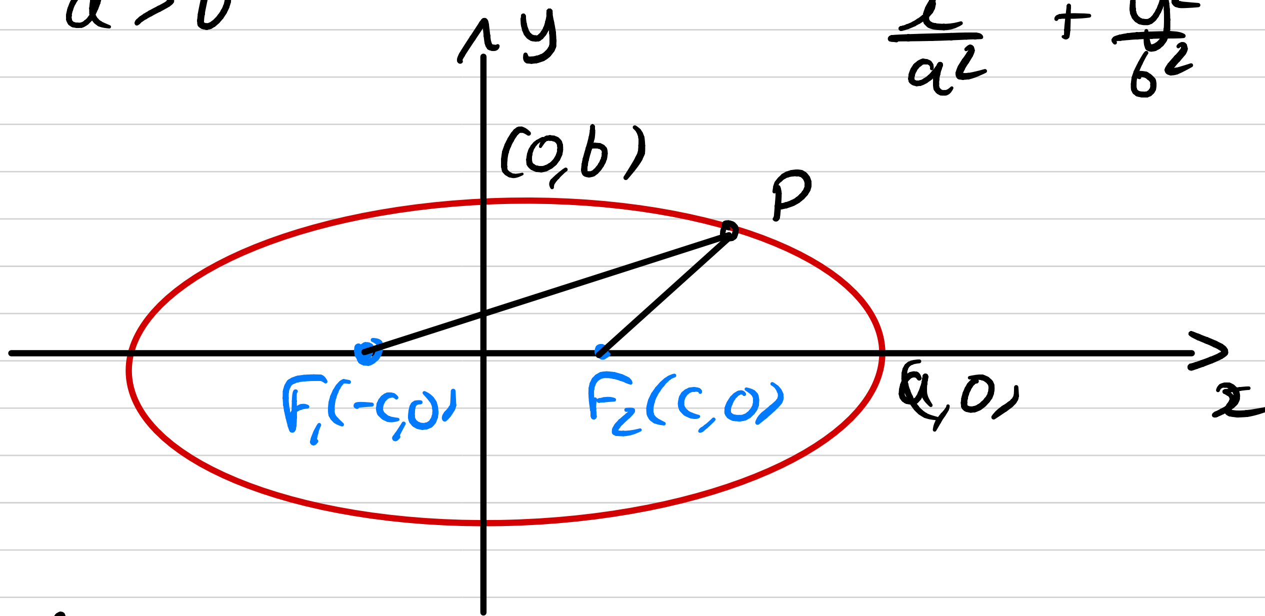
(i) <u>Parabola</u>

$$y = \frac{x^2}{4p}$$



$d(P, F) = d(P, M)$

(ii) Ellipse $a > b$ $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

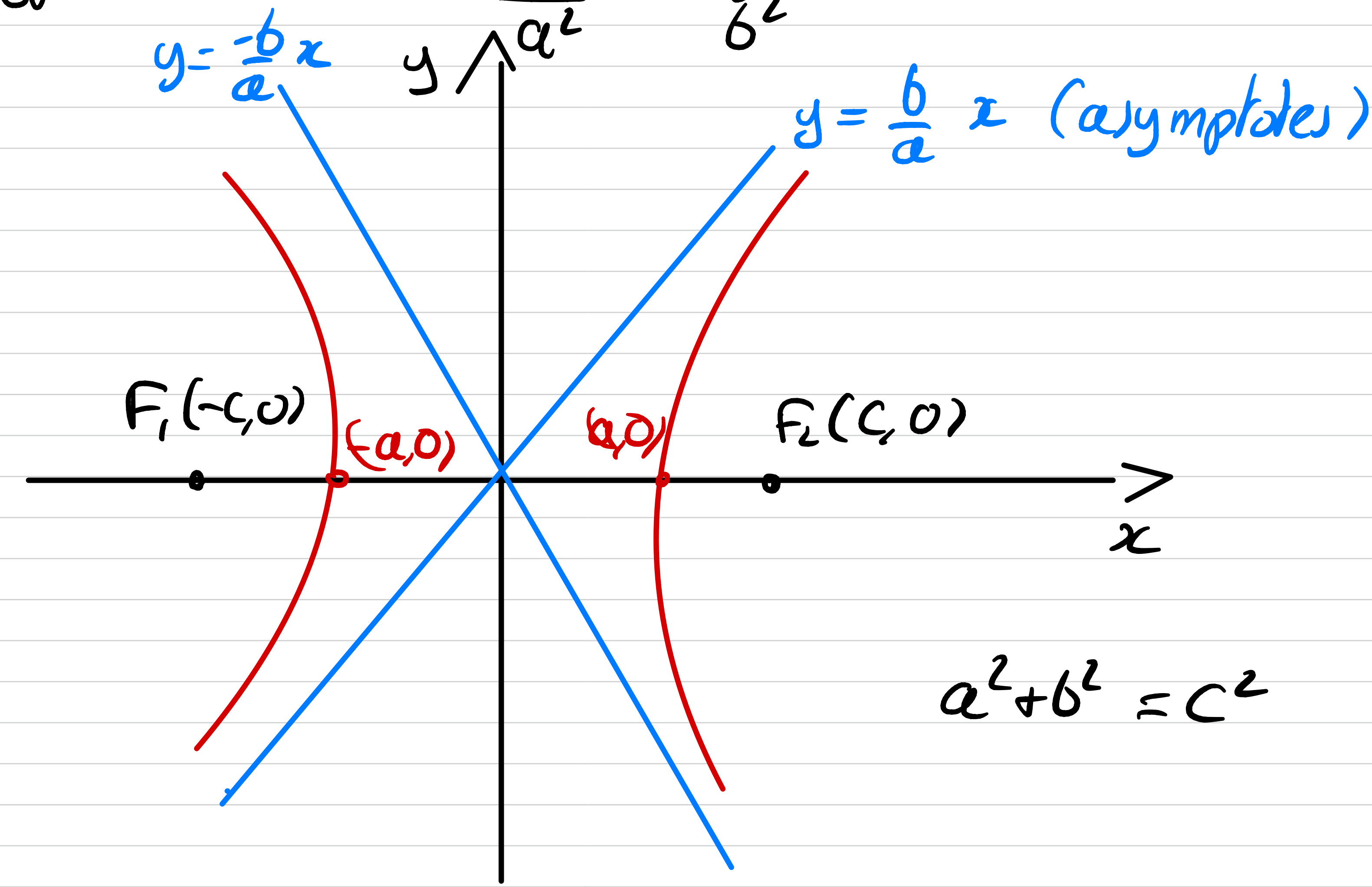


If $a^2 = b^2 + c^2$

$$d(P, F_1) + d(P, F_2) = 2a$$

(iii) Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



$a^2 + b^2 = c^2$

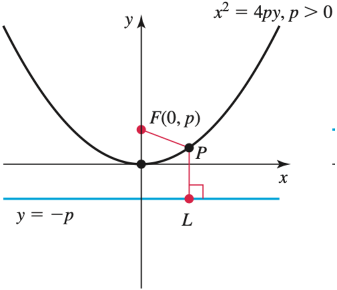
2d conic sections: parabola

Prototype of standard equation:

$$y = \frac{x^2}{4p}$$

Geometric definition:

$$\{P;\ \mathrm{dist}(P, F) = \mathrm{dist}(P, L)\}$$



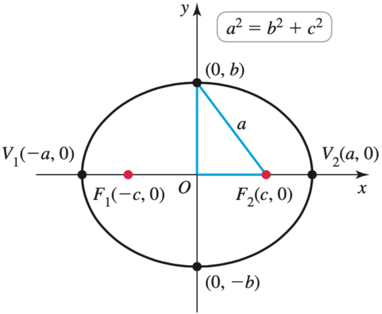
2d conic sections: ellipse

Prototype of standard equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Geometric definition: With $a^2 = b^2 + c^2$,

$$\{P;\ \text{dist}(P, F_1) + \text{dist}(P, F_2) = 2a\}$$



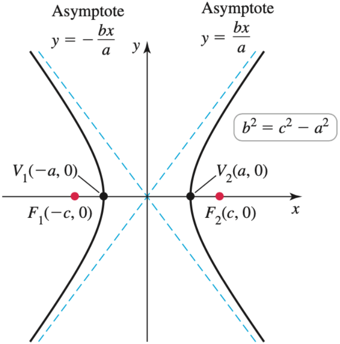
2d conic sections: hyperbola

Prototype of standard equation:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Geometric definition: With $a^2 = c^2 - b^2$,

$$\{P;\ \mathrm{dist}(P, F_2) - \mathrm{dist}(P, F_1) = \pm 2a\}$$



Hyperboloid of one sheet (1)

Equation:

$$\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$$

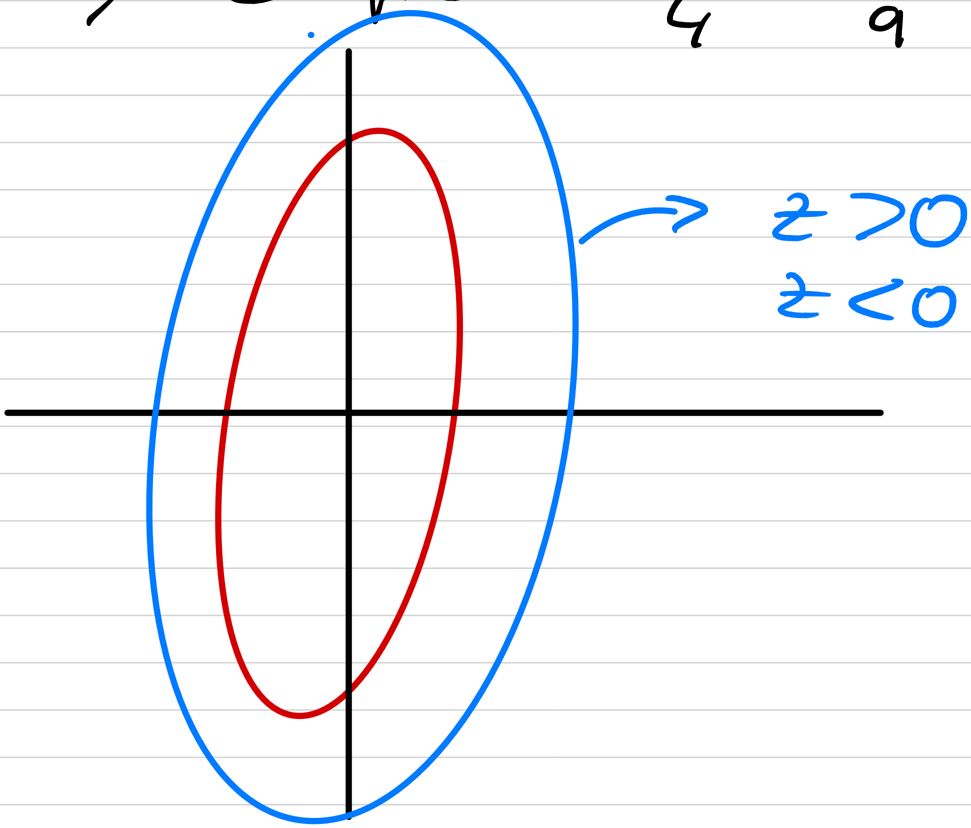
Intercepts:

$$(0, \pm 3, 0), \qquad (\pm 2, 0, 0)$$

xy-trace z is fixed. Then we get

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 + z^2$$

If $z = 0$, ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$ $a = 2$, $b = 3$



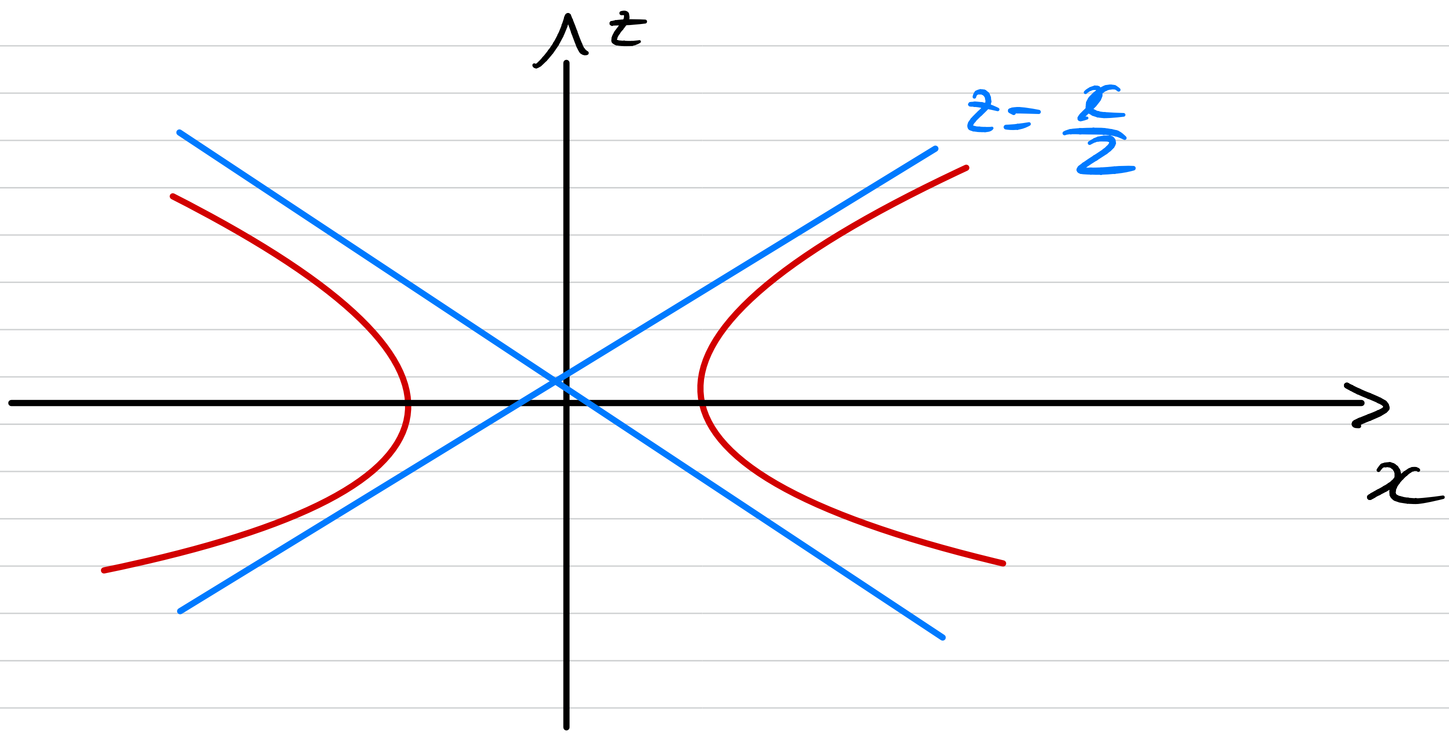
xz - trace

If $y = 0$

$$\frac{x^2}{4} - z^2 = 1$$

hyperbola

Asymptotes: $z = \pm \frac{x}{2}$



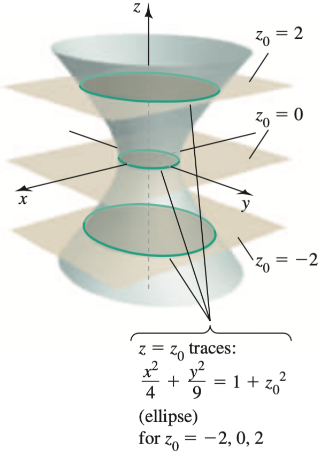
yz - trace

hyperbola

Hyperboloid of one sheet (2)

Traces in xy-planes: Ellipses of the form

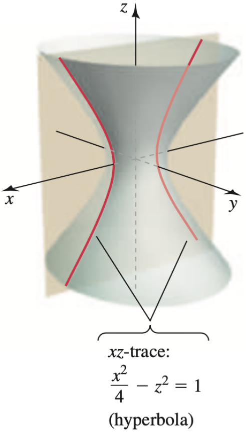
$$\frac{x^2}{4} + \frac{y^2}{9} = 1 + z^2$$



Hyperboloid of one sheet (3)

Traces in xz-planes: For $y = 0$, hyperbola

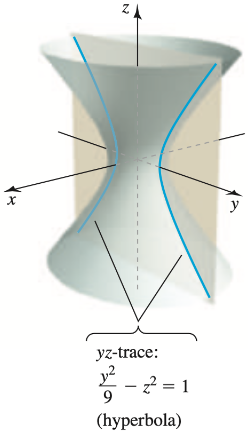
$$\frac{x^2}{4} - z^2 = 1$$



Hyperboloid of one sheet (4)

Traces in yz-planes: For $x = 0$, hyperbola

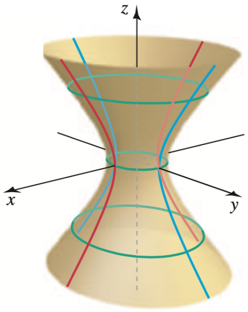
$$\frac{y^2}{9} - z^2 = 1$$



Hyperboloid of one sheet (5)

Equation:

$$\frac{x^2}{4} + \frac{y^2}{9} - z^2 = 1$$



Hyperbolic paraboloid (1)

Equation:

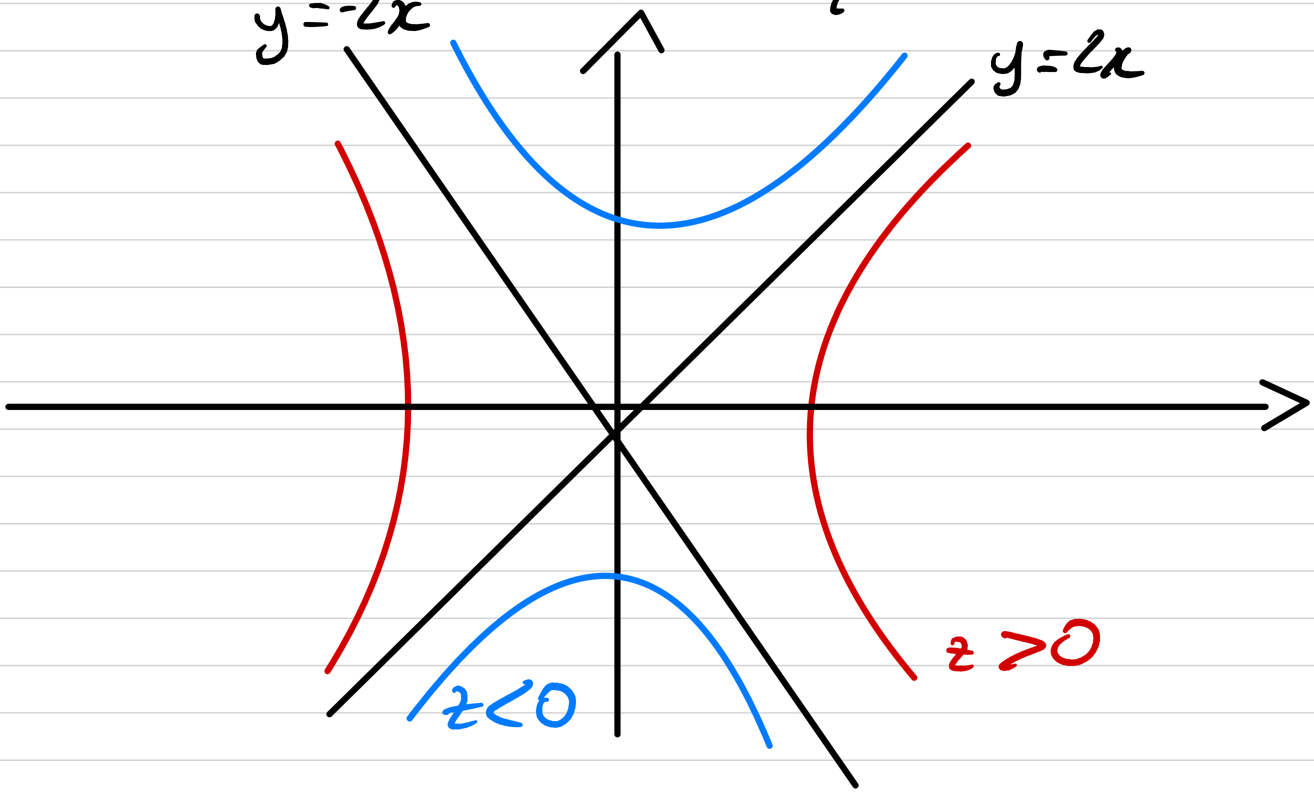
$$z = x^2 - \frac{y^2}{4}$$

Intercept:

$$(0, 0, 0)$$

xy-trace

If $z > 0$ fixed $\quad x^2 - \frac{y^2}{4} = z \quad$ hyperbola

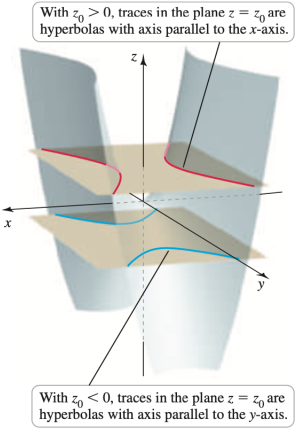


If $z < 0$ fixed $\quad \frac{y^2}{4} - x^2 = (-z)$

Hyperbolic paraboloid (2)

Traces in xy-planes: Hyperbolas (axis according to $z > 0$, $z < 0$) of the form

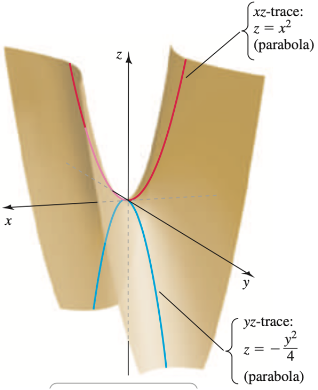
$$x^2 - \frac{y^2}{4} = z_0$$



Hyperbolic paraboloid (3)

Traces in xz-planes: For $y = y_0$, upward parabola

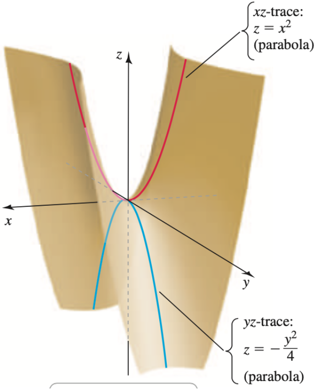
$$z = x^2 - \frac{y_0^2}{4}$$



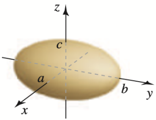
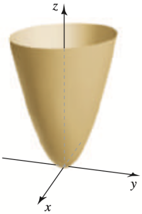
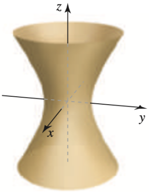
Hyperbolic paraboloid (4)

Traces in yz-planes: For $x = x_0$, downward parabola

$$z = -\frac{y^2}{4} + x_0^2$$



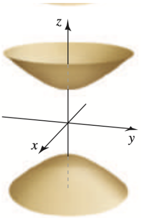
Summary of quadric surfaces (1)

Name	Standard Equation	Features	Graph
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$	All traces are ellipses.	
Elliptic paraboloid	$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$	Traces with $z = z_0 > 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are parabolas.	
Hyperboloid of one sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$	Traces with $z = z_0$ are ellipses for all z_0. Traces with $x = x_0$ or $y = y_0$ are hyperbolas.	

Summary of quadric surfaces (2)

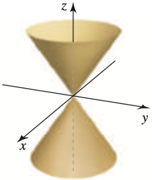
Hyperboloid of two sheets $\quad -\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

Traces with $z = z_0$ with $|z_0| > |c|$ are ellipses. Traces with $x = x_0$ and $y = y_0$ are hyperbolas.



Elliptic cone $\quad \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{z^2}{c^2}$

Traces with $z = z_0 \neq 0$ are ellipses. Traces with $x = x_0$ or $y = y_0$ are hyperbolas or intersecting lines.



Hyperbolic paraboloid $\quad z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

Traces with $z = z_0 \neq 0$ are hyperbolas. Traces with $x = x_0$ or $y = y_0$ are parabolas.