Outline

Functions with values in $\mathbb{R}^3$

Scalar-valued functions: We are used to functions like

$$f(t) = 3t^2 + 5 \quad \Longrightarrow \quad f(1) = 8 \in \mathbb{R}$$

Vector-valued functions: In this course we consider

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \quad \Longrightarrow \quad \mathbf{r}(t) \in \mathbb{R}^3$$

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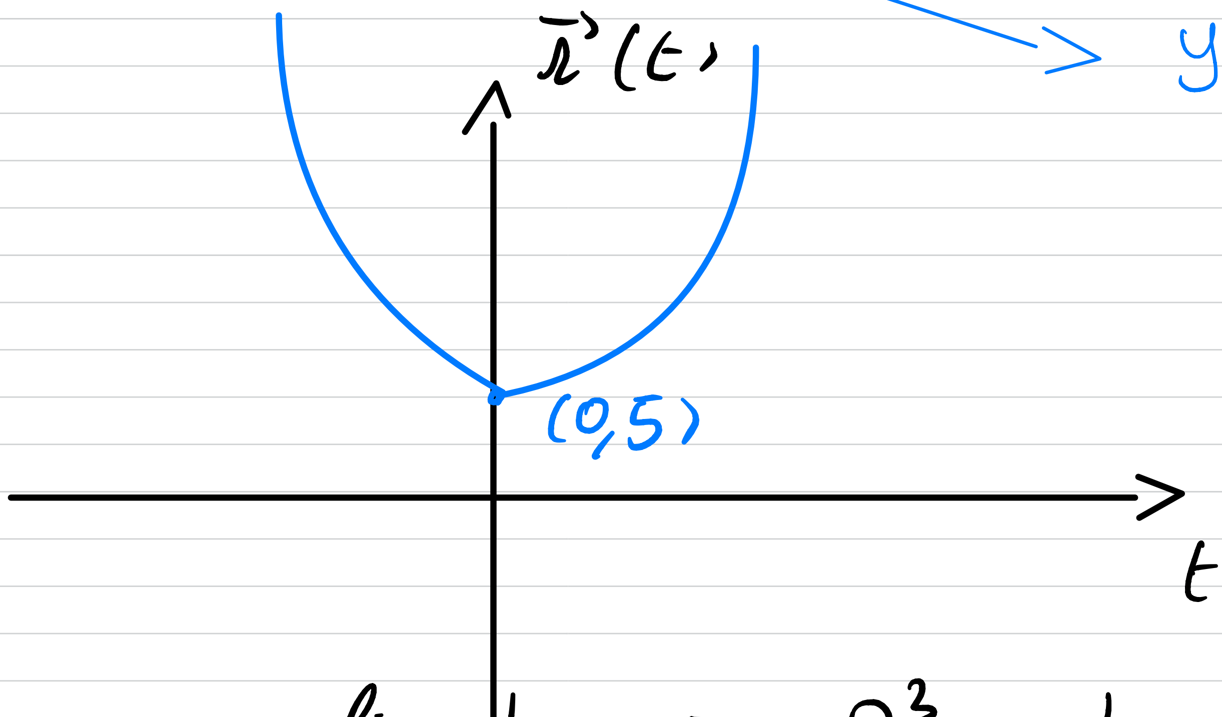
Go back to $\mathbb{R}^2$

If $f(t) = 3t^2 + 5$, one can form a vector in $\mathbb{R}^2$ indexed by t

$$\vec{r}(t) = \langle t, 3t^2+5 \rangle$$

$x(t)$, $y(t)$



Now : generalizations in $\mathbb{R}^3$, where

$$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$$

Lines as vector-valued functions (1)

Problem: Consider the line passing through

$$P(1, 2, 3) \quad \text{and} \quad Q(4, 5, 6)$$

Find a vector-valued function for this line

Parametric equation for the line

$P(1,2,3)$ $Q(4,5,6)$

Compute $\overrightarrow{PQ} = \langle 4-1, 5-2, 6-3 \rangle = \langle 3, 3, 3 \rangle$

The corresponding direction is $\vec{v} = \langle 1, 1, 1 \rangle$

Expression for $\vec{r}(t)$

$$\vec{r}(t) = P + t\vec{v}$$

$$= \langle 1, 2, 3 \rangle + t \langle 1, 1, 1 \rangle$$

$$= \langle 1+t , 2+t , 3+t \rangle$$

Lines as vector-valued functions (2)

Parallel vector:

$$\mathbf{v} = (3, 3, 3), \quad \text{simplified as} \quad \mathbf{v} = (1, 1, 1)$$

Equation for the line:

$$\mathbf{r}(t) = \langle 1 + t, 2 + t, 3 + t \rangle$$

Examples of points:

$$\mathbf{r}(0) = \langle 1, 2, 3 \rangle, \quad \mathbf{r}(1) = \langle 2, 3, 4 \rangle, \quad \mathbf{r}(2) = \langle 3, 4, 5 \rangle$$

Spiral (1)

Problem: Graph the curve defined by

$$\mathbf{r}(t) = \left\langle 4\cos(t), \sin(t), \frac{t}{2\pi} \right\rangle$$

Eq for spiral

$\vec{r}(t) = \langle 4\cos(t),\ \sin(t),\ t/2\pi \rangle$

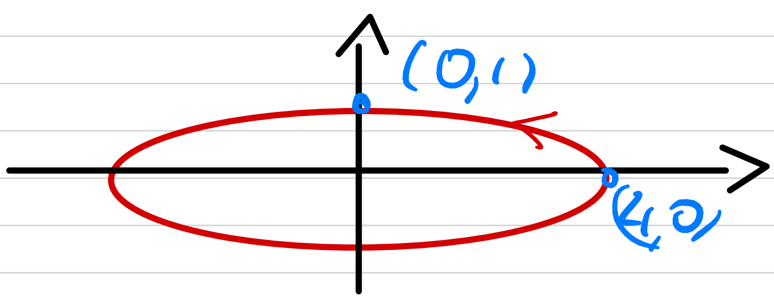
If we look at the projection on the xy-plane we get

$\langle \overbrace{4\cos(t)}^{x(t)},\ \overbrace{\sin(t)}^{y(t)} \rangle$

$\frac{x(t)}{4} = \cos(t)$

Since $\cos^2 + \sin^2 = 1$, we have

$\left(\frac{x(t)}{4}\right)^2 + y^2(t) = 1$ → eq. for an ellipse

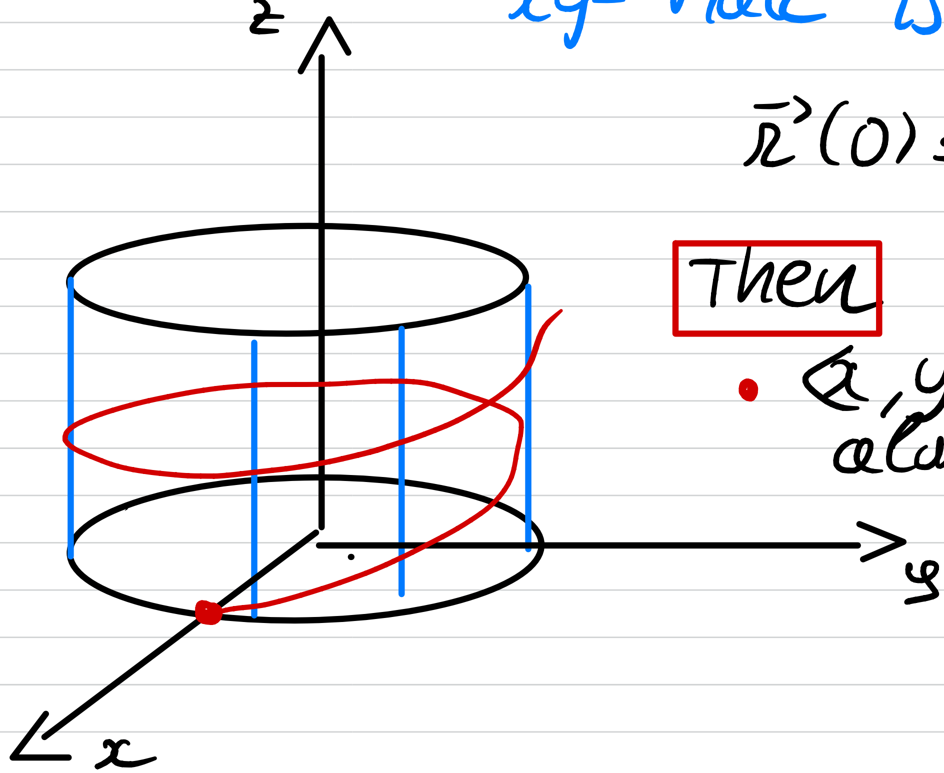


In $\mathbb{R}^3$, for every t we still have

$$\left(\frac{x(t)}{4}\right)^2 + (y(t))^2 = 1$$

Thus $\vec{r}(t)$ is located on the surface

$$\frac{x^2}{16} + y^2 = 1 \longrightarrow$$ cylinder, whose xy-trace is an ellipse



$\vec{r}(0) = \langle 4, 0, 0 \rangle$

Then

- $\langle x, y \rangle$ moves along the ellipse
- z moves linearly

Spiral (2)

Projection on xy-plane: Set $z = 0$. We get

$$\langle 4\cos(t), \sin(t) \rangle$$

This is an ellipse, counterclockwise, starts at $(4, 0, 0)$

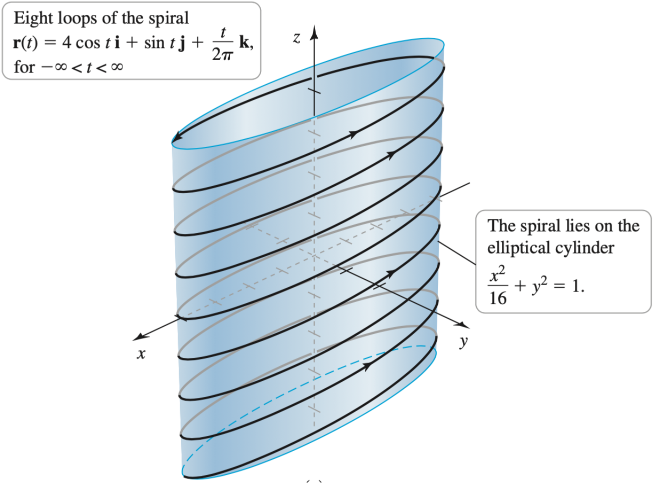
Related surface: We have

$$\frac{x^2}{4} + y^2 = 1$$

Thus curve lies on an elliptic cylinder

Upward direction: The z-component is $\frac{t}{2\pi}$
$\hookrightarrow$ Spiral on the cylinder

Spiral (3)



Eight loops of the spiral
$\mathbf{r}(t) = 4\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \frac{t}{2\pi}\mathbf{k}$,
for $-\infty < t < \infty$
z
x
y
The spiral lies on the elliptical cylinder
$\frac{x^2}{16} + y^2 = 1$.

Domain of vector-valued functions

Definition: The domain of $t \mapsto \mathbf{r}(t)$ is
$\hookrightarrow$ The intersection of the domains for each component

Example: If

$$\mathbf{r}(t) = \left\langle \sqrt{1-t^2}, \sqrt{t}, \frac{1}{\sqrt{5+t}} \right\rangle,$$

then the domain of $\mathbf{r}$ is

$$[0, 1]$$

Domain for the function

$$\vec{r}'(t) = \left\langle \underbrace{\sqrt{1-t^2}}_{x(t)},\ \underbrace{\sqrt{t}}_{y(t)},\ \underbrace{\frac{1}{\sqrt{5+t}}}_{z(t)} \right\rangle$$

When is $x(t)$ defined?

We want $t^2 \leq 1 \iff t \in [-1,1]$

For $y(t)$,

$y(t)$ defined if $t \in [0,\infty)$

For $z(t)$

$z(t)$ defined if $t \in (-5,\infty)$

Thus $\vec{r}'(t)$ def if $t \in [-1,1] \cap [0,\infty) \cap (-5,\infty)$ $= [0,1]$

Limits and continuity (1)

Function: We define

$$\mathbf{r}(t) = \left\langle \cos(\pi t), \sin(\pi t), e^{-t} \right\rangle$$

Questions:

1. Graph $\mathbf{r}$
2. Evaluate $\lim_{t\to 2} \mathbf{r}(t)$
3. Evaluate $\lim_{t\to\infty} \mathbf{r}(t)$
4. At what points is $\mathbf{r}$ continuous?

Limits for $\vec{r}(t) = \langle \cos(\pi t), \sin(\pi t), e^{-t} \rangle$

(all nice functions)

$$\lim_{t\to 2} \vec{r}(t) = \langle \lim_{t\to 2} \cos(\pi t), \lim_{t\to 2} \sin(\pi t), \lim_{t\to 2} e^{-t} \rangle$$

$$= \langle 1, 0, e^{-2} \rangle$$

$\lim_{t\to\infty} \vec{r}(t)$ does not exist. We just know that $\lim_{t\to\infty} z(t) = 0$. The limiting curve is the unit circle in the xy-plane

What are the points of continuity for $\vec{r}(t)$?

↳ all 3 functions are continuous, thus $\vec{r}(t)$ is continuous everywhere

Limits and continuity (2)

Answers

1. $\lim_{t\to 2} \mathbf{r}(t) = \langle 1, 0, e^{-2} \rangle$
2. No limit. As $t \to \infty$
   $\hookrightarrow$ $\mathbf{r}(t)$ approaches the unit circle in xy-plane
3. $\mathbf{r}$ is continuous everywhere