MIDTERM 2 - FALL 19

1. Let $\mathbf{A}$ be an $n \times n$ nonsingular matrix. Which of the following statements must be true?

(i) $\det\mathbf{A} = 0$.
(ii) $\text{rank}(\mathbf{A}) = n$.
(iii) $\mathbf{Ax} = \mathbf{0}$ has infinitely many solutions.
(iv) $\mathbf{Ax} = \mathbf{b}$ has a unique solution for every vector $\mathbf{b} \in \mathbb{R}^n$.
(v) $\mathbf{A}$ must be row equivalent to the $n \times n$ identity matrix $\mathbf{I}_n$.

(i) Nonsingular means A invertible, true iff $\det(A) \neq 0$. Thus (i) false

(ii) If $A = [v_1, \dots, v_n]$ is nonsingular, then $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{R}^n$. Hence $\text{Rank}(A) = n$
Thus (ii) true

(iii) If $\text{Rank}(A) = n$, then $\dim(\text{Null}(A)) = 0$ which means that 0 is the unique solution to $Ax = 0$. Thus (iii) false

(iv) If A invertible then $x = A^{-1}b$ is the unique solution to $Ax = b$. Thus (iv) true

(v) According to Jordan's decomposition, if A is invertible it is equivalent to I_n. Thus (v) true

Answer: E

2. Consider the initial value problem: $t(t-10)y'' + y' - \frac{1}{t-3}y = \ln(t-5)$, $y(\boxed{6}) = 0$, $y'(6) = 1$.

x_0

Find the largest interval for which the above initial value problem has a unique solution.

A. $(0, 5)$
B. $(0, 10)$
C. $(5, +\infty)$
D. $(10, +\infty)$
E. $(5, 10)$

Write the problem in standard form:

$$y'' + \underbrace{\frac{1}{t(t-10)}}_{p_1(t)} y' - \underbrace{\frac{1}{t(t-3)(t-10)}}_{p_2(t)} = \underbrace{\frac{\ln(t-5)}{t(t-10)}}_{g(t)}$$

Intervals of continuity

p_1: $(-\infty, 0) \cup (0, 10) \cup (10, \infty)$

p_2: $(-\infty, 0) \cup (0, 3) \cup (3, 10) \cup (10, \infty)$

g: $(5, 10) \cup (10, \infty)$

Largest interval We have $x_0 \in (5, 10)$, which is the largest interval on which there is a unique solution

E

3. Which of the following subset S is a subspace of V?

(i) $V = \mathbb{R}^3$ and S is the set of vectors (x, y, z) satisfying $x + 2y - 3z = 0$.
(ii) $V = M_2(\mathbb{R})$ and S is the set of 2×2 matrices with determinant $\neq 0$.
(iii) $V = P_2$ and S is the set of polynomials of the form $ax^2 - bx$, where $a, b \in \mathbb{R}$.
(iv) $V = M_n(\mathbb{R})$ and S is the set of $n \times n$ nonsymmetric matrices.

A. (i) and (iii) only
B. (i) and (iv) only
C. (ii) and (iii) only
D. (i) (iii) and (iv)
E. (i) (ii) and (iii)

Generally speaking, subspaces are obtained as solutions of systems like $Ax = 0$ (linear + homog.)

(i) This is a linear homogeneous equation. Thus V is a subspace

(ii) The determinant is not linear and we are considering the equation $\det(A) \neq 0$. Thus V is not a subspace

(iii) Here $V = \{ax^2 + bx + c \,;\, c = 0\}$. This is a linear homogeneous equation. Thus V is a subspace

(iv) Here $V = \{M \in M_n \,;\, M^T \neq M\}$. Thus V is not a subspace.

A

4. Determine the general solution to $(D+1)(D-1)^2(D^2+2D+2)y=0$.

A. $c_1e^{-x}+c_2e^{x}+e^{-x}(c_3\cos x+c_4\sin x)$
B. $c_1e^{-x}+c_2e^{x}+c_3xe^{x}+e^{-x}(c_4\cos x+c_5\sin x)$
C. $c_1e^{-x}+c_2e^{x}+e^{x}(c_3\cos x+c_4\sin x)$
D. $c_1e^{-x}+c_2xe^{-x}+c_3e^{x}+e^{-x}(c_4\cos x+c_5\sin x)$
E. $c_1e^{-x}+c_2e^{x}+c_3xe^{x}+e^{x}(c_4\cos x+c_5\sin x)$

The characteristic polynomial is

$$P(r) = (r+1)(r-1)^2((r+1)^2+1)$$

We get the following summary for the roots

Root	Multiplicity
-1	1
1	2
$-1+i$	1
$-1-i$	1

Fundamental solutions

$y_1 = e^{-x}$ $\quad y_2 = e^{x}$ $\quad y_3 = xe^{x}$

$y_4 = e^{-x}\cos(x)$ $\quad y_5 = e^{-x}\sin(x)$

B

5. Let $A = \begin{array}{c} \begin{array}{ccccc} v_1 & & v_3 & v_4 & \end{array} \\ \begin{bmatrix} \boxed{1} & 2 & \boxed{3} & \boxed{4} & 5 \\ \boxed{1} & 2 & \boxed{3} & \boxed{2} & 1 \\ \boxed{2} & 4 & \boxed{0} & \boxed{2} & -2 \\ \boxed{0} & 0 & \boxed{0} & \boxed{1} & 2 \end{bmatrix} \end{array}$, which of the following set is a basis of the column space of A?

Row-echelon form for A

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 2 & 1 \\ 2 & 4 & 0 & 2 & -2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \underset{\sim}{\overset{A_{13}(-2)}{A_{12}(-1)}} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & -2 & -4 \\ 0 & 0 & -6 & -6 & -12 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\underset{\sim}{\overset{M_3(-1/6)}{M_2(-1/2)}} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \underset{\sim}{P_{23}} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

$$\underset{\sim}{A_{34}(-1)} \begin{pmatrix} \boxed{1} & 2 & 3 & 4 & 5 \\ 0 & 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

leading 1

Basis for Col(A)

If we write

$A = [\, v_1 \;\; v_2 \;\; v_3 \;\; v_4 \;\; v_5 \,]$

then a basis is given by $\{v_1, v_3, v_4\}$

D

6. The general solution of $y^{(4)} - 8y'' + 16y = 0$ is

A. $c_1e^{2x} + c_2xe^{2x} + c_3e^{-2x} + c_4xe^{-2x}$
B. $c_1e^{2x} + c_2e^{-2x}$
C. $c_1e^{4x} + c_2e^{-4x}$
D. $c_1e^{2x} + c_2e^{-2x} + c_3$
E. $c_1e^{4x} + c_2xe^{4x} + c_3e^{-4x} + c_4xe^{-4x}$

The auxiliary polynomial is $P(r) = r^4 - 8r^2 + 16$. Set $z = r^2$. Then $P(r) = Q(z)$ with

$$Q(z) = z^2 - 8z + 16 = (z-4)^2$$

Roots We have

$$P(r) = (r^2 - 4)^2 = (r-2)^2 (r+2)^2$$

Hence the roots are ± 2, each with multiplicity 2

Fundamental solutions

$y_1 = e^{2x}$ $\quad$ $y_2 = x\, e^{2x}$

$y_3 = e^{-2x}$ $\quad$ $y_4 = x\, e^{-2x}$

A

7. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$, which of the following set is a basis of the null space of A?

Solving $Ax=0$ Since $R_2 = 2R_1$, it is immediate to see that $Ax=0$ is reduced to

$$x_1 + x_2 + x_3 = 0$$

We set $x_2 = s$, $x_3 = t$. Then $x_1 = -s-t$.

Solution set We get $Null(A) = S$ with

$$S = \{ (-s-t, s, t) \; ; \; s,t \in \mathbb{R} \}$$

$$= \text{Span} \left\{ \underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\text{Basis for } Null(A)} \right\}$$

A

8. Let $y(t)$ be the solution to the initial value problem $y'' + 3y' - 4y = 6e^{2t}$, $y(0) = 2$, $y'(0) = 3$, find $y(1)$.

A. e
B. e^2
C. $e + e^2$
D. $e - e^2$
E. $2e - e^2$

Homogeneous system Its polynomial is

$$P(r) = r^2 + 3r - 4 = (r-1)(r+4)$$

Fundamental solutions: e^t, e^{-4t}

Particular solution Of the form $y_p = a e^{2t}$

Then

$$\left.\begin{aligned} \times(-4)\quad y_p &= a e^{2t} \\ \times 3\quad y_p' &= 2a e^{2t} \\ \times 1\quad y_p'' &= 4a e^{2t} \end{aligned}\right\} \Rightarrow \begin{aligned} &y_p'' + 3y_p' - 4y_p \\ &= (4+6-4)\, a e^{2t} \\ &= 6a\, e^{2t} \end{aligned}$$

If we want the rhs to be $6e^{2t}$, we take $a = 1$ and

$$y_p = e^{2t}$$

General solution

$$y = c_1 e^{t} + c_2 e^{-4t} + e^{2t}$$

$$y' = c_1 e^{t} - 4c_2 e^{-4t} + 2e^{2t}$$

Initial value

If $y(0) = 2$ $\quad y'(0) = 3$

we get

$$\begin{cases} c_1 + c_2 + 1 = 2 \\ c_1 - 4c_2 + 2 = 3 \end{cases}$$

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 - 4c_2 = 1 \end{cases}$$

Thus $c_1 = 1$, $c_2 = 0$

We get a unique solution:

$$y = e^{t} + e^{2t}$$

Hence $y(1) = e + e^2$

C

9. Given that $y_1(t) = t$ is a solution to $t^2y'' - ty' + y = 0$, $t > 0$, find a second linearly independent solution $y_2(t)$.

A. t^2
B. $t \ln t$
C. $\ln t$
D. $t^2 \ln t$
E. te^t

This is about variation of parameter, not part of the program (Lesson 29)

10. Let $\lambda = 3$ be an eigenvalue of $A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Then the geometric multiplicity of $\lambda = 3$ is

A. 0
B. 1
C. 2
D. 3
E. 4

This is about eigenvalues, not part of the program (Lesson 30)

11. Consider a spring-mass system whose motion is governed by $y''+y=4\sin(t)$, $y(0)=2$, $y'(0)=0$. Find the solution of the above initial value problem.

A. $y(t)=2\cos(t)-\sin(t)$
B. $y(t)=\cos(t)+2\sin(t)$
C. $y(t)=\cos(t)+\sin(t)-t\cos(t)$
D. $y(t)=2\cos(t)+2\sin(t)-2t\cos(t)$
E. $y(t)=2\cos(t)-2\sin(t)+t\cos(t)$

Homogeneous equation $\quad y''+y=0$, with fundamental solutions

$$y_1=\cos(t) \qquad y_2=\sin(t)$$

Particular solution Since $\sin(t)$ is a fundamental solution (multiplicity 1), we look for y_p under the form

$$y_p=a\,t\cos(t)+b\,t\sin(t)$$
$$y_p'=-a\,t\sin(t)+b\,t\cos(t)+a\cos(t)+b\sin(t)$$
$$y_p''=-a\,t\cos(t)-b\,t\sin(t)$$
$$-2a\sin(t)+2b\cos(t)$$

Hence $\quad y_p+y_p''=-2a\sin(t)+2b\cos(t)$

If we want the rhs to be $4\sin(t)$ we get $\quad a=-2 \quad b=0$. Thus

$$y_p=-2t\cos(t)$$

Note: here we see that D is the only possible answer

General solution We get

$$y = c_1 \cos(t) + c_2 \sin(t) - 2t\cos(t)$$

$$y' = -c_1 \sin(t) + c_2 \cos(t) - 2\cos(t) + 2t\sin(t)$$

Initial condition With $y(0)=2$, $y'(0)=0$, we get

$$c_1 = 2$$

$$c_2 - 2 = 0 \Rightarrow c_2 = 2$$

The unique solution is

$$y = 2\cos(t) + 2\sin(t) - 2t\cos(t)$$

D