MA 262 Fall 17 Midterm 1

Pb1 Equation $y' = \frac{y}{x\ln(x)}$ → separable

Eq $\iff \frac{dy}{y} = \frac{1}{x}\frac{1}{\ln(x)}\,dx$

$\iff \ln(|y|) = \ln(|\ln(x)|) + c_1$

$\iff y = c_2 \ln(x)$ (d)

Pb2 Equation $y' + xy = e^{-\frac{x^2}{2}}$, $y(0)=2$ → linear

I such that $I' = xI \iff \frac{I'}{I} = x$

We take $\ln(I) = \frac{x^2}{2} \iff I = e^{\frac{x^2}{2}}$

Thus Eq $\iff (y e^{\frac{x^2}{2}})' = 1$

$\iff y e^{\frac{x^2}{2}} = x + c$

$\iff y = x e^{-\frac{x^2}{2}} + c$

With initial data $y(0)=2$, we get $c=2$

Solution: $y = (x+2)e^{-\frac{x^2}{2}}$

$= \frac{x+2}{e^{x^2/2}}$ (a)

Pb3 I satisfies $\frac{I'}{I} = \frac{2}{x}$. Take $I = x^2$

(a)

Pb4 Equation $(y^{-1}+x^{-1})\,dx + (xy^{-2}+2y^{-1})\,dy = 0$

Consider $I(x,y) = x^r y^s$, and multiply by I:

$$x^r y^s (y^{-1}+x^{-1})\,dx + x^r y^s (xy^{-2}+2y^{-1})\,dy = 0$$

$$\Leftrightarrow \underbrace{[x^r y^{s-1} + x^{r-1} y^s]}_{M}\,dx + \underbrace{[x^{r+1} y^{s-2} + 2x^r y^{s-1}]}_{N}\,dy = 0$$

This equation is exact iff $M_y = N_x$. Here

$$M_y = (s-1)x^r y^{s-2} + s\,x^{r-1} y^{s-1}$$

$$N_x = (r+1)x^r y^{s-2} + 2r\,x^{r-1} y^{s-1}$$

Thus

$$M_y = N_x \Leftrightarrow \begin{cases} r+1 - (s-1) = 0 \\ s = 2r \end{cases}$$

$$\Leftrightarrow \begin{cases} r - s = -2 \\ s = 2r \end{cases} \Leftrightarrow r = 2 \quad s = 4$$

I is an integrating factor iff $r=2$, $s=4$ (c)

Pb5 Equation $\overbrace{(4e^{2x}+2xy-y^2)}^{M}dx + \overbrace{(x-y)^2}^{N}dy = 0$

$M_y = 2x - 2y$
$= 2(x-y)$ $\quad N_x = 2(x-y)$

$\Rightarrow$ equation is exact

$\phi(x,y) = \int M\,dx = 2e^{2x} + x^2y - y^2x + h(y)$

Then

$\phi_y = N \Leftrightarrow x^2 - 2yx + h'(y) = x^2 - 2xy + y^2$

$\Leftrightarrow h'(y) = y^2$

$\Leftrightarrow h(y) = \frac{1}{3}y^3 + c_1$

We get a general solution:

$2e^{2x} + x^2y - y^2x + \frac{1}{3}y^3 = c$

Initial data: $x=0$, $y=3$. This yields

$2 + 9 = c \Rightarrow c = 11$

Solution

$2e^{2x} + x^2y - y^2x + \frac{1}{3}y^3 = 11$ $\quad$ (e)

Pb6 Equation : $y'' = -\frac{2}{1-y}(y')^2 \quad y(0)=3 \quad y'(0)=4$

Set $v = \frac{dy}{dx}$. Then y'' satisfies

$$y'' = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = v\frac{dv}{dy}$$

Thus

$$Eq \iff v\frac{dv}{dy} = \frac{-2}{1-y}v^2 \quad \to \text{ separable } 1^{st} \text{ order}$$

$$\iff \frac{1}{v}dv = \frac{2}{y-1}dy$$

$$\iff \ln|v| = 2\ln|y-1| + c_1$$

$$\iff v = c_2 (y-1)^2$$

We have found that $\frac{dy}{dx} = v$ verifies

$$\frac{dy}{dx} = c_2 (y-1)^2 \quad \to \text{ separable}$$

$$\iff \frac{dy}{(y-1)^2} = c_2\, dx$$

$$\iff -\frac{1}{(y-1)} = c_2 x + c_3$$

$$\iff y-1 = \frac{1}{c_3 x + c_4}$$

NB: Here the answer can only be (b)

$$\iff y = 1 + \frac{1}{c_3 x + c_4} \quad \left(\Rightarrow y' = \frac{-c_3}{c_3 x + c_4}\right)$$

$$y(0)=3 \Rightarrow 1 + \frac{1}{c_4} = 3 \Rightarrow c_4 = \frac{1}{2}$$

$$y'(0) = 4 \Rightarrow -\frac{c_3}{c_4} = 4, \quad c_3 = -2 \qquad (b)$$

$$\Rightarrow y = \frac{2x-3}{2x-1}$$

Pb 7. The reduced row echelon form of $A^\#$ is

$$A^\# \sim \begin{bmatrix} 1 & a & b & 0 & c & 0 \\ 0 & 0 & 0 & 1 & d & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore $r \equiv \text{rank}(A) = 2$, $r^\# \equiv \text{rank}(A^\#) = 3$ and

(i) A is not a square matrix $\rightarrow$ cannot be invertible

(ii) $r < r^\# \Rightarrow Ax = b$ is not consistent

(iii) $r < 3 \Rightarrow Ax = 0$ has ∞ number of solutions

$\hookrightarrow$ (d)

Pb 8

$$A = \begin{bmatrix} a & 2 & 6 \\ a & a & 3 \\ a & a & a \end{bmatrix} \overset{A_{12}(-1)}{\underset{A_{13}(-1)}{\sim}} \begin{bmatrix} a & 2 & 6 \\ 0 & a-2 & 3-6 \\ 0 & a-2 & a-6 \end{bmatrix} \overset{A_{23}(-1)}{\sim} \begin{bmatrix} a & 2 & 6 \\ 0 & a-2 & 3-6 \\ 0 & 0 & a-3 \end{bmatrix}$$

Thus

(i) $\det(A) = a(a-2)(a-3) \rightarrow$ depends on a, not on b
(ii) $\det(A) = 0$ iff $a \in \{0, 2, 3\} \rightarrow A$ invertible iff $a \notin \{0, 2, 3\}$
(iii) row reduced echelon has 3 leading 1s if $a \notin \{0, 2, 3\}$

(d)

Pb 9. $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 2 & -3 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ 2 & 1 & 0 & 2 \end{bmatrix}$

Expand wrt 3^{rd} column, which has 2 0's.
We get

$$\det(A) = -3 \begin{vmatrix} 2 & -3 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 2 \\ 2 & -3 & 2 \\ 2 & 1 & 2 \end{vmatrix} = 32$$

(c)

Pb 10 We have seen:

$\det(AB) = \det(A)\det(B)$ (II)

$(AB)^{-1} = B^{-1}A^{-1} \neq A^{-1}B^{-1}$

Only (II) is true (a)

Pb 11 With Gauss-Jordan

$$A^{\#} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \overset{A_{12}(-1)}{\sim} \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}$$

$$\overset{\substack{A_{21}(-1) \\ A_{23}(-1)}}{\sim} \begin{pmatrix} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 \end{pmatrix} \overset{M_3(-1)}{\sim} \begin{pmatrix} 1 & 0 & -1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{pmatrix}$$

$$\overset{\substack{A_{31}(1) \\ A_{32}(-2)}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{array}\right)$$

(c)

$\hookrightarrow A^{-1}$

Pb 12. We have seen

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Here $A \in M_{33}$ and $\det(A) = 3 \neq 0$.

Thus

$$\det(A^{-1}) = \frac{1}{(\det(A))^3} \det(\operatorname{adj}(A))$$

$$\Leftrightarrow \quad \det(\operatorname{adj}(A)) = (\det(A))^3 \det(A^{-1})$$

In addition, $\det(A^{-1}) = \frac{1}{\det(A)}$. Hence

$$\det(\operatorname{adj}(A)) = (\det(A))^2 = 3^2 = 9$$

(c)