FINAL - FALL 20

1. The phase portrait to the right corresponds to a linear system of the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$ in which the matrix $\mathbf{A}$ has two linearly independent eigenvectors. Determine the nature of the eigenvalues of the system.

   Click here to view page 1 of Gallery of Typical Phase Portraits for the System x'=Ax: Nodes[7]
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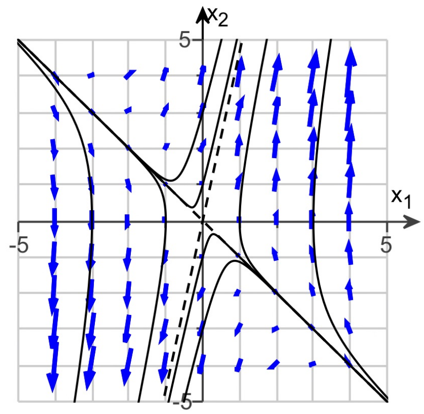
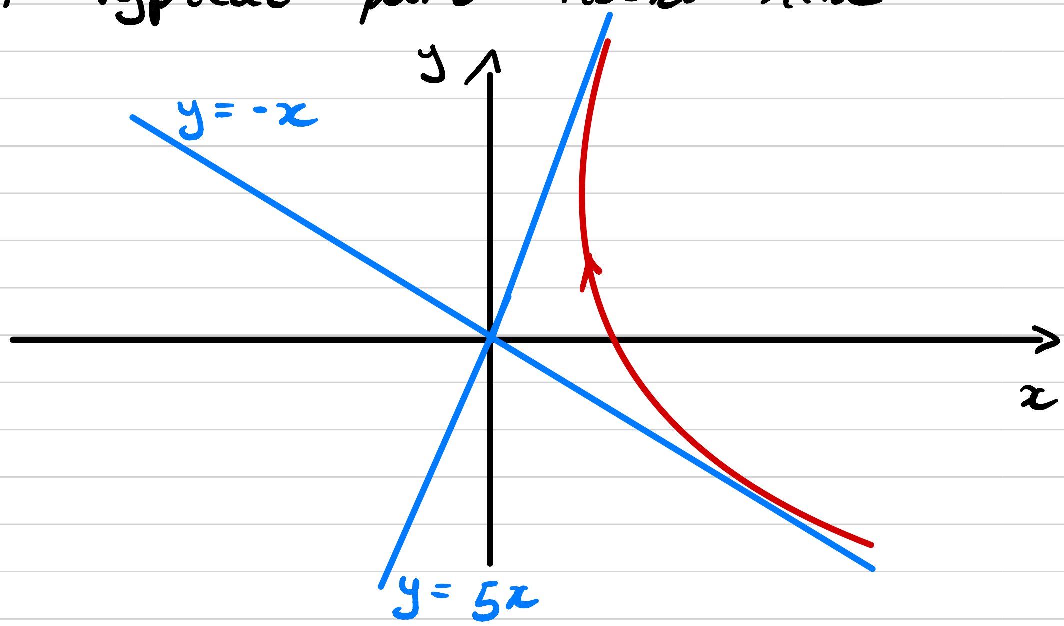


A typical path looks like



This corresponds to a function of the form

$$x(t) = c_1 e^{\lambda_1 t} \underbrace{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{v_1} + c_2 e^{\lambda_2 t} \underbrace{\begin{pmatrix} 1 \\ 5 \end{pmatrix}}_{v_2},$$

with $\lambda_1 < 0 < \lambda_2$. Saddle, 2 distinct eigvl

2. Transform the given system of differential equations into an equivalent system of first-order differential equations.

$$x'' + 5x' + 5x + 2y = 0$$
$$y'' + 3y' + 2x - 2y = \sin t$$

Change of variable We set

$x_1 = x$ $x_2 = x'$ $y_1 = y$ $y_2 = y'$

System We get

$$x_1' = x_2$$
$$x_2' + 5x_2 + 5x_1 + 2y_1 = 0$$
$$y_1' = y_2$$
$$y_2' + 3y_2 + 2x_1 - 2y_1 = \sin(t)$$

3. Find the general solutions of the system.

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 5 & 0 & 0 \\ -1 & 6 & 1 \\ 0 & 0 & 5 \end{bmatrix}}_{A} \mathbf{x}$$

We have

$$\det(A - \lambda I) = (5-\lambda)^2(6-\lambda)$$

Eigenvalues We get

$$\det(A - \lambda I) = (5-\lambda)^2(6-\lambda)$$

Hence $\lambda_1 = 5$ double eigenvalue

$\lambda_2 = 6$ simple eigenvalue

Eigenvector for λ_1

$$(A - 5I)v = 0 \iff \begin{pmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} v = 0$$

$$\iff -v^1 + v^2 + v^3 = 0$$

One can take

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

We get 2 eigenvectors for the double eigenvalue

Eigenvector for λ_2 We have

$(A-6Id)v=0 \iff \begin{pmatrix} -1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} v = 0$

Solutions are of the form $\begin{pmatrix} 0 \\ r \\ 0 \end{pmatrix}$. We choose

$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

General solution Of the form

$x(t) = c_1 e^{5t} v_1 + c_2 e^{5t} v_2 + c_3 e^{6t} v_3$

$= c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{6t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

4. What can be said about the following statements?

I) If A and B are square matrices, and det(B) is not equal to zero and B^{-1} is the inverse of B, then $BAB^{-1} - \lambda I = B(A - \lambda I)B^{-1}$ and so the matrices A and BAB^{-1} have the same eigenvalues.

II) If A is a square matrix and A^T is the transpose of A, then $\det(A - \lambda I) = \det\left(A^T - \lambda I\right)$ and so A and A^T have the same eigenvalues.

III) If A is a square matrix and det(A) is not equal to zero. If A^{-1} is the inverse of A and if λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

IV) If a 4x4 matrix A is defective, then it must have one eigenvalue of multiplicity three.

I We have

$$BAB^{-1} - \lambda I = BAB^{-1} - \lambda BIB^{-1}$$
$$= B(A - \lambda I)B^{-1}$$

Hence $\det(BAB^{-1} - \lambda I)$

$$= \det(B(A-\lambda I)B^{-1})$$
$$= \det(B)\det(A-\lambda I)\det(B^{-1})$$
$$= \det(A-\lambda I)$$

Thus A and BAB^{-1} have the same eigenvalues. I True

II $\det(A^T - \lambda I) = \det(A^T - \lambda I^T)$

$$= \det((A-\lambda I)^T) = \det(A-\lambda I)$$

Thus A and A^T have the same eigenvalues

II True

III If λ is an eigenvalue for A, there exists a nontrivial $u \in \mathbb{R}^n$ s.t

$$A u = \lambda u$$

$\times A^{-1}$

$$\Leftrightarrow \quad u = \lambda A^{-1} u$$

$$\Leftrightarrow \quad A^{-1} u = \frac{1}{\lambda} u$$

Hence $\frac{1}{\lambda}$ is an eigenvalue for A^{-1}

III True

IV A can have an eigenvalue with multiplicity 2 and 1 eigenvector only.

IV False

5. Let y(x) satisfy the following initial value problem:

$y''(x) + y(x) = \textbf{tan}(x)$

$y(0) = 0$ and $y'(0) = -1$

Then $y\left(\frac{\pi}{4}\right)$ (which is the value of y(x) when $x = \frac{\pi}{4}$) is equal to:

Strategy

This is a nonhomogeneous linear differential equation of order 2. Since tan is not one of the functions for which the undetermined coefficient method applies, we will use variation of parameters

Solutions for the hom. part The fundamental solutions of $y'' + y = 0$ are

$$y_1 = \cos(x) \qquad y_2 = \sin(x)$$

Particular solution Of the form

$y_p = u_1 y_1 + u_2 y_2$ with

$$\begin{cases} \cos(x)\, u_1' + \sin(x)\, u_2' = 0 \\ -\sin(x)\, u_1' + \cos(x)\, u_2' = \tan(x) \end{cases}$$

$$\Leftrightarrow \quad A\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}, \quad A = \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}$$

Solving the system   The system is

$$A\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}, \quad A = \begin{pmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{pmatrix}$$

Since $\det(A) = 1$, Cramer's rule yields

$$u_1' = \begin{vmatrix} 0 & \sin(x) \\ \tan(x) & \cos(x) \end{vmatrix} = -\frac{\sin^2(x)}{\cos(x)}$$

$$u_2' = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \tan(x) \end{vmatrix} = \sin(x)$$

Integrating   we get

$$u_1 = \int u_1'\,dx = \int -\frac{(1-\cos^2(x))}{\cos(x)}\,dx$$

$$= -\int \sec(x)\,dx + \int \cos(x)\,dx$$

$$= -\ln(|\sec(x) + \tan(x)|) + \sin(x) \quad (+c)$$

$$u_2 = \int u_2'\,dx = \int \sin(x)\,dx = -\cos(x) \quad (+c)$$

Thus

$$y_p = \left(-\ln(|\sec(x) + \tan(x)|) + \sin(x)\right)\cos(x) - \cos(x)\sin(x)$$

General solution We have found

$$y = c_1 \cos(x) + c_2 \sin(x) + y_p$$
$$= c_1 \cos(x) + c_2 \sin(x) + y_1 u_1 + y_2 u_2$$

Initial data We are given

$$y(0) = 0 \quad , y'(0) = -1$$

Moreover $y_p(0) = 0$. Hence

$$y(0) = 0 \implies c_1 = 0$$

Therefore

$$y' = c_2 \cos(x) + \overbrace{y_1 u_1' + y_2 u_2'}^{=0 \text{ from system}} + y_1' u_1 + y_2' u_2$$

From the expressions of u_1, u_2 we have $u_1(0) = 0$, $u_2(0) = -1$. Hence

$$y'(0) = -1 \iff c_2 - \cos(0) \times 1 = -1$$
$$\iff c_2 = 0$$

Unique solution

$$y = \left(-\ln\left(|\sec(x) + \tan(x)|\right) + \sin(x)\right)\cos(x)$$
$$- \cos(x)\sin(x)$$
$$= -\cos(x)\ln\left(|\sec(x) + \tan(x)|\right)$$

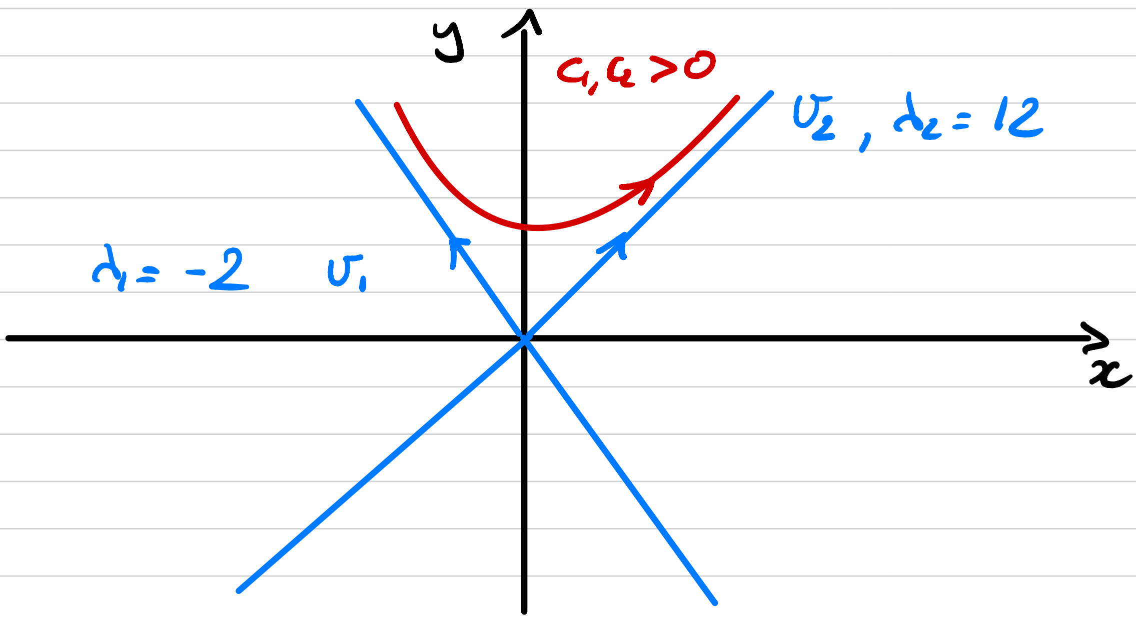
Hence, since $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$,

$$\boxed{y\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\ln\left(1+\sqrt{2}\right)}$$

6. Categorize the eigenvalues and eigenvectors of the coefficient matrix **A** according to the accompanying classifications and sketch the phase portrait of the system by hand. Then use a computer system or graphing calculator to check your answer.

System of equations	**Matrix equation**
$x_1' = 5x_1 + 7x_2$ $x_2' = 7x_1 + 5x_2$	$\mathbf{x}' = \begin{bmatrix} 5 & 7 \\ 7 & 5 \end{bmatrix}\mathbf{x}$
Eigenvalues	**Eigenvectors**
$\lambda_1 = -2, \lambda_2 = 12$	$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

The eigenvalues are real, distinct, with opposite signs. A typical graph is given by



We have classified this situation as saddle point

7. Three 234-gal fermentation vats are connected as indicated in the figure, and the mixtures in each tank are kept uniform by stirring. Denote by $x_i(t)$ the amount (in pounds) of alcohol in tank T_i at time t ($i = 1, 2, 3$). Suppose that the mixture circulates between the tanks at the rate of 18 gal/min. Derive the equations.

$$\begin{aligned} 13x_1' &= -x_1 \qquad + x_3 \\ 13x_2' &= x_1 - x_2 \\ 13x_3' &= \qquad x_2 - x_3 \end{aligned}$$

Let V_i = Volume tank $i \equiv V = 234$ gal

r = flow rate = 18 gal/min

Then

x_1' = flow in − flow out

$= \dfrac{x_3}{V} \times r - \dfrac{x_1}{V} \times r$ Set $L = \dfrac{V}{r} = 13$

$\Leftrightarrow L x_1' = -x_1 + x_3$

Along the same lines for x_2, x_3 we get

$$\begin{aligned} L x_1' &= -x_1 \qquad + x_3 \\ L x_2' &= x_1 - x_2 \\ L x_3' &= \qquad x_2 - x_3 \end{aligned}$$

8. Let y(t) be the solution of the following equalton representing a spring-mass system:

$y''(t) + 4y'(t) + 5y(t) = 0$

$y(0) = A$ and $y'(0) = B$

with $A \neq 0$ and $B \neq 0$. Then $\frac{y(\pi)}{y(3\pi)}$ (this is the quotient of the values of $y(\pi)$ and $y(3\pi)$) is equal to.

Characteristic polynomial

$$P(r) = r^2 + 4r + 5 = (r+2)^2 + 1$$

Roots : $-2 \pm i$

General solution

$$y(t) = e^{-2t}\left(c_1 \cos(t) + c_2 \sin(t)\right)$$

Initial condition $y(0) = A$, $y'(0) = B$. Thus $c_1 = A$. Moreover $\sin(3\pi) = \sin(\pi) = 0$, hence c_2 is not relevant in the computation of $y(\pi)$, $y(3\pi)$. In the end we get

$$y(\pi) = -e^{-2\pi} A \qquad y(3\pi) = -e^{-6\pi} A$$

Hence

$$\boxed{\frac{y(\pi)}{y(3\pi)} = \frac{-e^{-2\pi} A}{-e^{-6\pi} A} = e^{4\pi}}$$

9. The appropriate form of a particular solution of the differential equation

$(D-1)^3(D-3)^4\left(D^2+1\right)\ y(x) = x^{\boxed{3}}\ e^{x} + x^4\ e^{3x} + x^2\ \mathbf{sin}\,(x)$

is of the form

$y_p(x) = x^3 p_1(x)\ e^{x} + x^4\ p_2(x)\ e^{3x} + x\, p_3(x)\ \mathbf{sin}\ (x) + x\, p_4(x)\ \mathbf{cos}\,(x),$

where $p_1(x)$ is a polynomial of degree d_1, $p_2(x)$ is a polynomial of degree d_2, $p_3(x)$ is a polynomial of degree d_3, and $p_4(x)$ is a polynomial of dgree d_4. Which of the following is true?

The characteristic polynomial has roots

Root	Multiplicity
1	$\boxed{3}$
3	4
$\pm i$	1

Hence y_p is of the fum

$$x^{\boxed{3}} p_1(x)\, e^{x} + x^4 p_2(x)\, e^{3x} + x\, p_3 \sin(x) + x\, p_4 \cos(x)$$

Where

Polynomial	Degree
p_1	$\boxed{3}$
p_2	4
p_3	2

10. Find the general solution of the given system. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the system.

$$\mathbf{x}' = \underbrace{\begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}}_{A}\mathbf{x}$$

Eigenvalue We have

$$\det(A - rI) = (r-3)(r-1) + 1 = r^2 - 4r + 4 = (r-2)^2$$

Hence $r = 2$ is a double eigenvalue.

Eigenvector We solve

$$(A - 2I)v = 0 \iff \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} v = 0 \iff v^2 = -v^1$$

We thus choose $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Generalized eigenvectors We have $(A-2I)^2 v = 0$

Thus one can choose $v_2 = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$. Then

$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ chosen in the answer keys

$$v_1 = (A - 2I)v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (= v)$$

General solution

$$x(t) = e^{2t}\left\{ c_1 v_1 + c_2 v_1 t + c_2 v_2 \right\}$$

$$= c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\underbrace{\begin{pmatrix} t+1 \\ -t \end{pmatrix}}$$

Becomes $\begin{pmatrix} t \\ -t+1 \end{pmatrix}$ if $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

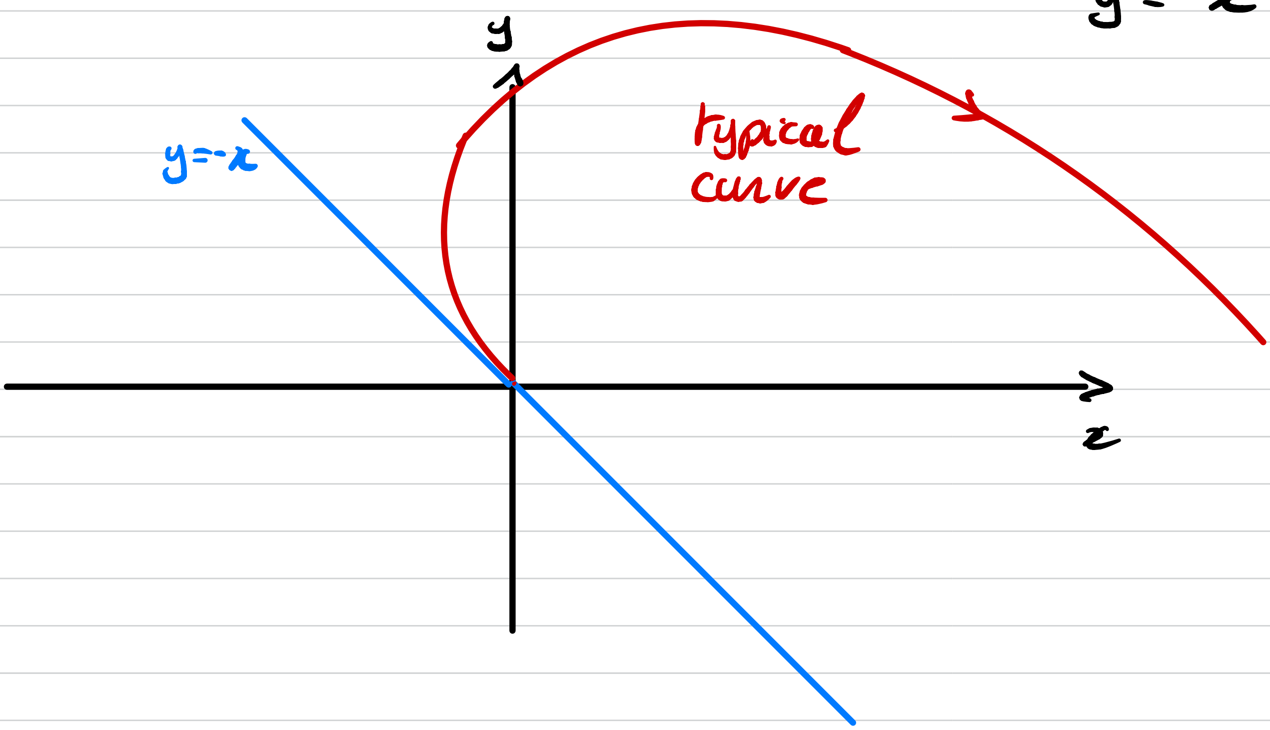
<u>Graph</u> We have found

$$x(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} t+1 \\ -t \end{pmatrix}$$

Hence

(i) As $t \to -\infty$, $x(t) \to 0$ with dominant direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(ii) As $t \to \infty$, $x(t) \to \infty$ with dominant direction $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and not close to the line $y = -x$



11. Apply the eigenvalue method to find a general solution of the given system. Find the particular solution corresponding to the given initial values. Use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

$x'_1 = 3x_1 + 4x_2$, $x'_2 = 3x_1 + 2x_2$, $x_1(0) = x_2(0) = 1$

The system is $x' = Ax$ with $A = \begin{pmatrix} 3 & 4 \\ 3 & 2 \end{pmatrix}$

Eigenvalues $\det(A - rI) = (r-3)(r-2) - 12$

$= r^2 - 5r - 6$

Roots: $r_1 = -1$, $r_2 = 6$

Eigenvectors (i) $(A + I)v = 0 \iff \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} v = 0$

$\iff v^2 = -v^1$. We take

$v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(ii) $(A - 6I)v = 0 \iff \begin{pmatrix} -3 & 4 \\ 3 & -4 \end{pmatrix} v = 0$

$\iff 4v^2 = 3v^1$

We take $v_2 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

General solution

$x(t) = c_1 e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{6t} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Initial condition The condition $x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ reads

$$\underbrace{\begin{pmatrix} 1 & 4 \\ -1 & 3 \end{pmatrix}}_{B} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Moreover $\det(B) = 7$. Hence following Cramer's rule we get

$$c_1 = \frac{1}{7} \begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = -\frac{1}{7}$$

$$c_2 = \frac{1}{7} \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} = \frac{2}{7}$$

Unique solution With our initial condition,

$$x(t) = -\frac{1}{7} e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{2}{7} e^{6t} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$