First order differential equations

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Differential equations and linear algebra - MA 262

Taken from *Differential equations and linear algebra*
Edwards, Penney, Calvis

Outline

1. Differential equations and mathematical models
2. Integrals as general and particular solutions
3. Slope fields and solution curves
4. Separable equations and applications
5. Linear equations
6. Substitution methods and exact equations
   - Homogeneous equations
   - Bernoulli equations
   - Exact differential equations
   - Reducible second order differential equations
7. Chapter review

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Newton's second law of motion

Quantities: For an object of mass m

- Force F
- Velocity $v(t)$ at time t
- Displacement $y = y(t)$

Newton's law:

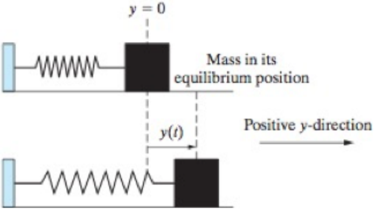
$$m \frac{dv}{dt} = F$$

Differential equation: Since $v = \frac{dy}{dt}$, we get

$$m \frac{d^2 y}{dt^2} = F$$

Spring force

Physical system: spring-mass with no friction



Hooke's law: The spring force is given by

$$F_s = -k\, y$$

Second order differential equation

Differential equation:

- Equation involving the derivatives of a function
- In particular the unknown is a function

Equation for spring-mass system:

According to Newton's and Hooke's laws

$$m \frac{d^2 y}{dt^2} = -ky \tag{1}$$

Second order differential equation (2)

Equation for spring-mass system (2): Set

$$\omega = \sqrt{\frac{k}{m}}$$

Then (1) is equivalent to

$$\frac{d^2y}{dt^2} + \omega^2 y = 0.$$

Solution: Of the form

$$y(t) = A\cos(\omega t - \phi).$$

Order of a differential equation

Definition: Order of a differential equation
= Order of highest derivative appearing in equation

Examples:

- Second law of motion, spring: second order
- First order: $y' = 4 - y^2$

General form of n-th order differential equation:

$$G(y, y', \ldots, y^{(n)}) = 0 \tag{2}$$

More vocabulary

Linear equations: of the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = F(x)$$

Initial value problem: A differential equation

$$G(y, y', \ldots, y^{(n)}) = 0,$$

plus initial values in order to get a unique solution:

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad \ldots \quad , y^{(n-1)}(x_0) = y_{n-1}$$

General solution: When no initial condition is specified
$\hookrightarrow$ Solution given in terms of constants $c_1, \ldots, c_n$

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Simplest case of differential equation

Equation: The rhs does not depend on y

$$y' = f(x)$$

General solution: For all $C \in \mathbb{R}$,

$$y(x) = \int f(x)\, dx + C$$

Family of solutions:

- We get a family indexed by $C \in \mathbb{R}$
- Two solutions for $C_1 \neq C_2$ are parallel

Example of parallel curves

Illustration:

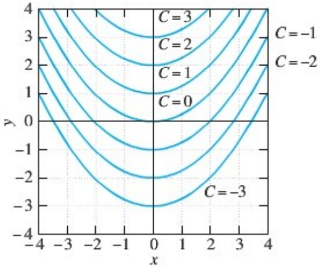


FIGURE 1.2.1. Graphs of $y = \frac{1}{4}x^2 + C$ for various values of C.

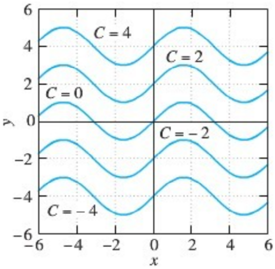


FIGURE 1.2.2. Graphs of $y = \sin x + C$ for various values of C.

Example of direct integration (1)

Equation: We want to solve

$$y' = 2x + 3$$

Example of direct integration (2)

General solution:

$$y = \int (2x + 3)\, dx = x^2 + 3x + C$$

Initial value problem

Particular solution: given by specifying an initial data

$$y' = f(x), \quad \text{and} \quad y(x_0) = y_0$$

Advantage:

An initial value problem yields a unique solution

Example of initial value problem (1)

Equation: We want to solve

$$y' = 2x + 3, \qquad \text{and} \qquad y(1) = 2$$

Example of initial value problem (2)

Unique solution: we get

$$y = \int (2x + 3)\, dx = x^2 + 3x - 2$$

Lunar lander problem (1)

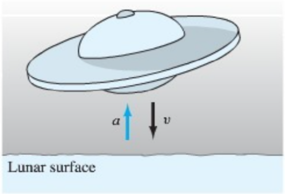
Situation:

- Lunar lander falling freely at speed $450 m/s$
- Retrorockets provide deceleration of $2.5 m/s^2$

Question:

At what height should we activate the retrorockets
$\hookrightarrow$ in order to ensure $v = 0$ at the surface?

Solution



Lunar lander problem (2)

Time origin: We set

- $t = 0$ when the retrorockets should be fired

Initial value problem: We want to solve

$$v' = 2.5, \quad \text{and} \quad v(0) = -450$$

Expression for v:

$$v(t) = 2.5t - 450$$

Time such that $v = 0$: We find

$$t = \frac{450}{2.5} = 180s$$

Lunar lander problem (3)

Expression for x:

$$x(t) = \int v(t)\, dt = 1.25t^2 - 450t + x_0$$

Aim: We wish to have

$$v = 0 \text{ when } x = 0, \text{ or otherwise stated } x = 0 \text{ for } t = 180$$

Solution: We find

$$x_0 = 40,500$$

Swimmer's problem (1)

Situation:

- River width parametrized by $-a \leq x \leq a$
- Velocity of the water flow is vertical and satisfies

$$v_R = v_0 \left(1 - \frac{x^2}{a^2}\right)$$

- Swimmer starts from $(-a, 0)$ with constant horizontal speed v_S

Question:

Find an equation for the function $y(x)$ of the swimmer
Particular case: $v_0 = 9\text{mi/h}$, $v_S = 3\text{mi/h}$ and $a = 1/2\text{mi}$

Swimmer's problem (2)

Equation: We have

$$\frac{dy}{dx} = \tan(\alpha) = \frac{v_0}{v_S}\left(1 - \frac{x^2}{a^2}\right)$$

Particular case: with $v_0 = 9\text{mi/h}$, $v_S = 3\text{mi/h}$ and $a = 1/2\text{mi}$ we get

$$\frac{dy}{dx} = 3\left(1 - 4x^2\right)$$

Thus

$$y(x) = 3x - 4x^3 + C$$

Swimmer's problem (3)

Initial value problem: The initial condition is

$$y\left(-\frac{1}{2}\right) = 0$$

Thus

$$y(x) = 3x - 4x^3 + 1$$

Downstream velocity at the end of the river:

$$y\left(\frac{1}{2}\right) = 2$$

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Existence and uniqueness result

Theorem 1.

General nonlinear equation:

$$y' = f(t, y), \qquad y(t_0) = y_0 \in \mathbb{R}. \tag{3}$$

Hypothesis:

- $(t_0, y_0) \in R$, where $R = (\alpha, \beta) \times (\gamma, \delta)$.
- f and $\frac{\partial f}{\partial y}$ continuous on R.

Conclusion:

One can find $h > 0$ such that there exists a unique function y $\hookrightarrow$ satisfying equation (3) on $(t_0 - h,\ t_0 + h)$.

Example of existence and uniqueness (1)

Equation considered:

$$y' = 3x\, y^{1/3}, \quad \text{and} \quad y(0) = a.$$

Example of existence and uniqueness (2)

Application of Theorem 1: we have

$$f(x, y) = 3x\, y^{1/3}, \qquad \frac{\partial f}{\partial y}(x, y) = xy^{-2/3}$$

Therefore if $a \neq 0$:

1. There exists rectangle R such that
   - $(0, a) \in R$
   - f and $\frac{\partial f}{\partial y}$ continuous on R
2. According to Theorem 1 there is unique solution on interval $(-h, h)$, with $h > 0$

Second example of existence and uniqueness (1)

Equation considered:

$$y' = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \text{and} \quad y(0) = -1.$$

Second example of existence and uniqueness (2)

Application of Theorem 1: we have

$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)}, \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2+4x+2}{2(y-1)^2}$$

Therefore:

1. There exists rectangle R such that
   - $(0,-1) \in R$
   - f and $\frac{\partial f}{\partial y}$ continuous on R
2. According to Theorem 1 there is unique solution on interval $(-h,h)$, with $h > 0$

Second example of existence and uniqueness (3)

Comparison with explicit solution: We will see that

$$y = 1 - \left((x+2)(x^2+2)\right)^{1/2}$$

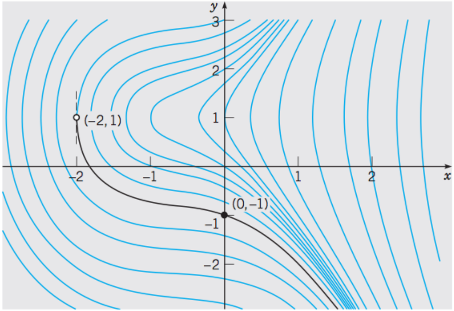
Interval of definition: $x \in (-2, \infty)$
$\hookrightarrow$ much larger than predicted by Theorem 1

Changing initial condition: consider $y(0) = 1$, on line $y = 1$. Then:

1. Theorem 1: nothing about possible solutions
2. Direct integration:
   - We find $y = 1 \pm (x^3 + 2x^2 + 2x)^{1/2}$
   - 2 possible solutions defined for $x > 0$

Second example of existence and uniqueness (3)

Interval of definition on integral curves:



Comments:

- Interval of definition delimited by vertical tangents

Example with non-uniqueness (1)

Equation considered:

$$y' = y^{1/3}, \quad \text{and} \quad y(0) = 0.$$

Application of Theorem 1: $f(y) = y^{1/3}$. Hence,

- $f : \mathbb{R} \to \mathbb{R}$ continuous on $\mathbb{R}$, differentiable on $\mathbb{R}^*$
- Theorem 1: gives existence, not uniqueness

Solving the problem: Separable equation, thus

- General solution: for $c \in \mathbb{R}$, $y = \left[\frac{2}{3}(t + c)\right]^{3/2}$
- With initial condition $y(0) = 0$,

$$y = \left(\frac{2t}{3}\right)^{3/2}$$

Example with non-uniqueness (2)

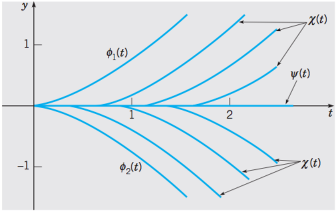
3 solutions to the equation:

$$\phi_1(t) = \left(\frac{2t}{3}\right)^{3/2}, \quad \phi_2(t) = -\left(\frac{2t}{3}\right)^{3/2}, \quad \psi(t) = 0.$$

Family of solutions: For any $t_0 \geq 0$,

$$\chi(t) = \chi_{t_0}(t) = \begin{cases} 0 & \text{for } 0 \leq t < t_0 \\ \pm\left(\frac{2(t-t_0)}{3}\right)^{3/2} & \text{for } t \geq t_0 \end{cases}$$

Integral curves:



Slope field for a gravity equation (1)

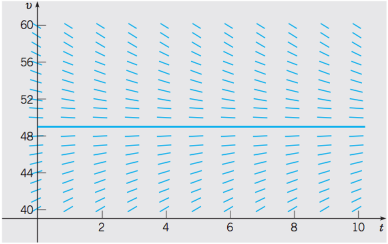
Gravity equation with friction

$$\frac{dv}{dt} = 9.8 - \frac{v}{5} \tag{4}$$

Slope field for a gravity equation (2)

Meaning of the graph:

$\hookrightarrow$ Values of $\frac{dv}{dt}$ according to values of v



Slope field for a gravity equation (3)

What can be seen on the graph:

- Critical value: $v_c = 49\text{ms}^{-1}$, solution to $9.8 - \frac{v}{5} = 0$
- If $v < v_c$: positive slope
- If $v > v_c$: negative slope

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General form of separable equation

General form of first order differential equation:

$$y' = f(x, y)$$

Definition 2.

A first order separable differential equation is of the form

$$h(y)\frac{dy}{dx} = g(x) \quad \Longleftrightarrow \quad h(y)\,dy = g(x)\,dx \tag{5}$$

Solving separable equations:
Integrate on both sides of (5).

Example of separable equation (1)

Equation:

$$(1 + y^2)\frac{dy}{dx} = x \cos(x)$$

Example of separable equation (2)

General solution: After integration by parts

$$y^3 + 3y = 3(x\sin(x) + \cos(x)) + c,$$

where $c \in \mathbb{R}$

Remark: Solution given in implicit form.

Example 2 of separable equation

Equation:

$$x\,dx + y\exp(-x)\,dy = 0, \qquad y(0) = 1$$

Unique solution:

$$y(x) = \left(2\exp(x) - 2x\exp(x) - 1\right)^{1/2}$$

Remark:
Radical vanishes for $x_1 \simeq -1.7$ and $x_2 \simeq 0.77$

Example 3 of separable equation (1)

Equation considered:

$$\frac{dy}{dx} = \frac{x^2}{1-y^2} \iff -x^2 + (1-y^2)\frac{dy}{dx} = 0. \tag{6}$$

Example 3 of separable equation (2)

Chain rule:

$$\frac{df(y)}{dx} = f'(y)\,\frac{dy}{dx}$$

Application of chain rule:

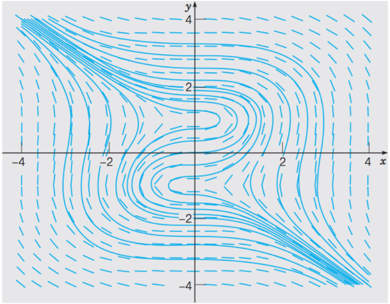
$$(1-y^2)\frac{dy}{dx} = \frac{d}{dx}\left(y - \frac{y^3}{3}\right), \quad \text{and} \quad x^2 = \frac{d}{dx}\left(\frac{x^3}{3}\right)$$

Example 3 of separable equation (3)

Equation for integral curves: We have, for $c \in \mathbb{R}$,

$$(6) \iff \frac{d}{dx}\left(-\frac{x^3}{3} + y - \frac{y^3}{3}\right) = 0 \iff -x^3 + 3y - y^3 = c$$

Some integral curves obtained by approximation:



General solution for separable equations

Proposition 3.

Equation considered:

$$M(x) + N(y)\,\frac{dy}{dx} = 0. \qquad (7)$$

Antiderivatives: let H_1, H_2 such that

$$H_1'(x) = M(x) \quad \text{and} \quad H_2'(y) = N(y).$$

Then general solution to (7) is given by:

$$H_1(x) + H_2(y) = c,$$

with $c \in \mathbb{R}$.

Solvable example of separable equation (1)

Equation considered:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y-1)}, \quad \text{and} \quad y(0) = -1. \tag{8}$$

Solvable example of separable equation (2)

Integration: for a constant $c \in \mathbb{R}$,

$$\begin{aligned}(8) \quad &\Longleftrightarrow \quad 2(y-1)\,dy = \left(3x^2 + 4x + 2\right)dx \\ &\Longleftrightarrow \quad y^2 - 2y = x^3 + 2x^2 + 2x + c\end{aligned}$$

Solving the equation: if $y(0) = -1$, we have $c = 3$ and

$$y = 1 \pm \left(x^3 + 2x^2 + 2x + 4\right)^{1/2}$$

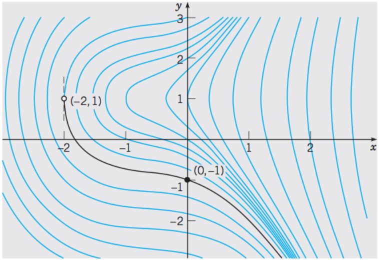
Solvable example of separable equation (3)

Determination of sign: Using $y(0) = -1$ again, we get

$$y = 1 - \left(x^3 + 2x^2 + 2x + 4\right)^{1/2} = 1 - \left((x+2)(x^2+2)\right)^{1/2}$$

Interval of definition: $x \in (-2, \infty)$
$\hookrightarrow$ boundary corresponds to vertical tangent on graph below

Integral curves:



Example of equation with implicit solution (1)

Equation considered:

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}.$$

General solution: for a constant $c \in \mathbb{R}$,

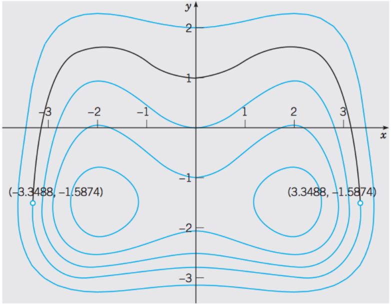
$$y^4 + 16y + x^4 - 8x^2 = c$$

Initial value problem: if $y(0) = 1$, we get

$$y^4 + 16y + x^4 - 8x^2 = 17$$

Example of equation with implicit solution (2)

Integral curves:



Interval of definition:

$\hookrightarrow$ boundary corresponds to vertical tangent on graph

Cooling cup example

Description of experiment:

- Cup of coffee cooling in a room

Notation:

- $T(t) \equiv$ temperature of cup
- $\tau \equiv$ temperature of room

Newton's law for thermic exchange:
Variations of temperature proportional to difference between T and τ

Equation:

$$\frac{dT}{dt} = -k\,(T - \tau)\,, \qquad T(0) = T_0.$$

Malthusian growth

Hypothesis:
Rate of change proportional to value of population

Equation: for $k \in \mathbb{R}$ and $P_0 \geq 0$,

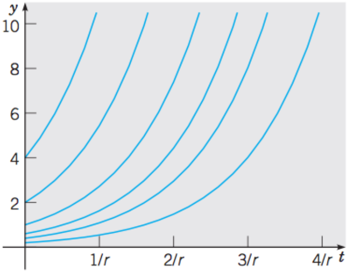
$$\frac{dP}{dt} = k\,P, \qquad P(0) = P_0$$

Solution:

$$P = P_0 \exp(kt)$$

Exponential growth (2)

Integral curves:



Limitation of model:

- Cannot be valid for large time t.

Logistic population model

Basic idea:

- Growth rate decreases when population increases.

Model:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{C}\right)P, \tag{9}$$

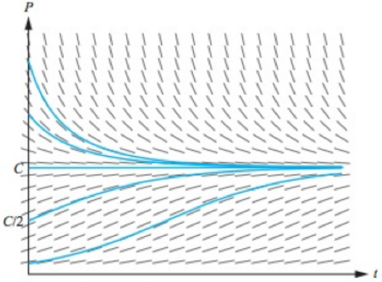
where

- $r \equiv$ reproduction rate
- $C \equiv$ carrying capacity

Logistic model: qualitative study

Information from slope field:

- Equilibrium at $P = C$
- If $P < C$ then $t \mapsto P$ increasing
- If $P > C$ then $t \mapsto P$ decreasing
- Possibility of convexity analysis



Logistic model: solution

First observation: Equation (9) is separable

Integration: Integrating on both sides of (9) we get

$$\ln\left(\left|\frac{P}{C-P}\right|\right) = rt + c_1$$

which can be solved as:

$$P(t) = \frac{c_2 C}{c_2 + e^{-rt}}$$

Initial value problem: If P_0 is given we obtain

$$P(t) = \frac{C\,P_0}{P_0 + (C - P_0)e^{-rt}}$$

Information obtained from the resolution

Asymptotic behavior:

$$\lim_{t \to \infty} P(t) = C$$

Prediction: If

- Logistic model is accurate
- P_0, r and C are known

Then we know the value of P at any time t

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General form of 1st order linear equation

General form 1:

$$\frac{dy}{dt} + p(t)y = g(t)$$

General form 2:

$$P(t)\frac{dy}{dt} + Q(t)y = G(t)$$

Remark:

2 forms are equivalent if $P(t) \neq 0$

Example with direct integration

Equation:

$$\left(4+t^2\right)\frac{dy}{dt}+2t\,y=4t$$

Equivalent form:

$$\frac{d}{dt}\left[\left(4+t^2\right)y\right]=4t$$

General solution: For a constant $c\in\mathbb{R}$,

$$y=\frac{2t^2+c}{4+t^2}$$

Method of integrating factor

General equation:

$$\frac{dy}{dt} + p(t)y = g(t) \tag{10}$$

Recipe for the method:

1. Consider equation (10)
2. Multiply the equation by a function μ
3. Try to choose μ such that equation (10) is reduced to:

$$\frac{d(\mu\, y)}{dt} = a(t) \tag{11}$$

4. Integrate directly equation (11)

Notation: If previous recipe works, μ is called integrating factor

Example of integrating factor (1)

Equation:

$$\frac{dy}{dt} + \frac{1}{2}\, y = \frac{1}{2}\, e^{t/3} \tag{12}$$

Example of integrating factor (2)

Multiplication by μ:

$$\mu(t)\frac{dy}{dt} + \frac{1}{2}\,\mu(t)\,y = \frac{1}{2}\,\mu(t)\,e^{t/3}$$

Integrating factor: Choose μ such that $\mu' = \frac{1}{2}\,\mu$, i.e $\mu(t) = e^{t/2}$

Solving the equation: We have, for $c \in \mathbb{R}$

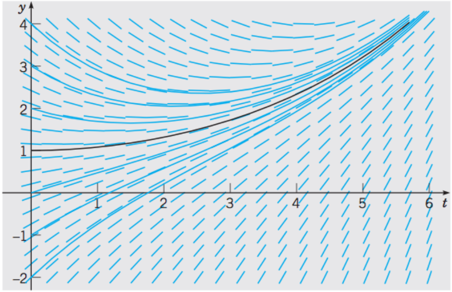
$$\begin{aligned}(12) \quad &\Longleftrightarrow \quad \frac{d\left(e^{t/2}y\right)}{dt} = \frac{1}{2}\,e^{\frac{5t}{6}} \\ &\Longleftrightarrow \quad y(t) = \frac{3}{5}\,e^{\frac{t}{3}} + c\,e^{-\frac{t}{2}}\end{aligned}$$

Example of integrating factor (3)

Solution for a given initial data: If we know $y(0) = 1$, then

$$y(t) = \frac{3}{5}\, e^{\frac{t}{3}} + \frac{2}{5}\, e^{-\frac{t}{2}}$$

Direction fields and integral curves:



General case with constant coefficient

Proposition 4.

Equation considered:

$$\frac{dy}{dt} + ay = g(t), \quad \text{and} \quad y(t_0) = y_0. \tag{13}$$

Hypothesis:

$$a \in \mathbb{R}, \qquad g : \mathbb{R}_+ \longrightarrow \mathbb{R} \text{ continuous.}$$

Then general solution to (13) is given by:

$$y(t) = e^{-at} \int_{t_0}^{t} e^{as} \, g(s) \, ds + c \, e^{-at}.$$

with $t_0 \geq 0$ and $c \in \mathbb{R}$.

Example with exponential growth

Equation:

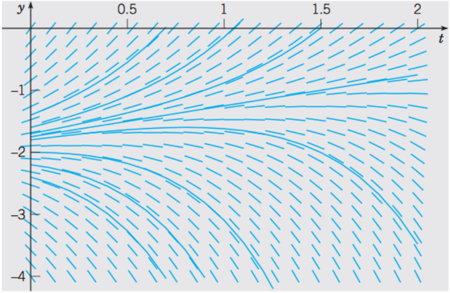
$$\frac{dy}{dt} - 2y = 4 - t$$

Example with exponential growth (2)

General solution: for $c \in \mathbb{R}$,

$$y(t) = -\frac{7}{4} + \frac{t}{2} + c\, e^{2t}$$

Direction fields and integral curves:



General first order linear case

Proposition 5.

Equation considered:

$$\frac{dy}{dt} + p(t)y = g(t), \tag{14}$$

Integrating factor:

$$\mu(t) = \exp\left(\int p(r)\, dr\right).$$

Then general solution to (14) is given by:

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)\, g(s)\, ds + c\right].$$

with $t_0 \geq 0$ and $c \in \mathbb{R}$.

Example with unbounded p (1)

Equation considered:

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2. \tag{15}$$

Example with unbounded p (2)

Equivalent form:

$$y' + \frac{2}{t} y = 4t, \qquad y(1) = 2.$$

Integrating factor:

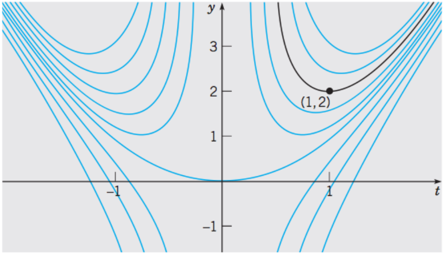
$$\mu(t) = t^2.$$

Solution:

$$y(t) = t^2 + \frac{1}{t^2} \tag{16}$$

Example with unbounded p (2)

Some integral curves:



Comments:

1. Example of solution which is not defined for all $t \geq 0$
2. Due to singularity of $t \mapsto \frac{1}{t}$
3. Integral curves for $t < 0$: not part of initial value problem
4. According to value of $y(1)$, different asymptotics as $t \to 0$
5. Boundary between 2 behaviors: function $y(t) = t^2$

Example with no analytic solution (1)

Equation considered:

$$2y' + t\,y = 2.$$

Example with no analytic solution (2)

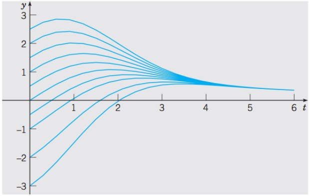
Integrating factor:

$$\mu(t) = \exp\left(\frac{t^2}{4}\right).$$

General solution:

$$y(t) = \exp\left(-\frac{t^2}{4}\right) \int_0^t \exp\left(\frac{s^2}{4}\right) ds + c \exp\left(-\frac{t^2}{4}\right).$$

Some integral curves obtained by approximation:



Existence and uniqueness: linear case

Theorem 6.

General linear equation:

$$y' + p(t)y = g(t), \qquad y(t_0) = y_0 \in \mathbb{R}. \tag{17}$$

Hypothesis:

- $t_0 \in I$, where $I = (\alpha, \beta)$.
- p and g continuous on I.

Conclusion:
There exists a unique function y satisfying equation (17) on I.

Remark: According to Theorem
$\hookrightarrow$ Solution fails to exists only when p or g are discontinuous

Maximal interval in a linear case

Equation considered: back to equation (15), namely

$$t\,y' + 2y = 4t^2, \qquad y(1) = 2.$$

Equivalent form:

$$y' + \frac{2}{t}\,y = 4t, \qquad y(1) = 2.$$

Application of Theorem 6:

- $g(t) = 4t$ continuous on $\mathbb{R}$
- $p(t) = \frac{2}{t}$ continuous on $(-\infty, 0) \cup (0, \infty)$ only
- $1 \in (0, \infty)$

We thus get unique solution on $(0, \infty)$

Maximal interval in a linear case (2)

Comparison with explicit solution: We have seen (cf (16)) that

$$y' + \frac{2}{t} y = 4t, \quad y(1) = 2 \quad \Longrightarrow \quad y(t) = t^2 + \frac{1}{t^2}.$$

This is defined on $(0, \infty)$ as predicted by Theorem 6.

Changing initial condition: consider

$$y' + \frac{2}{t} y = 4t, \qquad y(-1) = 2.$$

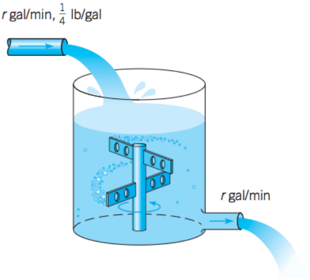
Then:

- Solution defined on $(-\infty, 0)$
- On $(-\infty, 0)$ we have $y(t) = t^2 + \frac{1}{t^2}$.

Salt concentration example

Description of experiment:

- At $t = 0$, Q_0 lb of salt dissolved in 100 gal of water
- Water containing $\frac{1}{4}$ lb salt/gal entering, with rate r gal/min
- Well-stirred mixture draining from tank, rate r



Salt concentration example (2)

Notation: $Q(t) \equiv$ quantity of salt at time t

Hypothesis: Variations of Q due to flows in and out,

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

Equation:

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}, \qquad Q(0) = Q_0$$

Equation, standard form:

$$\frac{dQ}{dt} + \frac{r}{100}\,Q = \frac{r}{4}, \qquad Q(0) = Q_0$$

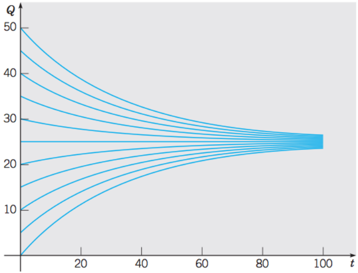
Salt concentration example (3)

Integrating factor: $\mu(t) = e^{\frac{rt}{100}}$

Solution:

$$Q(t) = 25 + (Q_0 - 25)\, e^{-\frac{rt}{100}}$$

Integral curves:



Salt concentration example (4)

Expression for Q:

$$Q(t) = 25 + (Q_0 - 25)\, e^{-\frac{rt}{100}}$$

Question: time to reach $q \in (Q_0, 25)$?

Answer: We find

$$Q(t) = q \quad \Longleftrightarrow \quad t = \frac{100}{r} \ln\left(\frac{Q_0 - 25}{q - 25}\right)$$

Application: If $r = 3$, $Q_0 = 50$ and $q = 25.5$, then:

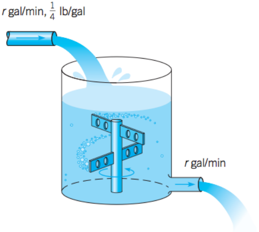
$$t = 130.4 \text{ min}$$

Chemical pollution example

Description of experiment:

- At $t = 0$, 10^7 gal of fresh water
- Water containing unwanted chemical component entering
  $\hookrightarrow$ with rate $5 \cdot 10^6$ gal/year
- Water flows out, same rate $5 \cdot 10^6$ gal/year
- Concentration of chemical in incoming water:

$$\gamma(t) = 2 + \sin(2t) \text{ g/gal}$$



Chemical pollution example (2)

Notation: $Q(t) \equiv$ quantity of chemical comp. at time t
$\hookrightarrow$ measured in grams

Remark: Volume is constant

Hypothesis: Variations of Q due to flows in and out,

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

Equation:

$$\frac{dQ}{dt} = 5 \cdot 10^6 \, \gamma(t) - 5 \cdot 10^6 \cdot \frac{Q}{10^7}, \qquad Q(0) = 0$$

Chemical pollution example (3)

Equation, standard form: We set $Q = 10^6 q$ and get

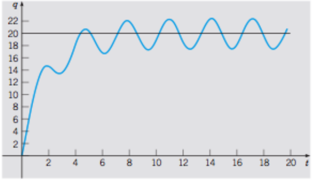
$$\frac{dq}{dt} + \frac{1}{2}\,q = 10 + 5\sin(2t), \qquad q(0) = 0$$

Integrating factor: $\mu(t) = e^{\frac{t}{2}}$

Solution:

$$q(t) = 20 - \frac{40}{17}\cos(2t) + \frac{10}{17}\sin(2t) - \frac{300}{17}e^{-\frac{t}{2}}$$

Integral curve:



Outline

1. Differential equations and mathematical models
2. Integrals as general and particular solutions
3. Slope fields and solution curves
4. Separable equations and applications
5. Linear equations
6. Substitution methods and exact equations
   - Homogeneous equations
   - Bernoulli equations
   - Exact differential equations
   - Reducible second order differential equations
7. Chapter review

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General form of homogeneous equation

General form of first order equation:

$$\frac{dy}{dx} = f(x, y) \tag{18}$$

General form of homogeneous equation:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

How to see if an equation is homogeneous: When in (18) we have

$$f(tx, ty) = f(x, y)$$

Heuristics to solve homogeneous equations:

- Go back to a separable equation

Solving homogeneous equations

Equation:

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right). \tag{19}$$

General method:

1. Set $y(x) = x\,V(x)$, and express y' in terms of x, V, V'.
2. Replace in equation (19) $\longrightarrow$ separable equation in V.
3. Solve the separable equation in V.
4. Go back to y recalling $y = x\,V$.

Theorem 7.

For equation (19), the function V satisfies

$$\frac{1}{F(V) - V}\,dV = \frac{1}{x}\,dx,$$

which is a separable equation

Example of homogeneous equation (1)

Equation:

$$\frac{dy}{dx} = \frac{4x + y}{x - 4y}$$

Example of homogeneous equation (2)

Equation for V:

$$\frac{1-4V}{4(1+V^2)}\,dV = \frac{1}{x}\,dx$$

Solution for V:

$$\frac{1}{4}\arctan(V) - \frac{1}{2}\ln(1+V^2) = \ln(|x|) + c_1$$

Solution for y:

$$\frac{1}{2}\arctan\left(\frac{y}{x}\right) - \ln\left(x^2+y^2\right) = c_2 \tag{20}$$

Example of homogeneous equation (3)

Polar coordinates: Set

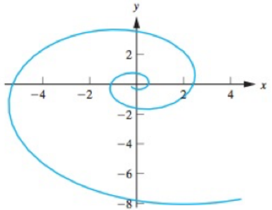
$$x = r\cos(\theta), \quad \text{and} \quad y = r\sin(\theta)$$

that is

$$r = \left(x^2 + y^2\right)^{1/2}, \quad \text{and} \quad \theta = \arctan\left(\frac{y}{x}\right)$$

Solution in polar coordinates: Equation (20) becomes

$$r = c_3 e^{\frac{\theta}{4}}$$



Another example of homogeneous equation (1)

Equation:

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}$$

Another example of homogeneous equation (2)

Equation for V:

$$\frac{1-V}{V^2-4}\,\frac{dV}{dx}=\frac{1}{x}$$

Solution for the V equation: for $c \in \mathbb{R}$,

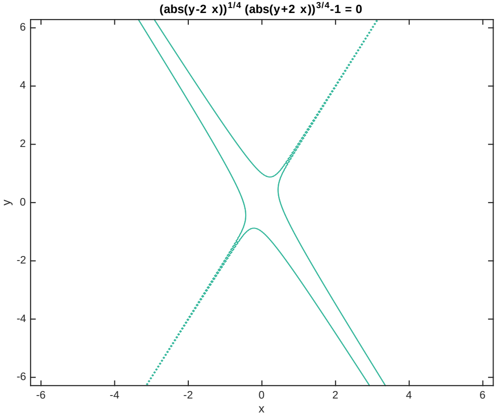
$$|V-2|^{1/4}|V+2|^{3/4}=\frac{c}{|x|}$$

Another example of homogeneous equation (3)

Solution for the y equation: for $c \in \mathbb{R}$,

$$|y - 2x|^{1/4} |y + 2x|^{3/4} = c$$

Graph for the implicit equation: observe symmetry w.r.t origin



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Jacob Bernoulli

Some facts about Bernoulli:

- Lifespan: 1654-1705, in Switzerland
- Discovers constant e
- Establishes divergence of $\sum \frac{1}{n}$
- Contributions in diff. eq
- Bernoulli:
  family of 8 prominent mathematicians
- Fierce math fights between brothers

Bernoulli equations

Definition 8.

A Bernoulli equation is of the form

$$y' + p(x)y = q(x)y^n \tag{21}$$

Recipe to solve a Bernoulli equation:

1. Divide equation (21) by y^n
2. Change of variable: $u = y^{1-n}$
3. The equation for u is a linear equation of the form

$$\frac{1}{1-n}u' + p(x)u = q(x)$$

Example of Bernoulli equation (1)

Equation:

$$y' + \frac{3}{x}y = \frac{12y^{2/3}}{(1+x^2)^{1/2}}$$

Example of Bernoulli equation (2)

Solution:

1. Divide the equation by $y^{2/3}$. We get

$$y^{-2/3}y' + \frac{3}{x}y^{1/3} = \frac{12}{(1+x^2)^{1/2}}$$

2. Change of variable $u = y^{1/3}$. We end up with the linear equation

$$u' + \frac{1}{x}u = \frac{4}{(1+x^2)^{1/2}}$$

Example of Bernoulli equation (3)

Solving the linear equation: Integrating factor given by

$$\mu(x) = x$$

Then integrating we get

$$u(x) = x^{-1}\left(4(1+x^2)^{1/2} + c\right)$$

Going back to y: We have $u = y^{1/3}$. Thus

$$y(x) = x^{-3}\left(4(1+x^2)^{1/2} + c\right)^3$$

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Example of exact equation

Equation considered:

$$2x + y^2 + 2xyy' = 0 \tag{22}$$

Remark: equation (22) neither linear nor separable

Additional function: Set $\phi(x, y) = x^2 + xy^2$. Then:

$$\frac{\partial \phi}{\partial x} = 2x + y^2, \quad \text{and} \quad \frac{\partial \phi}{\partial y} = xy.$$

Example of exact equation (2)

Expression of (22) in terms of ϕ: we have

$$(22) \quad \Longleftrightarrow \quad \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0$$

Solving the equation: We assume $y = y(x)$. Then

$$(22) \quad \Longleftrightarrow \quad \frac{d\phi}{dx}(x, y) = 0 \quad \Longleftrightarrow \quad \phi(x, y) = c,$$

for a constant $c \in \mathbb{R}$.

Conclusion: equation solved under implicit form

$$x^2 + xy^2 = c.$$

Recall: separable equations

Proposition 9.

Equation considered:

$$M(x)\,dx + N(y)\,dy = 0. \qquad (23)$$

Antiderivatives: let H_1, H_2 such that

$$H_1'(x) = M(x) \quad \text{and} \quad H_2'(y) = N(y).$$

Then general solution to (23) is given by:

$$H_1(x) + H_2(y) = c,$$

with $c \in \mathbb{R}$.

General exact equation

Proposition 10.

Equation considered:

$$M(x,y)\,dx + N(x,y)\,dy = 0. \qquad (24)$$

Hypothesis: there exists $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial \phi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N(x,y).$$

Conclusion: general solution to (24) is given by:

$$\phi(x,y) = c, \quad \text{with} \quad c \in \mathbb{R},$$

provided this relation defines $y = y(x)$ implicitely.

Criterion for exact equations

Notation: For $f : \mathbb{R}^2 \to \mathbb{R}$, set $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$

Theorem 11.

Let:

- $R = \{(x, y);\ \alpha < x < \beta,$ and $\gamma < y < \delta\}$.
- M, N, M_y, N_x continuous on R.

Then there exists ϕ such that:

$$\phi_x = M, \quad \text{and} \quad \phi_y = N \quad \text{on } R,$$

if and only if M and N satisfy:

$$M_y = N_x \quad \text{on } R$$

Computation of function ϕ

Aim: If $M_y = N_x$, find ϕ such that $\phi_x = M$ and $\phi_y = N$.

Recipe in order to get ϕ:

1. Write ϕ as antiderivative of M with respect to x:

$$\phi(x, y) = a(x, y) + h(y), \quad \text{where} \quad a(x, y) = \int M(x, y)\, dx$$

2. Get an equation for h by differentiating with respect to y:

$$h'(y) = N(x, y) - a_y(x, y)$$

3. Finally we get:

$$\phi(x, y) = a(x, y) + h(y).$$

Computation of ϕ: example (1)

Equation considered:

$$\underbrace{y\cos(x) + 2xe^{y}}_{M} + \underbrace{\left(\sin(x) + x^{2}e^{y} - 1\right)}_{N} y' = 0. \tag{25}$$

Computation of ϕ: example (2)

Step 1: verify that $M_y = N_x$ on $\mathbb{R}^2$.

Step 2: compute ϕ according to recipe. We find

$$\phi(x, y) = y\sin(x) + x^2 e^y - y$$

Solution to equation (25):

$$y\sin(x) + x^2 e^y - y = c.$$

Computation of ϕ: counter-example

Equation considered:

$$\underbrace{3xy + y^2}_{M} + \underbrace{\left(x^2 + xy\right)}_{N} y' = 0. \qquad (26)$$

Step 1: verify that $M_y \neq N_x$.

Step 2: compute ϕ according to recipe. We find

$$h'(y) = -\frac{x^2}{2} - xy \quad \longrightarrow \quad \text{still depends on } x!$$

Conclusion: Condition $M_y = N_x$ necessary.

Solving an exact equation: example

Equation considered:

$$\underbrace{2x - y}_{M} + \underbrace{(2y - x)}_{N} y' = 0, \quad y(1) = 3. \tag{27}$$

Step 1: verify that $M_y = N_x$ on $\mathbb{R}^2$.

Step 2: compute ϕ according to recipe. We find

$$\phi(x, y) = x^2 - xy + y^2.$$

Solution to equation (27): recalling $y(1) = 3$, we get

$$x^2 - xy + y^2 = 7.$$

Solving an exact equation: example (2)

Expressing y in terms of x: we get

$$y = \frac{x}{2} \pm \left(7 - \frac{3x^2}{4}\right)^{1/2}.$$

Recalling $y(1) = 3$, we end up with:

$$y = \frac{x}{2} + \left(7 - \frac{3x^2}{4}\right)^{1/2}.$$

Interval of definition:

$$x \in \left(-2\sqrt{\frac{7}{4}};\, 2\sqrt{\frac{7}{4}}\right) \simeq (-3.05\,;\, 3.05)$$

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Objective

General 2nd order differential equation:

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right)$$

Aim: See cases of 2nd order differential equations
$\hookrightarrow$ which can be solved with 1st order techniques

2nd order eq. with missing dependent variable

Case 1: Equation of the form

$$\frac{d^2y}{dx^2} = F\left(x, \frac{dy}{dx}\right)$$

Method for case 1: the function $v = y'$ solves

$$\frac{dv}{dx} = F(x, v).$$

Then compute $y = \int v(x)\, dx$.

Example (1)

Equation:

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{dy}{dx} + x^2\cos(x)\right), \quad x > 0.$$

Example (2)

Change of variable: $v = y'$ solves the linear equation

$$v' - x^{-1} v = x\cos(x)$$

Integrating factor:

$$I(x) = \exp\left(-\int x^{-1}\,dx\right) = x^{-1}$$

Solving for v:

$$v = x\sin(x) + cx$$

Solving for y:

$$y = \int v = -x\cos(x) + \sin(x) + c_1 x^2 + c_2$$

2nd order eq. with missing independent variable

Case 2: Equation of the form

$$\frac{d^2y}{dx^2} = F\left(y, \frac{dy}{dx}\right)$$

Method for case 2: We set $v = \frac{dy}{dx}$. Then observe that

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\,\frac{dy}{dx} = v\,\frac{dv}{dy}$$

Thus v solves the 1st order equation

$$v\,\frac{dv}{dy} = F(y, v).$$

Example(1)

Equation:

$$\frac{d^2y}{dx^2} = -\frac{2}{1-y}\left(\frac{dy}{dx}\right)^2.$$

Example (2)

Change of variable: $v = y'$ solves the 1st order separable equation

$$v \frac{dv}{dy} = -\frac{2}{1-y} v^2$$

Solving for v:

$$v(y) = c_1 (1-y)^2$$

Example (3)

Separable equation for y:

$$\frac{dy}{dx} = c_1(1-y)^2$$

Solving for y:

$$y = \frac{c_1 x + (c_2 - 1)}{c_1 x + c_2}$$

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Review table

Type	**Standard form**	**Technique**
Separable	$p(y)y' = q(x)$	Separate variables and integrate
Linear	$y' + p(x)y = q(x)$	Integrating factor $\mu = e^{\int p(x)dx}$
Homog.	$y' = f(x,y)$ where $f(tx,ty) = f(x,y)$	Set $y = xv$ v solves separable equation
Bernoulli	$y' + p(x)y = q(x)y^n$	Divide by y^n, set $u = y^{1-n}$ u solves linear equation
Exact	M dx + N dy =0 with $M_y = N_x$	Solution $\phi(x,y) = c$, where ϕ integral of M and N

Example (1)

Equation:

$$\frac{dy}{dx} = -\frac{8x^5 + 3y^4}{4xy^3}$$

Example (2)

Type of method:

- Not separable, not homogeneous, not linear
- Bernoulli, under the form

$$y' + \frac{3}{4x}y = -2x^4 y^{-3}$$

- Not exact under the form

$$\underbrace{\left(8x^5 + 3y^4\right)}_{M} dx + \underbrace{4xy^3}_{N} dy$$