Gaussian elimination

General system

m equations

n unknown $x_1, x_2, \dots, x_n$

Coefficients

a_{ij} → j-th unknown

→ i-th equation

Example of system

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ \quad\;\; x_2 - x_3 = 2 \\ \quad\;\; x_2 + x_3 = 6 \end{cases}$$

Then

$$a_{12} = 1$$

$$a_{23} = -1$$

$$b_3 = 6$$

Rmk: This is a system in $\mathbb{R}^3$

3 unknown

x_1, x_2, x_3

In $\mathbb{R}^3$. The equation

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b,$$

is the equation of a plane in $\mathbb{R}^3$

System of 3 equations with 3 ~~un~~ unknown

↳ Intersection of 3 planes

Augmented matrix

$$A^{\#} = \left(A \;\middle|\; \begin{matrix} b_1 \\ \vdots \\ b_m \end{matrix} \right)$$

System in $\mathbb{R}^3$

$$\begin{aligned} x_1 &+ 3x_2 - 4x_3 = 1 \\ 2x_1 &+ 5x_2 - x_3 = 5 \\ x_1 & + 6x_3 = 3 \end{aligned}$$

Augmented matrix

$$A^{\#} = \left(\underbrace{\begin{matrix} 1 & 3 & -4 \\ 2 & 5 & -1 \\ 1 & 0 & 6 \end{matrix}}_{A} \quad \underbrace{\begin{matrix} 1 \\ 5 \\ 3 \end{matrix}}_{b} \right)$$

Compact notation for the system

$$\underbrace{Ax}_{} = b$$

↳ "Product" to be defined

Operations on systems

$$x_1 + 3x_2 - 4x_3 = 1$$
$$2x_1 + 5x_2 - x_3 = 5$$
$$x_1 + 6x_3 = 3$$

Permutation P_{12}

$$2x_1 + 5x_2 - x_3 = 5$$
$$x_1 + 3x_2 - 4x_3 = 1$$
$$x_1 + 6x_3 = 3$$

System unchanged

Multiplication $M_3(-2)$

$$\cancel{2x_1 + 5x_2 - x_3 = 5}$$
$$x_1 + 3x_2 - 4x_3 = 1$$
$$2x_1 + 5x_2 - x_3 = 5$$
$$-2x_1 - 12x_3 = -6$$

System unchanged

Add rows $A_{13}(2) \to 2R_1 + R_3$

$$
\underbrace{\begin{aligned}
x_1 + 3x_2 - 4x_3 &= 1 \\
2x_1 + 5x_2 - x_3 &= 5 \\
3x_1 + 6x_2 - 2x_3 &= \cancel{1}\ 5
\end{aligned}}
$$

system unchanged

Same operation, on matrices

$$A^{\#} = \begin{pmatrix} 1 & 3 & -4 & 1 \\ 2 & 5 & -1 & 5 \\ 1 & 0 & 6 & 3 \end{pmatrix}$$

$$\overset{A_{13}(2)}{\sim} \begin{pmatrix} 1 & 3 & -4 & 1 \\ 2 & 5 & -1 & 5 \\ 3 & 6 & -2 & 5 \end{pmatrix}$$

Next aim: get a simpler system by performing this kind of operation

↳ Matrix in row-echelon form

Example of row-echelon matrix

1st leading coeff

$$\begin{pmatrix} \boxed{1} & 4 & -3 & 2 \\ 0 & \boxed{3} & -2 & 6 \\ 0 & 0 & 0 & \boxed{5} \end{pmatrix}$$

Triangular shape for leading coefficients

Another example of row-echelon

$$A^{\#} = \begin{pmatrix} \boxed{1} & 1 & -1 & 4 \\ 0 & \boxed{1} & -3 & 5 \\ 0 & 0 & \boxed{1} & 2 \end{pmatrix}$$

Point: the corresponding system is easy to solve

$$x_3 = 2$$

$$x_2 - 3x_3 = 5$$

$$\Rightarrow x_2 = 5 + 3x_3 = 5+6 = 11$$

$$x_1 + x_2 - x_3 = 4$$

$$\Rightarrow x_1 = 4 - x_2 + x_3 = 4 - 11 + 2$$

$$\Rightarrow x_1 = -5$$

Unique solution: $(-5, 11, 2) \in \mathbb{R}^3$

Next step:

Reduce any system to a row-echelon form by using the 3 operations

P, M, A

1st pivot position

Reduction to row-echelon

$$A^{\#} = \begin{pmatrix} \boxed{3} & 2 & -5 & 2 \\ 1 & 1 & -1 & 1 \\ 1 & 0 & -3 & 4 \end{pmatrix}$$

$$\overset{P_{12}}{\sim} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 3 & 2 & -5 & 2 \\ 1 & 0 & -3 & 4 \end{pmatrix}$$

2nd pivot

$$\overset{\substack{A_{13}(-1) \\ A_{12}(-3)}}{\sim} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & \boxed{-1} & -2 & -1 \\ 0 & -1 & -2 & 3 \end{pmatrix}$$

$$\overset{M_2(-1)}{\sim} \begin{pmatrix} 1 & 1 & -1 & 1 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & -1 & -2 & 3 \end{pmatrix}$$

$$\overset{A_{23}(1)}{\sim} \begin{pmatrix} \boxed{1} & 1 & -1 & 1 \\ 0 & \boxed{1} & 2 & 1 \\ 0 & 0 & 0 & \boxed{4} \end{pmatrix}$$

Row echelon

Solving the system

last equation:

$$0 = 4$$

↳ No solution!