Operations on matrices / Inverse

Operations on matrices

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

Then

$$A + B = \begin{pmatrix} 1+2 & -2+1 \\ 0+1 & 2-1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

$$2A = \begin{pmatrix} 2\times 1 & 2\times(-2) \\ 2\times 0 & 2\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -4 \\ 0 & 4 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

can be def

since

A has 2 columns

B has 2 rows

$$= \begin{pmatrix} 1\times 2 + (-2)\times 1 & 1\times 1 + (-2)\times(-1) \\ 0\times 2 + 2\times 1 & 0\times 1 + 2\times(-1) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 \\ 2 & -2 \end{pmatrix}$$

$$BA = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -2 \\ 1 & -4 \end{pmatrix} \neq AB$$

Identity matrix

$$I_3 = \mathrm{Diag}(1,1,1)$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_n = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

Rmk I_n plays the role of 1 for matrix multiplication

zero matrix

$$O_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example of dot product

$$a = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \qquad b = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$$

Then

$$a \cdot b = 1 \times 0 + (-3)(2) + (1)(-1)$$

$$= -7$$

Lower / Upper triangular matrix

↳ 0's below / above the diagonal

Inverse of a matrix

We have seen how to define AB

$A+B$ ...

Question: could we define something like "$\frac{1}{A}$"

not a proper notation

Proper notation: A^{-1}

Inverse

Recall: For a number $x \in \mathbb{R}$, x^{-1} is defined by

$$x \times (x)^{-1} = 1$$

For matrices: $A(A^{-1}) = Id$

A, A^{-1}

Rmk In $\mathbb{R}$, x is invertible iff $x \neq 0$
For matrices, the situation is more complicated. We will see that

A invertible iff "det(A)" $\neq 0$

2x2 case

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$\det(A) = ad - bc$

If $\det(A) \neq 0$, then

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

switch elements on diagonal

switch signs on anti-diagonal

Computing an inverse

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}$$

Then $\det(A) = 1 \times 2 - (0 \times (-2))$

$$= 2 \neq 0$$

↳ A invertible

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix}$$

Check: $A A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$n \times n$ matrix

Prop

A is invertible
iff its reduced row-echelon form is I_n

Inverse with Gauss-Jordan

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$\det(A) = 1-2 = -1 \neq 0$

$\Rightarrow A$ invertible

$$A^{\#} = \begin{pmatrix} \boxed{1} & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\overset{A_{12}(-1)}{\sim} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & \boxed{-1} & -1 & 1 \end{pmatrix}$$

$$\overset{M_2(-1)}{\sim} \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \end{pmatrix}$$

$$\overset{A_{21}(-2)}{\sim} \begin{pmatrix} \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} & \underbrace{\begin{matrix} -1 & 2 \\ 1 & -1 \end{matrix}}_{A^{-1}} \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

check that this is true using Thm 13

Solving a system with A^{-1}

$Ax = b$ with

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

We have seen that one way to solve the system is to write

$$Ax = b \iff x = A^{-1} b$$

$$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \times 3 + 2 \times 2 \\ 1 \times 3 - 1 \times 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

<u>Unique solution:</u> $(x_1, x_2) = (1, 1)$