$\mathbb{R}^3$ and $\mathbb{R}^n$

Examples for addition in $\mathbb{R}^3$

(2) $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad v = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

Then $\vec{u} + \vec{v} = \begin{pmatrix} 1+4 \\ 2+5 \\ 3+6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} \overset{1}{\in} \mathbb{R}^3$

$\vec{v} + \vec{u} = \begin{pmatrix} 4+1 \\ 5+2 \\ 6+3 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$

Thus $u + v = v + u$

5 Consider

$$v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Then

$$-v = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

Check:

$$v + (-v) = \begin{pmatrix} 1 - 1 \\ 2 - 2 \\ 3 - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \vec{0}$$

Examples for scalar multiplication

(7) $u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ $\quad v = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

$r = 2$

Then

$$w = u + v = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$$

$$rw = 2 \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 18 \end{pmatrix}$$

and $ru + rv = 2 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + 2 \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

$$= \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \begin{pmatrix} 8 \\ 10 \\ 12 \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \\ 18 \end{pmatrix}$$

Thus $ru + rv = r(u+v)$

Rmk If $v_1, v_2 \in \mathbb{R}^3$, then

$$\underbrace{0}_{c_1} \times v_1 + \underbrace{0}_{c_2} \times v_2 = 0$$

Example of linear dep.

$$u = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad v = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$$

Question: what is the solution set for

$$c_1 u + c_2 v = 0 \quad ?$$

i.e

$$c_1 \begin{pmatrix} 3 \\ -2 \end{pmatrix} + c_2 \begin{pmatrix} -6 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{cases} 3c_1 - 6c_2 = 0 \\ -2c_1 + 4c_2 = 0 \end{cases}$$

In matrix form we get

$$\underbrace{\begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A ⇝ matrix whose columns are [u, v]

Row-echelon form for A

$$A = \begin{pmatrix} 3 & -6 \\ -2 & 4 \end{pmatrix}$$

$$\overset{M_1(1/3)}{\sim} \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$$

$$\overset{A_{12}(2)}{\sim} \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

Thus the system can be read as

$$c_1 - 2c_2 = 0$$

Thus one possibility is

$$c_2 = 1 \qquad c_1 = 2c_2 = 2$$

Therefore

$$2u + v = 0$$

Thus u, v linearly dependent

Note we had

$$u = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad v = \begin{pmatrix} -6 \\ 4 \end{pmatrix}$$

Thus one could easily see that

$$v = -2u$$

$\Rightarrow u, v$ linearly dependent.

Another example

$$u = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad v = \begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

Then $c_1 u + c_2 v = 0$

$$\iff A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

with $A = [\, u \;\; v \,]$

$$= \begin{pmatrix} 3 & 5 \\ -2 & 7 \end{pmatrix}$$

Computing the row-ech. form of A, we get (check) that

$$A \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \text{ iff } \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow u, v$ lin. indep.

Example for 3 vectors.

$u = (1, 2, -3)$ $v = (3, 1, -2)$

$w = (5, -5, 6)$

Then

$$c_1 u + c_2 v + c_3 w = 0$$

$$\Longleftrightarrow \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & -5 \\ -3 & -2 & 6 \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0$$

Row-echelon form for A

$$A \overset{\substack{A_{13}(3) \\ A_{12}(-2)}}{\sim} \begin{pmatrix} 1 & 3 & 5 \\ 0 & -5 & -15 \\ 0 & 7 & 21 \end{pmatrix}$$

$$\overset{\substack{M_3(1/7) \\ M_2(-1/5)}}{\sim} \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

free variable

$$\sim \begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

Solution set: we set $c_3 = s$

Then

$$c_2 = -3c_3 = -3s$$

$$c_1 = -3c_2 - 5c_3 = 9s - 5s$$

$$= 4s$$

Solution set:

$$S = \{ (4s, -3s, s) ; s \in \mathbb{R} \}$$

Conclusion: we can take

$c_1 = 4$, $c_2 = -3$, $c_3 = 1$

Then

$$c_1 u + c_2 v + c_3 w = 0$$

Thus $\{u, v, w\}$ linearly dependent

Rmk on Prop 4

$(v_1, v_2, v_3) \in \mathbb{R}^3$ are lin. ind

$\iff \det([v_1, v_2\ v_3]) \neq 0$

$\iff A = [v_1, v_2\ v_3]$ invertible

Example

$$\begin{array}{c} \begin{array}{ccc} u & v & w \end{array} \\ \begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & -5 \\ -3 & -2 & 6 \end{vmatrix} \end{array} = 0$$

$\Rightarrow$ u, v, w are lin dep.

(But we don't know which c_1, c_2, c_3 make $c_1 u + c_2 v + c_3 w = 0$)

2nd example

$$\begin{array}{c}\begin{vmatrix} 1 & 3 & 5 \\ 2 & 1 & -5 \\ -3 & -2 & 0 \end{vmatrix} = 30 \neq 0 \\ \begin{array}{ccc} u & v & w \end{array}\end{array}$$

Thus $\{u, v, w\}$ lin indep.

Addition in $\mathbb{R}^4$

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \qquad v = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

Then $u + v = \begin{pmatrix} 3 \\ 5 \\ 7 \\ 9 \end{pmatrix}$ $\qquad 3u = \begin{pmatrix} 3 \\ 6 \\ 9 \\ 12 \end{pmatrix}$

Claim $\mathbb{P}_2$ is a vect. space

$\mathbb{P}_2 =$ set of pol. with degree ≤ 2

$= \{ ax^2 + bx + c \ ; \ a, b, c \in \mathbb{R} \}$

Addition

If $p_1(x) = a_1 x^2 + b_1 x + c_1$

$p_2(x) = a_2 x^2 + b_2 x + c_2$

Then $[p_1 + p_2](x)$

$= (a_1 + a_2) x^2 + (b_1 + b_2) x$

$+ (c_1 + c_2)$

$[3p_1](x) = (3a_1) x^2 + (3b_1) x + 3c_1$