Bases

Example in $\mathbb{R}^2$

$$v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -3 \\ -9 \end{pmatrix}$$

Question: do we have $\{v_1, v_2\}$ lin indep ?

Here one can see that

$$v_2 = -3 v_1$$

$\Rightarrow$ $\{v_1, v_2\}$ lin. dep.
with nontrivial linear comb.

$$3v_1 + v_2 = 0$$

However $\{v_1, v_2\}$ lin. indep if

$$v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$

since $v_2 \neq \alpha v_1$

Lin. dep in $\mathbb{R}^3$

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 8 \\ 1 \\ 1 \end{pmatrix}$$

Are those lin dep?

Method 1: Compute

$\det([v_1 \; v_2 \; v_3])$

$$= \begin{vmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} \overset{\text{Check}}{=} 0$$

$\Rightarrow \{v_1, v_2, v_3\}$ lin dep.

Method 2: Solve the system

$$c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$$

Related matrix

$$A = \begin{array}{ccc} v_1 & v_2 & v_3 \\ \end{array}$$
$$A = \begin{pmatrix} 1 & 2 & 8 \\ 2 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

row-echelon

$$\sim \begin{pmatrix} \boxed{1} & 2 & 8 \\ 0 & \boxed{1} & 3 \\ 0 & 0 & \boxed{0} \end{pmatrix}$$

free var, called t

Here we have 2 leading 1's and 3 variables $\Rightarrow$ 1 free var.

$\Rightarrow$ there exits a non trivial solution to $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$\Rightarrow$ lin dep. for $\{v_1, v_2, v_3\}$

If we solve the system, we get

$$S = \{(-2t, -3t, t) ; t \in \mathbb{R}\}$$

For $t = 1$ we get

$$(c_1, c_2, c_3) = (-2, -3, 1)$$

Thus

$$-2v_1 - 3v_2 + v_3 = 0$$

$\Rightarrow \{v_1, v_2, v_3\}$ lin dep.

Definition of $\mathbb{P}_1$

$\mathbb{P}_1 = \{$ polynomials with degree $\leq 1\}$

$= \{ p(t) = at + b, \ a, b \in \mathbb{R} \}$

lin dep in $\mathbb{P}_1$

$p_1(t) = 1 \qquad p_2(t) = t$

$p_3(t) = 4 - t$

Question: $\{p_1, p_2, p_3\}$ lin dep?

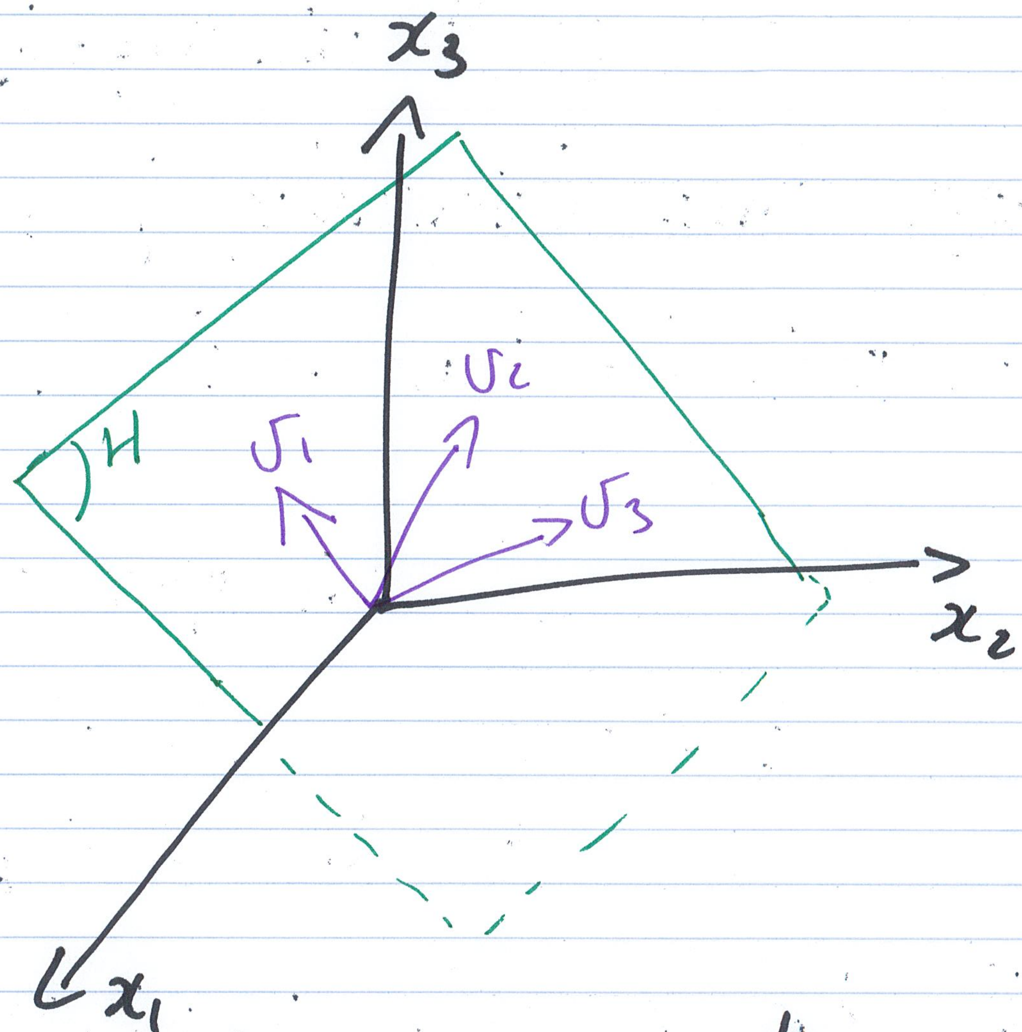
$$p_3(t) = 4 - t$$

$$= 4 \times \overbrace{1}^{p_1(t)} - 1 \times \overbrace{t}^{p_2(t)}$$

$$= 4\,p_1(t) - p_2(t)$$

We get $p_3 = 4p_1 - p_2$ $\quad \{p_1, p_2, p_3\}$ lin. dep.

$\iff 4p_1 - p_2 - p_3 = 0$



One can write

$H = \text{Span}\{v_1, v_2, v_3\}$

linearly independent

However, it is more efficient to write

$H = \text{Span}\{v_1, v_3\}$

↳ use basis / dimension

~~Exam~~ Natural Canonical basis in $\mathbb{R}^3$

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then $\{e_1, e_2, e_3\}$ basis of $\mathbb{R}^3$

(a) Question: are $\{e_1, e_2, e_3\}$ lin. indep?

$\text{Det}([e_1 \; e_2 \; e_3])$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

Thus $\{e_1, e_2, e_3\}$ lin. indep

(b) Take

$$v = \begin{pmatrix} 5 \\ -6 \\ \pi \end{pmatrix} \in \mathbb{R}^3$$

Then $v = 5\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + (-6)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \pi\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$= 5e_1 - 6e_2 + \pi e_3$

General case

$$v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1 e_1 + v_2 e_2 + v_3 e_3$$

$\in \operatorname{Span}\{e_1, e_2, e_3\}$

Thus

$\operatorname{Span}\{e_1, e_2, e_3\} = \mathbb{R}^3$

$\Rightarrow \{e_1, e_2, e_3\}$ basis of $\mathbb{R}^3$

Canonical basis for $M_2(\mathbb{R})$

$$M_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \; a, b, c, d \in \mathbb{R} \right\}$$

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Then $\{e_1, e_2, e_3, e_4\}$ canonical basis for $M_2(\mathbb{R})$.

Basis for $H \subset \mathbb{R}^3$. Take

$$v_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 6 \\ 16 \\ -5 \end{pmatrix}$$

$H = \text{Span}\{v_1, v_2, v_3\}$

Question: Is v_3 a linear comb. of v_1, v_2?

"$\Longleftrightarrow$" v_1, v_2, v_3 lin. dep.

$\Longleftrightarrow$ $\det([v_1\ v_2\ v_3]) = 0$

Here $\det([v_1\ v_2\ v_3])$

$$= \begin{vmatrix} 0 & 2 & 6 \\ 2 & 2 & 16 \\ -1 & 0 & -5 \end{vmatrix} \overset{\text{Check}}{=} 0$$

We get:

v_3 lin. comb. of v_1, v_2

$\overset{\text{Thm B-①}}{\Longrightarrow}$ $H = \text{Span}\{v_1, v_2, v_3\}$

$= \text{Span}\{v_1, v_2\}$

Is $\{v_1, v_2\}$ a basis for H?

True if $\{v_1, v_2\}$ lin. indep.

$\Longleftrightarrow$ $v_2 \neq \alpha\, v_1$

Question: Do we have $v_2 = \alpha v_1$?

$$v_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} \quad \xrightarrow{\alpha = 1} \quad v_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} \qquad 0 \neq -1 \qquad \times\ \alpha = 1$$

Thus $v_2 \neq \alpha\, v_1$

Conclusion:

$\{v_1, v_2\}$ is a basis for H

Def

$Col(A) = \text{Span}\{v_1, \ldots, v_n\}$

where $A = [v_1 \; v_2 \; \cdots v_n]$

Example of basis in $\mathbb{R}^3$

$$H = \text{Span}\left\{ \underbrace{\begin{pmatrix}1\\4\\7\end{pmatrix}}_{v_1} \underbrace{\begin{pmatrix}2\\5\\8\end{pmatrix}}_{v_2} \underbrace{\begin{pmatrix}3\\6\\9\end{pmatrix}}_{v_3} \underbrace{\begin{pmatrix}4\\7\\10\end{pmatrix}}_{v_4} \right\}$$

Question: Basis for H?

Consider

$$A = [\, v_1 \quad v_2 \quad v_3 \quad v_4 \,]$$

we wish to find a basis for $\text{Col}(A)$

$$A = \begin{pmatrix} \boxed{\begin{matrix}1\\4\\7\end{matrix}} & \boxed{\begin{matrix}2\\5\\8\end{matrix}} & \begin{matrix}3\\6\\9\end{matrix} & \begin{matrix}4\\7\\10\end{matrix} \end{pmatrix}$$

Row-echelon

$$\sim \begin{pmatrix} \boxed{1} & 2 & 3 & 4 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Basis for H:

$$\left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$$