General solutions of diff eq

Equation $y'' + y' + \frac{37}{4} y = 0$

Aux eq: $P(r) = 0$ with

$$P(r) = r^2 + r + \frac{37}{4}$$

$$\Delta = 1^2 - 4 \times 1 \times \frac{37}{4} = -36$$

Roots: $r_1 = \frac{-1 + \sqrt{-36}}{2}$

$$r_1 = -\frac{1}{2} + 3i$$

$$\left(r_2 = -\frac{1}{2} - 3i \right) \to \text{not used to solve the eq}$$

According to recipe - item 2, the gen. solution is

$$y = c_1 e^{-\frac{1}{2}t} \cos(3t) + c_2 e^{-\frac{1}{2}t} \sin(3t)$$

$(fg)' = f'g + fg'$

Initial conditions: $y(0)=2, \quad y'(0)=8$

$$y = \underbrace{e^{-\frac{1}{2}t}}_{=1 \text{ for } t=0}\left(c_1 \underbrace{\cos(3t)}_{=1 \text{ if } t=0} + c_2 \underbrace{\sin(3t)}_{=0 \text{ for } t=0}\right)$$

$$y' = e^{-\frac{1}{2}t}\left(\left(-\tfrac{1}{2}c_1+3c_2\right)\cos(3t) + \left(-3c_1-\tfrac{1}{2}c_2\right)\underbrace{\sin(3t)}_{=0 \text{ for } t=0}\right)$$

Thus

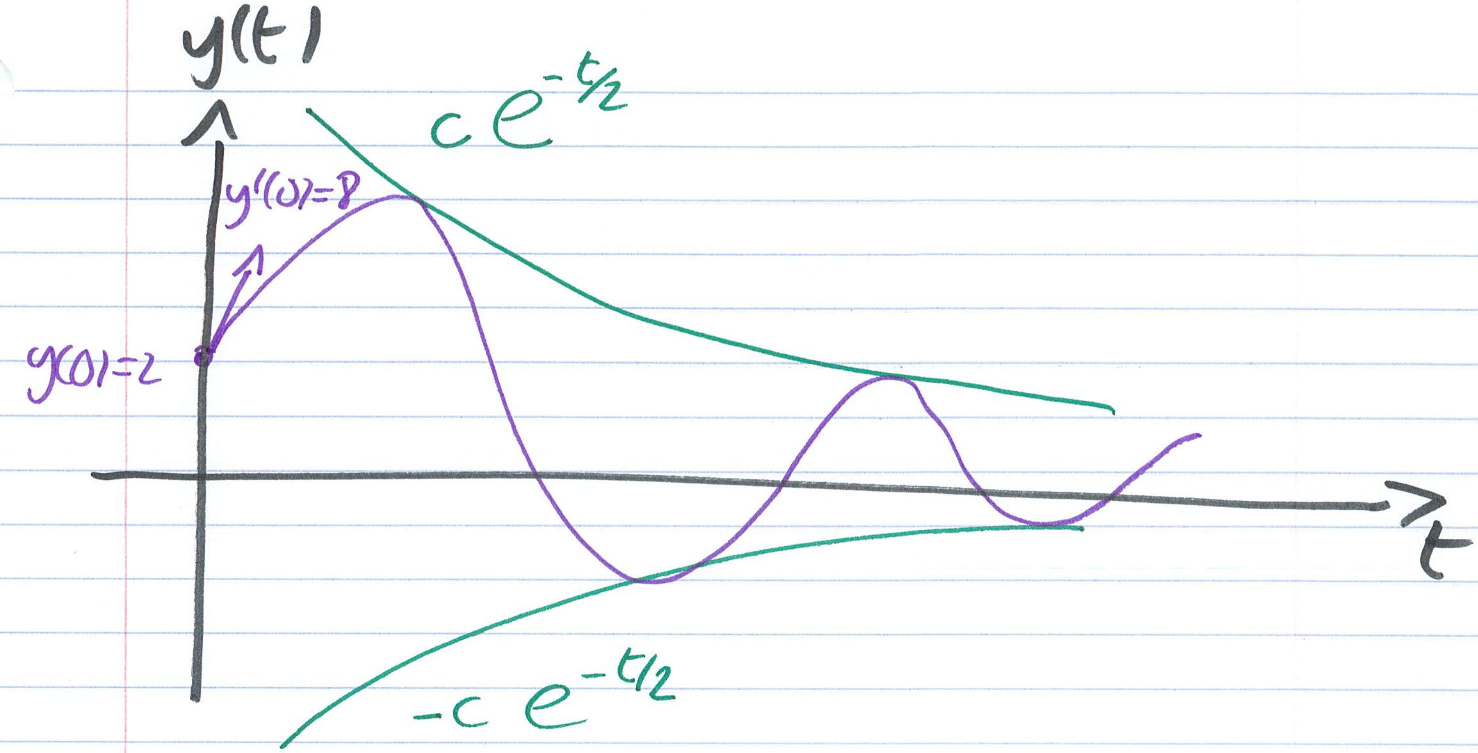
$$y(0)=2 \Rightarrow c_1 = 2$$

$$y'(0)=8 \Rightarrow 3c_2 = 8+\tfrac{1}{2}c_1 = 9$$

$$\Rightarrow c_2 = 3$$

Unique solution

$$y = \underbrace{e^{-\frac{1}{2}t}}_{\text{Damped exp.}}\underbrace{\left(2\cos(3t) + 3\sin(3t)\right)}_{\text{oscillating function}}$$


y(t)
$ce^{-t/2}$
$y'(0)=8$
$y(0)=2$
t
$-ce^{-t/2}$

Example with double root

$$y'' - y' + \frac{1}{4} y = 0$$

$$P(r) = r^2 - r + \frac{1}{4} = \left(r - \frac{1}{2}\right)^2$$

Double root: $r = \frac{1}{2}$

General solution

$$y = c_1 e^{\frac{1}{2}t} + c_2 t e^{\frac{1}{2}t}$$

Initial condition: $y(0) = 2 \quad y'(0) = \frac{1}{3}$

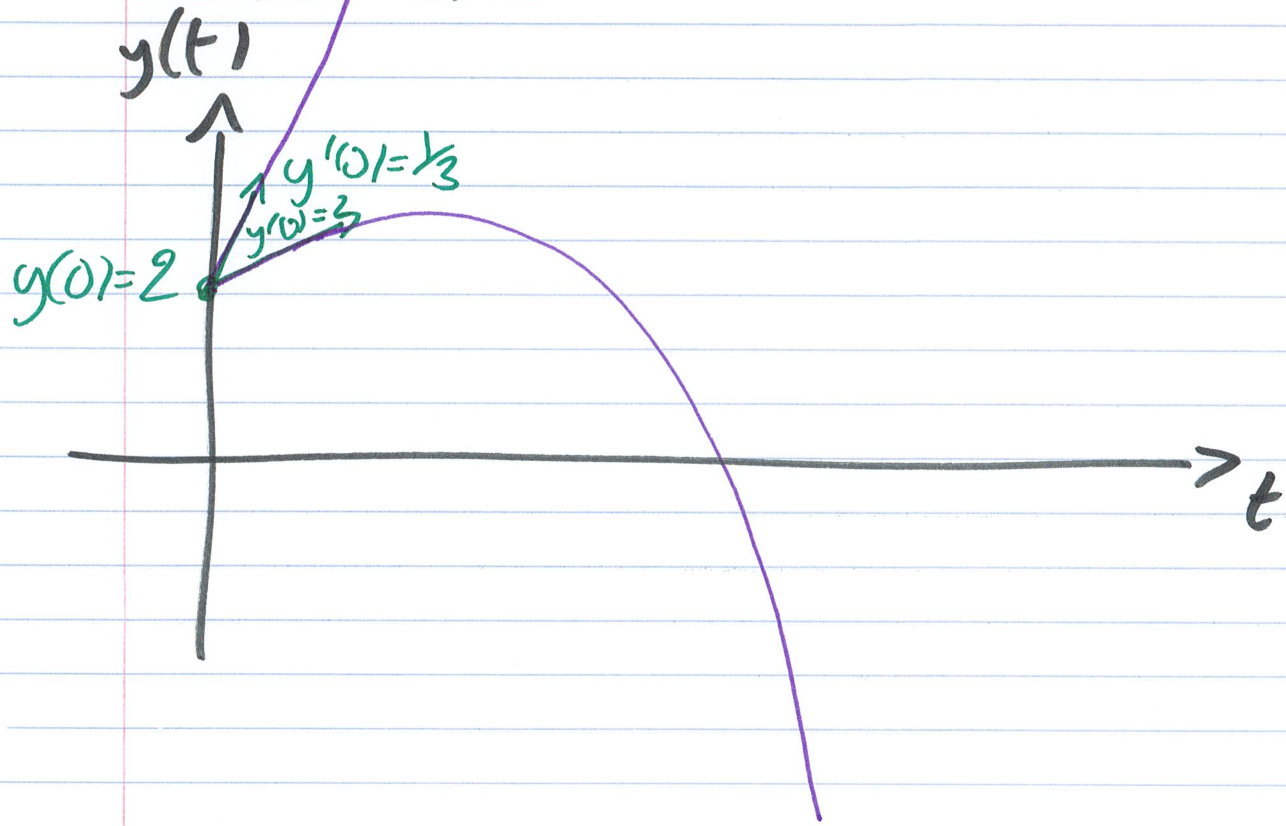
we get $c_1 = 2 \qquad c_2 = -\frac{2}{3}$

Unique sol: $y = \left(2 - \frac{2}{3}t\right) e^{+t/2}$

$-\infty$ as $t \to \infty$ $\qquad$ $\to +\infty$ as $t \to \infty$

$$\lim_{t \to \infty} \left(2 - \frac{2}{3}t\right) e^{t/2} = -\infty$$

Shape of y



<u>Rmk</u>: With initial cond

$y(0)=2$, $y'(0)=0$,

we get $y(t) \xrightarrow[t\to\infty]{} +\infty$

From non standard to standard form

$$a^{\cancel{(n)}}_n(x)\, y^{(n)} + \cdots + a_1 y' + a_0 y = b$$

$$\Big\downarrow \times \frac{1}{a_n}$$

$$y^{(n)} + \underbrace{\frac{a_{n-1}}{a_n}}_{p_1} y^{(n-1)} + \cdots + \underbrace{\frac{a_1}{a_n}}_{p_{n-1}} y' + \underbrace{\frac{a_0}{a_n}}_{p_n} y = \underbrace{\frac{b}{a_n}}_{g}$$

This works if $a_n \neq 0$

Rmk

For 1st order eq. we need 1 init. cond
" 2nd " " " " 2 " "
" n-th " " " " n " "

Summary of Thm 6

If $p_1, \ldots, p_n, g$ are continuous, one can safely solve the diff eq.

Example of 3rd order eq

$$\underbrace{x(x-1)}_{a_3}\, y''' - \underbrace{3x}_{a_2} y'' + \overbrace{6x^2}^{a_1} y' - \overbrace{\cos(x)}^{a_0} y = \underbrace{\sqrt{x+5}}_{b}$$

$$y(\overset{x_0}{-2}) = \overset{\gamma_0}{2}, \quad y'(\overset{x_0}{-2}) = \overset{\gamma_1}{1}, \quad y''(\overset{x_0}{-2}) = \overset{\gamma_2}{-1}$$

Write eq in standard form $\left(\times \frac{1}{x(x-1)}\right)$

$$y''' - \overbrace{\frac{3}{x-1}}^{p_1} y'' + \overbrace{\frac{6x}{x-1}}^{p_2} y' - \overbrace{\frac{\cos(x)}{x(x-1)}}^{p_3} y = \underbrace{\frac{\sqrt{x+5}}{x(x-1)}}_{g}$$

p_1, p_2 cont on $(-\infty, 1) \cup (1, \infty)$

p_3 cont on $(-\infty, 0) \cup (0,1) \cup (1, \infty)$

g cont on $(-5, 0) \cup (0,1) \cup (1, \infty)$

Here $x_0 = -2 \in (-5, 0)$

$\overset{\text{Thm 6}}{\Longrightarrow}$ Unique solution on $(-5, 0)$

Notation "differentiate once"

$(D-2)(D-3)\,y$

This can be interpreted as

$$(D-2)(D-3)\,y$$
$$= (D^2 - 5D + 6)\,y$$
$$= y'' - 5y' + 6y$$

Equation

$$(D-1)(D-2)(D+3)\,y = 0$$

With this expression, we directly have 3 solutions:

$$y_1 = e^{x}, \quad y_2 = e^{+2x}, \quad y_3 = e^{-3x}$$

Are y_1, y_2, y_3 lin indep?

↳ Wronskian

$y_1 = e^x \quad y_2 = e^{2x} \quad y_3 = e^{-3x}$

$$W[y_1, y_2, y_3](x)$$

$$\overset{\text{Def 7}}{=} \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

$$= \begin{vmatrix} e^x & e^{2x} & e^{-3x} \\ e^x & 2e^{2x} & -3e^{-3x} \\ e^x & 4e^{2x} & 9e^{-3x} \end{vmatrix}$$

$$= \left(e^x \times e^{2x} \times e^{-3x}\right) = 1$$

$$\times \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 1 & 4 & 9 \end{vmatrix} = 20 \neq 0$$

Gen solution of diff eq: (Thm 8)

$y = c_1 e^x + c_2 e^{2x} + c_3 e^{-3x}$

Non hom. eq.

$$\underbrace{y^{(n)} + a_1 y^{(n-1)} + \cdots + a_0 y}_{\equiv Ly} = F(x)$$

Example of non ~~lin.~~ hom. eq.

$$y'' + y' - 6y = 8 \cancel{x} e^{5x}$$

Hom. eq: $y'' + y' - 6y = 0$

We have seen that the fund sol. are

$$y_1 = e^{2x} \quad , \quad y_2 = e^{-3x}$$

Claim: $y_p = \frac{1}{3} e^{5x}$ solves the eq.

$y_p = \frac{1}{3} e^{5x}$ $\times(-6)$

$y_p' = \frac{5}{3} e^{5x}$ $\times 1$

$y_p'' = \frac{25}{3} e^{5x}$ $\times 1$

$\frac{1}{3}(-6+5+25) e^{5x}$

$= 8 e^{5x}$

Thus

$y_p'' + y_p' - 6y_p = 8 e^{5x}$

General solution is (Thm 9)

$y = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{3} e^{5x}$