Mechanical vibrations

Eq $(D^2+6D+13)^2 y = 0$

Aux polynomial

$$P(r) = (r^2+6r+13)^2$$

Complete the square:

$$P(r) = ((r+3)^2+4)^2$$

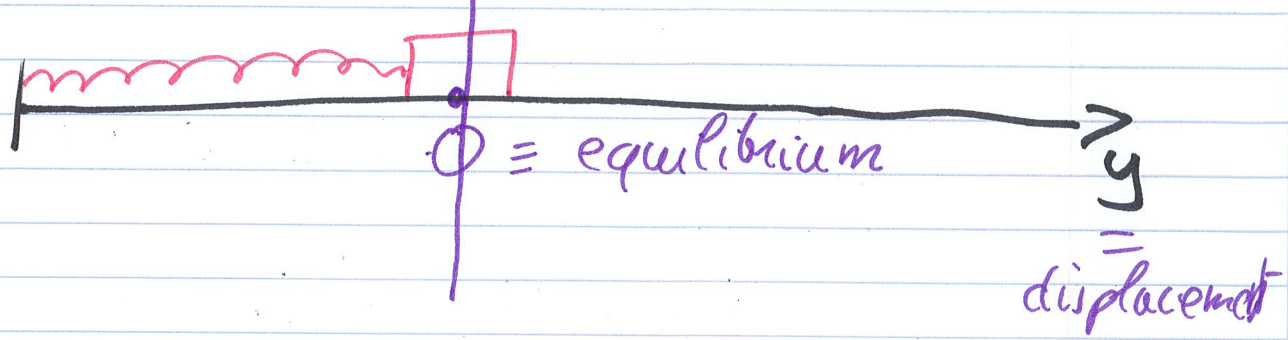
Roots	Multiplicity
$-3+2i$	2
$-3-2i$	2

$$y_4 = xe^{-3x}\sin(2x)$$

Fund solutions

$$y_1 = e^{-3x}\cos(2x)$$
$$y_2 = e^{-3x}\sin(2x)$$
$$y_3 = xe^{-3x}\cos(2x)$$

Forces in the spring-mass



Force due to the spring

$F_s = -ky$ (Hooke's law)

Force due to friction

$F_d = -b\,y'$

Notes: this is the standard model

Newton's law

$$m.y'' = \Sigma F = F_s + F_d$$

$$m y'' = -ky - by'$$

$$m y'' + by' + ky = 0$$

↳ 2^{nd} order linear eq
with constant coeff.

Particular case: no friction,
i.e $b=0$. we get

$$m y'' + ky = 0$$

No friction case

$$my'' + ky = 0 \qquad \frac{k}{m} = \omega^2 \Leftrightarrow \omega = \sqrt{\frac{k}{m}}$$

$$\Leftrightarrow \quad y'' + \frac{k}{m} y = 0$$

$$\Leftrightarrow \quad y'' + \omega^2 y = 0$$

$$\Leftrightarrow \quad y'' = -\omega^2 y$$

Rmk: If $y'' = -y$,
then $y = C_1 \cos(t) + C_2 \sin(t)$

$$\Leftrightarrow \quad y = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$

Check: if $y = \cos(\omega t)$

$y' = -\omega \sin(\omega t)$

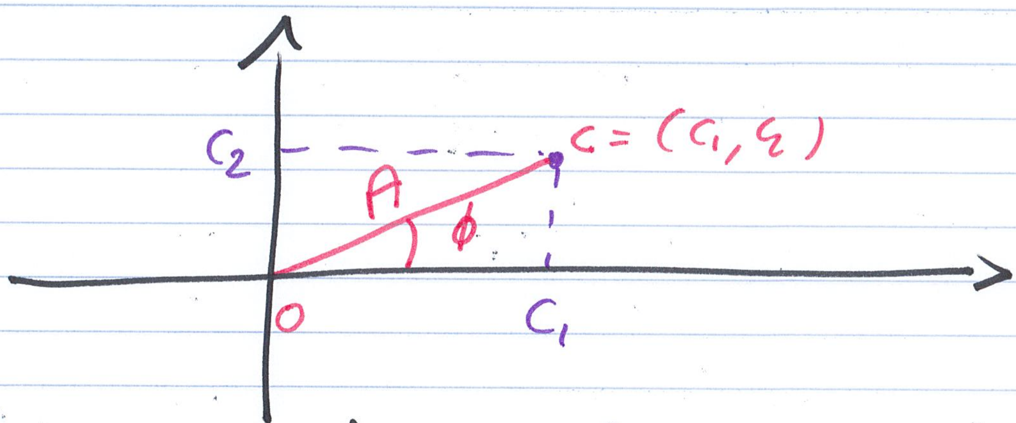
$y'' = -\omega^2 \cos(\omega t)$

$= -\omega^2 y$

Conclusion: y is an oscillating fct with freq ω

Physical expression for y

$$y = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$



We write (c_1, c_2) in polar coordinates

$$c_1 = A \cos(\phi) \qquad c_2 = A \sin(\phi)$$

or $$A = \sqrt{c_1^2 + c_2^2} \qquad \tan(\phi) = \frac{c_2}{c_1}$$

$$\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a-b)$$

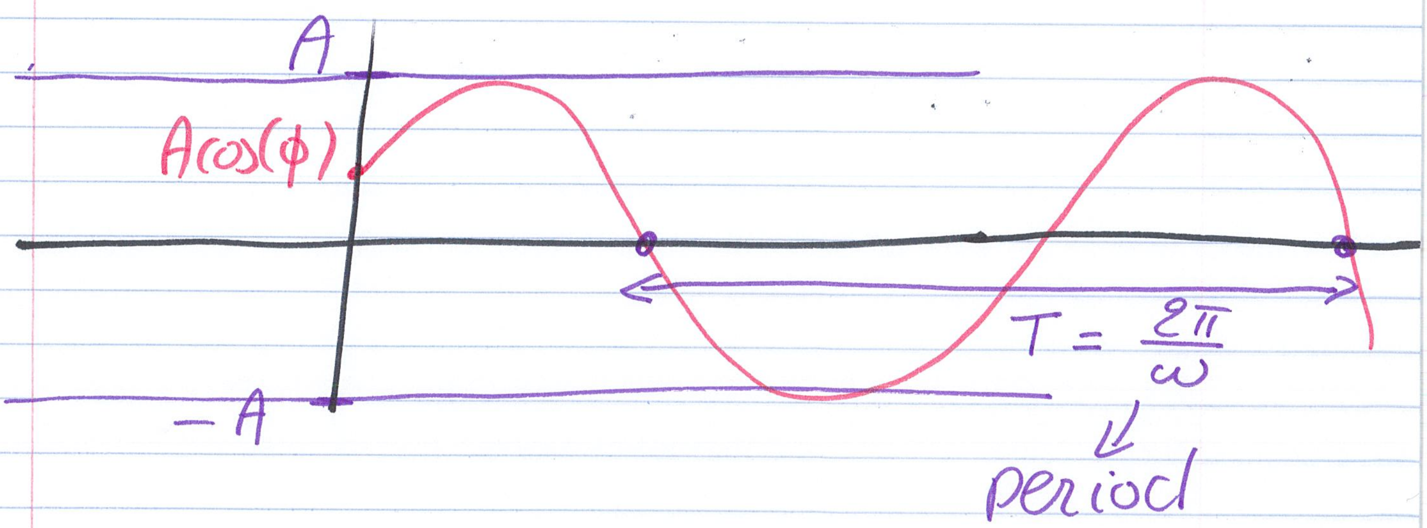
Then

$$y = \overset{A\cos(\phi)}{C_1}\cos(\omega t) + \overset{A\sin(\phi)}{C_2}\sin(\omega t)$$

$$= A\cos(\omega t)\cos(\phi) + A\sin(\omega t)\sin(\phi)$$

$$= A\{\cos(\omega t)\cos(\phi) + \sin(\omega t)\sin(\phi)\}$$

$$= \underset{\text{amplitude}}{A}\cos(\omega t - \underset{\text{phase}}{\phi})$$



Expression for the period

$\omega = \sqrt{k/m}$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}$$

Interpretation

m large $\Rightarrow$ longer period

k large $\Rightarrow$ shorter period

Damped case

$$m y'' + b y' + k y = 0$$

Polynomial

$$m r^2 + b r + k = 0$$

Quadratic formula

$$\Delta = b^2 - 4mk$$

roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4mk}}{2m}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4mk}}{2m}$$

Study: according to sign(Δ)

Rmk (i) If $\Delta \geq 0$, then
$\sqrt{b^2 - 4mk} < b$

Thus $r_1, r_2 < 0$

$\Rightarrow y = \sum$ damped exp

(ii) If $\Delta < 0$, then

$$Re(r_1) = \frac{-b}{2m} < 0$$

$\Rightarrow y = \sum$ damped + oscillating exp

In all cases

$$\lim_{t \to \infty} y(t) = 0$$

(smaller and smaller oscillations for the spring)

Small b case (underdamped)

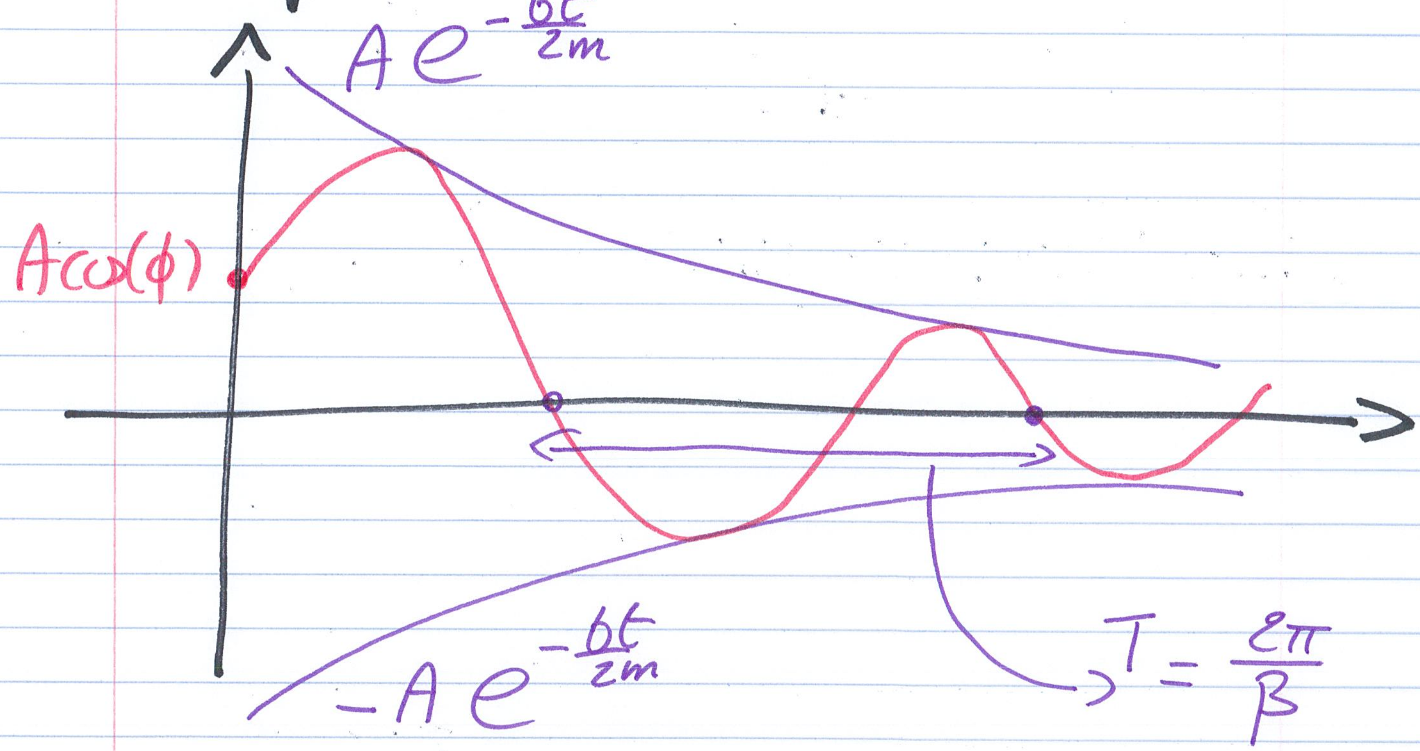
we have seen

$$y = \left(c_1 \cos(\beta t) + c_2 \sin(\beta t)\right) e^{-\frac{bt}{2m}}$$

$$\beta = \frac{(4km - b^2)^{\frac{1}{2}}}{2m} \quad (>0 \text{ if } b \text{ small})$$

$$= A \cos(\beta t - \phi)\, e^{-\frac{bt}{2m}}$$

Graph



Undamped case

Assume we are given a mass $m = \frac{1}{2}$ kg. We assume that if we apply a force $F = 25$ N to the spring, then the extension will be $1/2$ m. Then write the eq. for the motion (no friction)

$$m y'' + k y = 0$$

In order to compute k, write Hooke's law

$$|F| = k y \Rightarrow k = \frac{|F|}{y}$$

$$= \frac{25}{1/2} = 50$$

Thus the eq is

$$\frac{1}{2} y'' + 50y = 0$$

$$\Longleftrightarrow \quad y'' + \overset{\omega^2}{100}\, y = 0$$

Gen sol:

$$y = c_1 \cos(10t) + c_2 \sin(10t)$$

or

$$y = A \cos(10t - \phi)$$

to be computed with initial conditions