Eigenvalues

Eigenvalue

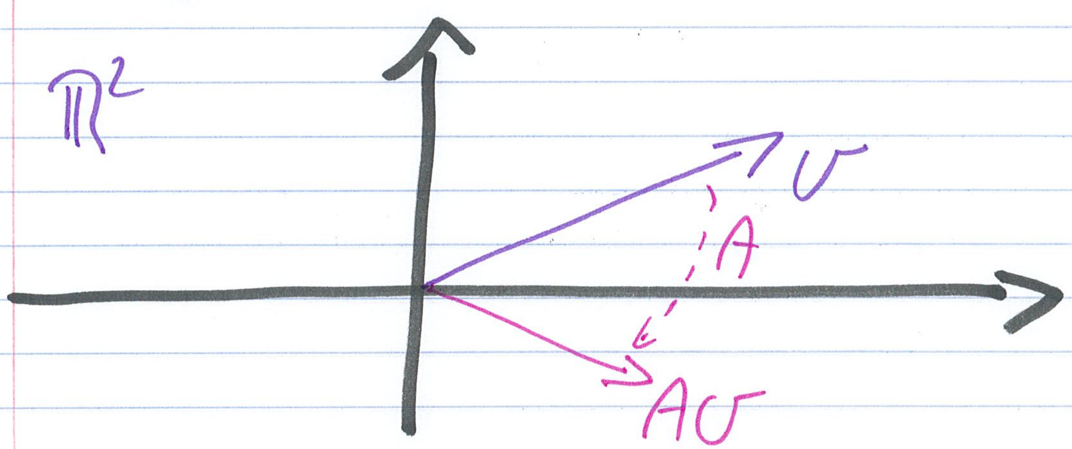
↳ Hilbert
Hardy

Proper value

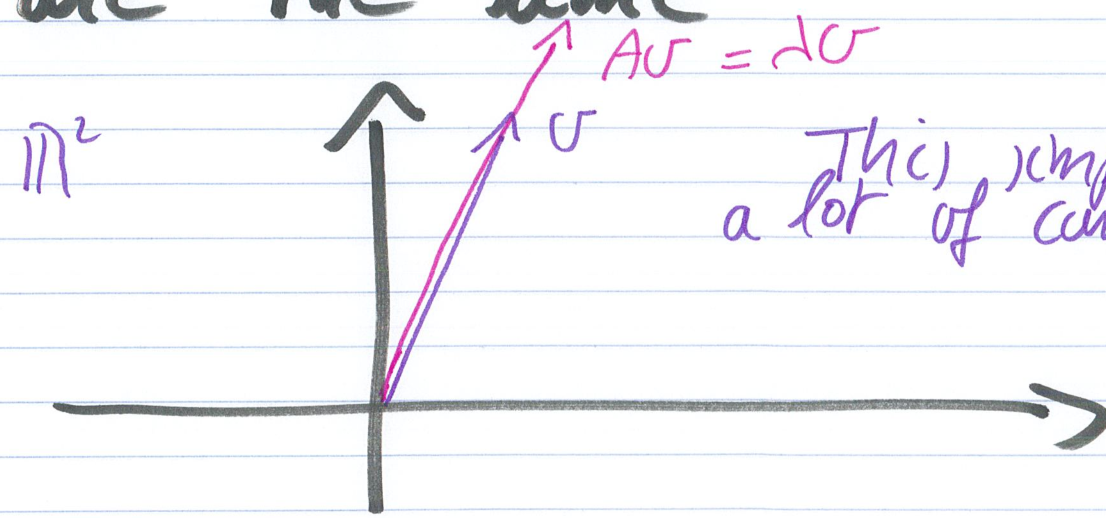
Ex

Interpretation of eigenvector

1. In general, the orientations of v and Av are different



2. For an eigenvector, the orientations of v and Av are the same



This simplifies a lot of computation

Example $A = \begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix}$

Claim: $\lambda = 7$ is eigenvalue

According to Def 1, this means that there is a nontrivial solution to the system

$$A v = 7 v$$

$$\Longleftrightarrow \quad (A - 7I) v = 0$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

satisfies $I v = v$

System B

$(A - 7I)v = 0$ → linear hom. system $Bv = 0$

$$\Leftrightarrow \left[\begin{pmatrix} 1 & 6 \\ 5 & 2 \end{pmatrix} - \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}\right] v = 0$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix}}_{B} v = 0$$

Row echelon

$$B = \begin{pmatrix} -6 & 6 \\ 5 & -5 \end{pmatrix} \underset{r_1(-\frac{1}{6})}{\overset{r_2(\frac{1}{5})}{\sim}} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$ → free var. $x_2 = r$

Solution set:

$$x_1 - x_2 = 0 \Leftrightarrow x_1 = x_2 = r$$

$$S = \{ r(1, 1) \; ; \; r \in \mathbb{R} \}$$

Eigenvector For $n = 1$, we get that

$$v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

satisfies

$$Av = 7v$$

Thus $\lambda = 7$ is an eigenvalue

Next aim

Method to get eigenvalues

Method Based on the fact that A (square matrix) not invertible iff $\det(A) \begin{cases} \neq 0 \\ = 0 \end{cases}$

Char eq. λ eigenvalue

iff $\underbrace{\det(A - \lambda I)}_{\text{polynomial of degree } n \text{ in } \lambda} = 0$

We are reduced to a computation of the roots of a polynomial

Example of eigenvalues

$$A = \begin{pmatrix} 5 & -4 \\ 8 & -7 \end{pmatrix}$$

1

$$\det(A - \lambda I)$$

$$= \begin{vmatrix} 5-\lambda & -4 \\ 8 & -7-\lambda \end{vmatrix}$$

$$= (5-\lambda)(-7-\lambda) + 32$$

$$= (\lambda-5)(\lambda+7) + 32$$

$$= \lambda^2 + 2\lambda - 3$$

trivial

Roots: $1 = \lambda_2$

$-3 = \lambda_1$

product $= -3$

② (i) Eigenvector for $\lambda_1 = -3$

$$A + 3I = \begin{pmatrix} 5 & -4 \\ 8 & -7 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 8 & -4 \\ \cancel{8} & \cancel{-4} \end{pmatrix}}_{B}$$

Note: for a 2×2 matrix, if λ eigenvalue, we can always forget the 2nd eq.

We wish to solve $Bv = 0$
This is true if

$$8v_1 - 4v_2 = 0$$

$$\Leftrightarrow 2v_1 - v_2 = 0$$

$$\Leftrightarrow v_2 = 2v_1$$

coordinate #1

⚠

Eigenvector : $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

vector #1

Rmk : all the eigenvect. are of the form $r\begin{pmatrix} 1 \\ 2 \end{pmatrix}, r \in \mathbb{R}$

$$A = \begin{pmatrix} 5 & -4 \\ 8 & -7 \end{pmatrix}$$

(ii) Eigenvector for $\lambda_2 = 1$

$$A - 1 \times I = \begin{pmatrix} 4 & -4 \\ 8 & -8 \end{pmatrix}$$

$(A - I)v = 0$ iff

$$4v_1 - 4v_2 = 0$$

$$\Leftrightarrow v_1 - v_2 = 0$$

$$\Leftrightarrow v_2 = v_1$$

!

Eigenvector: for $v_1 = 1$,

$$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Rmk v_1, v_2 are lin indep

$\Rightarrow \{v_1, v_2\}$ basis for $\mathbb{R}^2$

A computation ~~A~~ with eigenvalues

$v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\lambda_1 = -3$ $\quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\lambda_2 = 1$

Thus $\quad v_1 - v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If we wish to compute

$$A^3 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = A^3 v_1 - A^3 v_2$$

$$= \lambda_1^3 v_1 - \lambda_2^3 v_2$$

$$= (-3)^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1^3 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= -27 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -28 \\ -55 \end{pmatrix}$$

easier to compute than calculating A^3

Example with double eigenvalue

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

1 Eigenvalues

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (\lambda - 1)^2$$

Roots : 1, double root

2 Eigenvectors

$$(A - 1 \times I) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$(A - I)v = 0 \iff 0 \times v_1 + v_2 = 0$

$\iff v_2 = 0 \rightarrow$ we get only 1 eigenvector $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\left.\begin{array}{l}\text{1 double root} \\ \text{only 1 eigenvector}\end{array}\right) \rightarrow$ A defective matrix

Complex eigenvalues

$$A = \begin{pmatrix} -2 & -6 \\ 3 & 4 \end{pmatrix}$$

1 $\det(A - \lambda I)$

$$= \begin{vmatrix} -2-\lambda & -6 \\ 3 & 4-\lambda \end{vmatrix}$$

$$= (\lambda+2)(\lambda-4) + 18$$

$$= \lambda^2 - 2\lambda + 10$$

$$= (\lambda-1)^2 + 9$$

roots: $\lambda_1 = 1 + 3i$

$\lambda_2 = 1 - 3i = \bar{\lambda}_1$

Eigenvector for $\lambda_1 = 1+3i$

$A - (1+3i)I$

$$= \begin{pmatrix} -3-3i & -6 \\ \cancel{3} & \cancel{3-3i} \end{pmatrix}$$

Thus

$$(A - (1+3i)I)v = 0$$

iff $(3+3i)v_1 + 6v_2 = 0$

$\iff (1+i)v_1 + 2v_2 = 0$

Check: $v_1 = \begin{pmatrix} -(1+i) \\ 1 \end{pmatrix}$