Solving homogeneous eq. with

$y' = F\left(\frac{y}{x}\right)$ $\rightarrow$ set $\frac{y(x)}{x} = V(x)$

$\Leftrightarrow y = xV$

Eq: $\quad y' = \dfrac{4x+y}{x-4y}$

First question: is this a hom. eq?

We compute

$$f(tx, ty) = \frac{4\cancel{t}x + \cancel{t}y}{\cancel{t}x - 4\cancel{t}y}$$

$$= \frac{4x+y}{x-4y} = f(x,y)$$

↳ hom. eq.

Another way to see that eq. is hom.

$$f(x,y) = \frac{\Sigma \text{ terms of order 1 in } x,y}{\Sigma \text{ terms of order 1 in } x,y}$$

Thus the eq. is homogeneous

Eq: $y' = \dfrac{(4x+y)\times\frac{1}{x}}{(x-4y)\times\frac{1}{x}}$

$\Leftrightarrow \quad y' = \dfrac{4 + y/x}{1 - 4\,y/x} = F\left(\frac{y}{x}\right)$

where

$F(v) = \dfrac{4+v}{1-4v}$ $\qquad (fg)' = f'g + fg'$

① Set $V = y/x$ a $\quad y = xV$

Then $y' = (xV)'$

$= xV' + V$

The eq. becomes $\qquad$ diff eq fn

$xV' + V = \dfrac{4+V}{1-4V}$ $\qquad \nearrow \; V$

(2) $xV' + V = \frac{4+V}{1-4V}$

$\iff xV' = \frac{4+V}{1-4V} - V$

$\iff xV' = \frac{4+V - V(1-4V)}{1-4V}$

$\iff xV' = \frac{4+\cancel{V}-\cancel{V}+4V^2}{1-4V}$

$\frac{dV}{dx}$

$\iff xV' = \frac{4(1+V^2)}{1-4V}$

$\iff \frac{(1-4V)}{1+V^2} dV = 4 \frac{dx}{x}$

$\hookrightarrow$ separable eq.

③ $$\frac{1-4V}{1+V^2}\,dV = 4\,\frac{dx}{x}$$

Integrate on both sides

(i) $$\int \frac{1-4V}{1+V^2}\,dV$$

$$= \int \frac{1}{1+V^2}\,dV - \frac{4}{2}\int \frac{\overbrace{2V}^{\frac{u'}{u} \text{ with } u=1+V^2}}{1+V^2}\,dV$$

$$= \arctan(V) - 2\ln(1+V^2) \ (+c)$$

(ii) $$\int \frac{dx}{x} = \ln|x| \ (+c)$$

Back to sep eq: we get

$$\arctan(V) - 2\ln(1+V^2) = \overbrace{4\ln|x|}^{=2\ln(x^2)} + c_1$$

(4) we have obtained

$$\arctan(v) - 2\ln(1+v^2) - 2\ln(x^2) = C_1$$

$$\Leftrightarrow \arctan\left(\frac{y}{x}\right) - 2\ln\left(1+\frac{y^2}{x^2}\right) - 2\ln(x^2) = C_1$$

$$\Leftrightarrow \arctan\left(\frac{y}{x}\right) - 2\ln(x^2+y^2) + 2\ln(x^2) - 2\ln(x^2) = C_1$$

$$\Leftrightarrow \arctan\left(\frac{y}{x}\right) - 2\ln(x^2+y^2) = C_1$$

↳ general form of the solution

$$\ln\left(1+\frac{y^2}{x^2}\right) = \ln\left(\frac{x^2+y^2}{x^2}\right)$$

$$= \ln(x^2+y^2) - \ln(x^2)$$

Eq: $y' + \frac{3}{x} y = \frac{12}{(1+x^2)^{\frac{1}{2}}} y^{2/3}$

$\hookrightarrow$ Bernoulli

① Divide by $y^{2/3}$. we get

$$3 \times \frac{1}{3} y^{-2/3} y' + \frac{3}{x} \underbrace{y^{1/3}}_{=u} = \frac{12}{(1+x^2)^{\frac{1}{2}}}$$

② Set $u = y^{1/3}$. Then

$$u' = \frac{1}{3} y^{-2/3} y'$$

The eq becomes

linear eq.

$$3u' + \frac{3}{x} u = \frac{12}{(1+x^2)^{\frac{1}{2}}}$$

③ linear eq

$$\overset{\times \frac{1}{3}}{\Longleftrightarrow} \quad u' + \frac{1}{x} u = \frac{4}{(1+x^2)^{\frac{1}{2}}}$$

Integrating factor:

$\int \frac{1}{x} dx \quad \ln(x)$

Solving the linear eq, we find

$$u(x) = \frac{1}{x}\left(4(1+x^2)^{\frac{1}{2}} + C \right)$$

Go back to y:

$$u = y^{\frac{1}{3}} \iff y = u^3$$

Thus general solution is

$$y = \frac{1}{x^3}\left(4(1+x^2)^{\frac{1}{2}} + C \right)^3$$

Exact equations: can we have situations for which we can solve

$$M(x,y)\,dx + N(x,y)\,dy = 0 \,?$$

Related questions:

(i) How do we know that there exists ϕ such that

$$\phi_x \leftarrow \frac{\partial \phi}{\partial x} = M \quad \text{and} \quad \frac{\partial \phi}{\partial y} = N \quad ?$$

(ii) If ϕ exists, how to compute it?

Iff $M_y = N_x$

eq. $\left(y\cos(x)+2xe^{y}\right)+\left(\sin(x)+x^{2}e^{y}-1\right)\frac{dy}{dx}=0$

$$\iff \underbrace{\left(y\cos(x)+2xe^{y}\right)}_{M}dx+\underbrace{\left(\sin(x)+x^{2}e^{y}-1\right)}_{N}dy=0$$

(i) Is this an exact eq?

Yes iff $M_y = N_x$. Here

$M_y = \cos(x) + 2xe^{y}$

$N_x = \cos(x) + 2xe^{y}$

Thus $M_y = N_x$ and the eq is exact. There exist ϕ s.t. $\phi_x = M$ and $\phi_y = N$

(ii) Next step: compute ϕ

Eg: $(y\cos(x)+2xe^{y})\,dx+(\sin(x)+x^{2}e^{y}-1)\,dy = 0$

① $\phi(x,y) = \int M\,dx$

$= \int (y\cos(x)+2xe^{y})\,dx$

$= y\sin(x)+x^{2}e^{y}+h(y)$

② We write $\phi_y = N$

$\Leftrightarrow \sin(x)+x^{2}e^{y}+h'(y)$

$= \sin(x)+x^{2}e^{y}-1$

$\Leftrightarrow h'(y) = -1$

③ Integrate. We get

$h(y) = \int(-1)\,dy = -y$ $(+C)$

Thus we have

General solution

$$y\sin(x)+x^2e^y-y=C$$