Outline

Objective

indep. variable

General 2nd order differential equation:

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right) \longrightarrow$$

separate chapter for higher order diff eq.

Aim: See cases of 2nd order differential equations
$\hookrightarrow$ which can be solved with 1st order techniques

2nd order eq. with missing dependent variable

Case 1: Equation of the form

$$\frac{d^2y}{dx^2} = F\left(x, \frac{dy}{dx}\right)$$

No dependence on y

Method for case 1: the function $v = y'$ solves

$$\frac{dv}{dx} = F(x, v).$$

Then compute $y = \int v(x)\, dx$.

General eq $\quad y'' = F(x, y')$

Method $\quad$ Set $\quad y' = v$

Then $\quad y'' = v'$

and eq becomes

$$v' = F(x, v)$$

First order diff eq.

Example (1)

Equation:

$$\frac{d^2y}{dx^2} = \frac{1}{x}\left(\frac{dy}{dx} + x^2\cos(x)\right), \quad x > 0.$$

Eq $\quad y'' = \frac{1}{x}\left(y' + x^2\cos(x)\right)$

$$\Leftrightarrow \quad y'' = \underbrace{\frac{1}{x} y' + x\cos(x)}_{F(x,\, y')}$$

Set $y' = v$. We get

$$v' = \frac{1}{x} v + x\cos(x) \longrightarrow \text{linear eq}$$

$$\Leftrightarrow \quad v' - \frac{1}{x} v = x\cos(x)$$

Integrating factor $\quad \mu = e^{-\int \frac{1}{x} dx} = e^{-\ln(x)} = \frac{1}{x}$

General solution $\quad v = x\sin(x) + c_1 x$

Solving for y $\qquad v = y'$, so

$$y = \int v \, dx$$

$$= \int (x \sin(x) + c_1 x) \, dx$$

$$\overset{\text{ibp}}{=} -x \cos(x) + \sin(x) + c_1 x^2 + c_2$$

with $c_1, c_2 \in \mathbb{R}$

Rmk $\quad$ This is a second order diff eq

$\Rightarrow$ The general solution depends on 2 parameters

Example (2)

Change of variable: $v = y'$ solves the linear equation

$$v' - x^{-1} v = x \cos(x)$$

Integrating factor:

$$I(x) = \exp\left(\int x^{-1}\, dx \right) = x^{-1}$$

Solving for v:

$$v = x \sin(x) + cx$$

Solving for y:

$$y = \int v = -x \cos(x) + \sin(x) + c_1 x^2 + c_2$$

2nd order eq. with missing independent variable

Case 2: Equation of the form

$$\frac{d^2y}{dx^2} = F\left(y, \frac{dy}{dx}\right)$$

Variable x is missing

Method for case 2: We set $v = \frac{dy}{dx}$. Then observe that

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\,\frac{dy}{dx} = v\,\frac{dv}{dy}$$

Thus v solves the 1st order equation

$$v\,\frac{dv}{dy} = F(y, v).$$

General eq $\qquad y'' = F(y, y')$

We set $v = \frac{dy}{dx}$. We assume $v = v(y)$

Then

Chain rule

$$\text{lhs} = y'' = \frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\overbrace{\frac{dy}{dx}}^{v} = v\frac{dv}{dy}$$

Thus eq becomes

$$v\frac{dv}{dy} = F(y, v) \longrightarrow$$ 1st order diff eq.

Eq $\quad y'' = \overbrace{-\frac{2}{1-y}(y')^2}^{F(y,\,y')}$

Set $v = \frac{dy}{dx}$. The eq. becomes

$$v\frac{dv}{dy} = -\frac{2}{1-y}v^2 \longrightarrow$$ 1st order diff eq. for $v = v(y)$

$$\overset{\times\frac{1}{v^2}}{\Longleftrightarrow} \quad \frac{1}{v}\,dv = \frac{-2}{1-y}\,dy \longrightarrow$$ separable

General solution for v & y

$$v = c_1(1-y)^2$$

$$\Longleftrightarrow \quad \frac{dy}{dx} = c_1(1-y)^2 \longrightarrow$$ separable eq

$$\Longrightarrow \quad \boxed{y = \frac{c_1x + (c_2-1)}{c_1x + c_2}}$$

Example(1)

Equation:

$$\frac{d^2y}{dx^2} = -\frac{2}{1-y}\left(\frac{dy}{dx}\right)^2.$$

Example (2)

Change of variable: $v = y'$ solves the 1st order separable equation

$$v \frac{dv}{dy} = -\frac{2}{1-y} v^2$$

Solving for v:

$$v(y) = c_1 (1-y)^2$$

Example (3)

Separable equation for y:

$$\frac{dy}{dx} = c_1(1-y)^2$$

Solving for y:

$$y = \frac{c_1 x + (c_2 - 1)}{c_1 x + c_2}$$