(17)

One eigenvalue of $\underbrace{\begin{bmatrix} 1 & 2 & 2 \\ -1 & 4 & 1 \\ 0 & 0 & 3 \end{bmatrix}}_{A}$ is $\lambda = 3$. A basis for the corresponding eigenspace is

Let $B = A - 3I = \begin{pmatrix} -2 & 2 & 2 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Then

$$B \sim \begin{pmatrix} 1 & -1 & -1 \\ -2 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \overset{A_{12}(2)}{\sim} \begin{pmatrix} \boxed{1} & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

leading 1

free variables

Thus the solution set of $Bv = 0$ is

$$S = \{ (s+t, s, t) ; \ s, t \in \mathbb{R} \}$$

$$= \left\{ s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} ; \ s, t \in \mathbb{R} \right\}$$

A basis for S is obtained by taking $(s,t) = (1,0)$ and $(s,t) = (1,1)$. We get

$$\boxed{\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}}$$

18

If $y(x)$ is a solution of $y'' - 2y' + y = 0$ satisfying $y(0) = 1$ and $y'(0) = -1$, then $y(\frac{1}{2}) =$

Polynomial $P(r) = r^2 - 2r + 1 = (r-1)^2$

$r = 1$ is a double root

General solution

$$y = e^t (c_1 t + c_2)$$

$$\Rightarrow y' = e^t (c_1 t + c_1 + c_2)$$

Initial data we have

$y(0) = 1 \Rightarrow c_2 = 1$

$y'(0) = -1 \Rightarrow c_1 + c_2 = -1 \Rightarrow c_1 = -2$

Thus the unique solution is

$$y = e^t (-2t + 1)$$

and

$$\boxed{y(\tfrac{1}{2}) = 0}$$

19

The general solution of the differential equation $y'' + 4y = -8\frac{1}{\sin x}$ is

Hom. equation with polynomial $P(r) = r^2 + 4$ and roots $r = \pm 2i$. Hence

$$y_c = c_1 \overbrace{\cos(2x)}^{y_1} + c_2 \overbrace{\sin(2x)}^{y_2}$$

Variation of the constant we get

$$y_p = u_1 y_1 + u_2 y_2 ,$$

where u_1', u_2' satisfy

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{8}{\sin(x)} \end{pmatrix}$$

det = 2

$$\Leftrightarrow \begin{pmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{8}{\sin(x)} \end{pmatrix}$$

Thus $u_1' = \frac{1}{2} \begin{vmatrix} 0 & \sin(2x) \\ -8/\sin(x) & 2\cos(2x) \end{vmatrix}$

$$= 4\frac{\sin(2x)}{\sin(x)} = 8\frac{\sin(x)\cos(x)}{\sin(x)} = 8\cos(x)$$

$$u_2' = \frac{1}{2}\begin{pmatrix} \cos 2x & 0 \\ -2\sin(2x) & -\frac{8}{\sin(x)} \end{pmatrix}$$

$$= -4\,\frac{\cos(2x)}{\sin(x)} = -4\,\frac{(1-2\sin^2(x))}{\sin(x)}$$

$$= \frac{-4}{\sin(x)} + 8\sin(x)$$

We have found

$$\boxed{\begin{aligned} u_1' &= 8\cos(x) \\ u_2' &= \frac{-4}{\sin(x)} + 8\sin(x) \end{aligned}}$$

Conclusion The general solution is

$$y = c_1\cos(2x) + c_2\sin(2x) + u_1\cos(2x) + u_2\sin(2x)$$

None of the solutions has the $c_1\cos(2x) + c_2\sin(2x)$ term

E

20

The general solution of $\mathbf{x}' = \overbrace{\begin{bmatrix} 3 & 5 \\ -1 & -1 \end{bmatrix}}^{A} \mathbf{x}$ has the form

Eigenvalues $\det(A - \lambda I) = (\lambda - 3)(\lambda + 1) + 5$

$= \lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1$

Thus $\lambda_1 = 1 + i$, $\lambda_2 = \bar{\lambda}_1$

Eigenvector

$(A - (1+i)I)\vec{v} = 0$

remove safely this equation

$\Leftrightarrow \begin{pmatrix} 2-i & 5 \\ -1 & -2-i \end{pmatrix} \vec{v} = 0$

$\Leftrightarrow v_1 = (-2 - i) v_2$

Take $\vec{v} = \begin{pmatrix} 2+i \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Fundamental solutions

$x_1(t) = e^t \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \cos(t) - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \sin(t) \right\}$

$x_2(t) = e^t \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \sin(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cos(t) \right\}$

Therefore we obtain

$$\boxed{\begin{aligned} x_1(t) &= e^t \begin{pmatrix} 2\cos(t) - \sin(t) \\ -\cos(t) \end{pmatrix} \\ x_2(t) &= e^t \begin{pmatrix} 2\sin(t) + \cos(t) \\ -\sin(t) \end{pmatrix} \end{aligned}}$$

21

If $y = u_1 y_1 + u_2 y_2$ where $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ is a particular solution of

$$y'' - 4y = 4\tan x$$

then u_1 and u_2 are determined by

Equation u_1', u_2' satisfy

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 4\tan x \end{pmatrix}$$

det = -4

$$\Leftrightarrow \begin{pmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 4\tan x \end{pmatrix}$$

Solution we get

$$u_1' = -\frac{1}{4} \begin{vmatrix} 0 & e^{-2x} \\ 4\tan x & -2e^{-2x} \end{vmatrix}$$

$$\boxed{u_1' = e^{-2x} \tan x}$$

$$u_2' = -\frac{1}{4} \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 4\tan x \end{vmatrix}$$

$$\boxed{u_2' = -e^{2x} \tan x}$$

(22) Find all values of a such that the following system of equations has exactly one solution.

$$\begin{cases} x + y - z = 2 \\ x + 2y + z = 3 \\ x + y + (a^2 - 5)z = a \end{cases}$$

Write the system under the form

$A\vec{x} = b$, with

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & a^2-5 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 3 \\ a \end{pmatrix}$$

Then the system admits a unique solution iff $\det(A) = 0$. We have

$$\det(A) \underset{\substack{A_{12}(-1) \\ A_{13}(-1)}}{=} \begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & a^2-4 \end{vmatrix}$$

$$= a^2 - 4$$

Hence $\det(A) \neq 0$ iff $\boxed{a \neq 2,\ a \neq -2}$

23

Determine all values of k such that the vectors $(1,-1,0)$, $(1,2,2)$, $(0,3,k)$ are a basis for $\mathbb{R}^3$.

Form the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 3 \\ 0 & 2 & k \end{pmatrix}$$

Then we have a basis iff $\det(A) \neq 0$.

Compute

$$\det(A) \overset{A_{12}(1)}{=} \begin{vmatrix} 1 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & 2 & k \end{vmatrix}$$

$$= 3(k-2)$$

Hence

$\det(A) \neq 0$ iff $\boxed{k \neq 2}$