SPRING 19 - MIDTERM2 - MA262 - REVIEW

Pb1 We have

$$\det(A) = 2 \qquad \det(B) = -3$$

Thus

$$\det(A^{-1}B^2) = \frac{(\det(B))^2}{\det(A)}$$

$$= \frac{9}{2}$$

Pb 2 In $\mathbb{R}^3$, the family $\{v_1, v_2, v_3\}$ forms a basis iff $\det([v_1\ v_2\ v_3]) \neq 0$

Here

$$\det[v_1\ v_2\ v_3]$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ z & 1 & 1 \end{vmatrix}$$

$$= 2z - 1$$

Thus $\{v_1, v_2, v_3\}$ is a basis of $\mathbb{R}^3$ iff

$$z \neq \frac{1}{2}$$

Pb 3

In order to solve the system

$$x_1 - 3x_2 + 6x_3 = 0,$$

we set 2 free variables:

$$x_2 = s \qquad x_3 = t.$$

Then

$$x_1 = 3s - 6t$$

Therefore D is given by x of the form

$$x = \begin{bmatrix} 3s-6t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix},$$

and $\dim(D) = 2$

Pb 4 For a matrix $A \in \mathbb{R}^{3\times 3}$ we have

(i) Sum of eigenvalues $= a_{11} + a_{22} + a_{33}$

Here we get 1

(ii) Product of eigenvalues $= \det(A)$

Here we get 2

Pb 5 We compute Mv, that is

$$Mv = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+i-2 \\ -1+i+3 \end{bmatrix} = \begin{bmatrix} -3+i \\ 2+i \end{bmatrix}$$

Observe that $(Mv)_2 = (2+i)v_2$. Also

$$(2+i)v_1 = (2+i)(-1+i)$$

$$= (-2-1) + i(2-1)$$

$$= -3+i = (Mv)_1$$

Thus we have

$$Mv = (2+i)v$$

Pb 6 Eq: $y'' - y' - 2y = 0$ $y(0) = 2$ $y'(0) = 0$

Aux polynomial $P(r) = r^2 - r - 2 = (r-2)(r+1)$

General solution: $y = c_1 e^{2x} + c_2 e^{-x}$
and thus : $y' = 2c_1 e^{2x} - c_2 e^{-x}$

Initial data

$$\begin{cases} 2 = y(0) = c_1 + c_2 \\ 0 = y'(0) = 2c_1 - c_2 \end{cases} \Leftrightarrow \begin{cases} c_1 + c_2 = 2 \\ 2c_1 - c_2 = 0 \end{cases}$$

We get

$$c_1 = \frac{2}{3} \qquad c_2 = \frac{4}{3}$$

Therefore we have a unique solution

$$y = \frac{2}{3} e^{2x} + \frac{4}{3} e^{-x}$$

Pb 7 Eq: $y'' - 2y' + 2y = e^{x}\cos(x) - 3e^{2x}$

Aux polynomial: $P(r) = r^2 - 2r + 2 = (r-1)^2 + 1$

Roots : $r = 1 \pm i$

Therefor y_p will be of the form

$$y_p = x\left(A\cos(x) + B\sin(x)\right)e^{x} + Ce^{2x}$$

Pb 8 Eq: $y'' - 3y' + 2y = t^{-1}e^{3t}$

Aux polynomial: $P(r) = (r^2 - 3r + 2) = (r-1)(r-2)$
Fund solutions: $y_1 = e^t \qquad y_2 = e^{2t}$

According to the variation of parameters method, v_1', v_2' solve the system

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^{-1}e^{3t} \end{bmatrix}$$

$$\iff \overbrace{\begin{bmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{bmatrix}}^{M} \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^{-1}e^{3t} \end{bmatrix}$$

We have $\det(M) = e^{3t}$. Thus according to Cramer's rule we have

$$v_1' = e^{-3t} \begin{vmatrix} 0 & e^{2t} \\ t^{-1}e^{3t} & 2e^{2t} \end{vmatrix}$$

$$= -t^{-1}e^{2t}$$

We obtain v_2' in a similar way

Pb.9 Eq: $y'' - \frac{1}{t} y' + \frac{1}{t^2} y = 0$

We set $y = tv$. Then

$\times\left(\frac{1}{t^2}\right)$ $y = tv$

$\times\left(-\frac{1}{t}\right)$ $y' = v + tv'$

$\times (1)$ $y'' = 2v' + tv''$

Thus

$y'' - \frac{1}{t} y' + \frac{1}{t^2} y = 0 \iff -v' + 2v' + tv'' = 0$

The equation for v is

$tv'' + v' = 0$

Pb 10 Eq: $y'' + by' + 2y = 0$

Then

(i) If $b = 0$, no damping
$\Rightarrow$ periodic solution

(ii) If $b^2 - 8 > 0 \Leftrightarrow b > \sqrt{8}$,
the system is overdamped

(iii) If $b^2 = \sqrt{8}$
the system is critically damped

(iv) If $b^2 < \sqrt{8}$
the system is underdamped