REVIEW PROBLEMS - MIDTERM 1

Pb 1

Equation $y' = \dfrac{y}{x \ln|x|}$ → separable

$$Eq \iff \frac{dy}{y} = \frac{1}{x}\,\frac{1}{\ln|x|}\,dx$$

$$\iff \ln(|y|) = \ln(|\ln|x||) + c_1$$

$$\iff y = c_2 \ln(x)$$

Pb 2

Equation $\overbrace{(3x^2 - 2xy + 2)}^{M}\,dx + \overbrace{(6y^2 - x^2 + 3)}^{N}\,dy = 0$

We have $M_y = -2x \qquad N_x = -2x$

Thus the equation is exact. Then

$$F(x,y) = \int M\,dx = x^3 - x^2 y + 2x + g(y)$$

Then $F_y = N$

$$\iff -x^2 + g'(y) = 6y^2 - x^2 + 3 \iff g'(y) = 6y^2 + 3$$

$$\iff g(y) = 2y^3 + 3y + c$$

Solution: $x^3 - x^2 y + 2x + 2y^3 + 3y$ ...

Pb 3

$$(x-2y)\,dx + 4x\,dy = 0$$

$$\Leftrightarrow \quad y = \frac{x-2y}{4x} = \frac{1}{4} - \frac{1}{2}\frac{y}{x}$$

Set $v = \frac{y}{x} \Rightarrow y' = xv' + v$. We get

$$xv' + v = \frac{1}{4} - \frac{1}{2}v$$

$$\Leftrightarrow xv' = \frac{1}{4}(1-6v)$$

$$\Leftrightarrow \frac{dv}{1-6v} = \frac{dx}{4x} \quad \rightsquigarrow \text{separable eq}$$

$$\Leftrightarrow \ln|1-6v| = \frac{1}{4}\ln|x| + c_1$$

$$\Leftrightarrow |1-6v| = c_2|x|^{\frac{1}{4}}$$

$$\Leftrightarrow 6v = 1 + c_3|x|^{\frac{1}{4}} \quad \text{with } c_3 \in \mathbb{R}$$

$$\Leftrightarrow v = \frac{1}{6} + c_4|x|^{\frac{1}{4}} \quad \text{with } c_4 \in \mathbb{R}$$

$$\Leftrightarrow y = \frac{x}{6} + c_4\, x|x|^{\frac{1}{4}}$$

Pb 4 Equation $y' - \frac{2}{x} y = x^2 y^2$ Bernoulli

$$\Leftrightarrow \quad y^{-2} y' - \frac{2}{x} y^{-1} = x^2$$

Set $u = y^{-1} \Rightarrow u' = -y^{-2} y'$. We get

$$-u' - \frac{2}{x} u = x^2$$

$$\Leftrightarrow \quad u' + \frac{2}{x} u = x^2 \quad \rightarrow \text{linear eq.}$$

Intg factor: $\mu = e^{\int \frac{2}{x} dx} = x^2$. Thus

$$(u x^2)' = x^2 \quad \Leftrightarrow \quad u x^2 = \frac{1}{3} x^3 + c_1$$

$$\Leftrightarrow \quad u = \frac{x^3 + c_2}{3 x^2}$$

$$\Leftrightarrow \quad y = \frac{3x^2}{x^3 + c_2}$$

Pb 5 $V = 100 \quad c_1 = \frac{1}{2} \quad r_1 = 2 \quad r_2 = 2$

Thus $A' = c_1 r_1 - \frac{A}{V} r_2 = \frac{1}{2} \times 2 - \frac{A}{100} \times 2$

$$\Leftrightarrow A' + \frac{1}{50} A = 1 \qquad (1)$$

Integrating factor $\mu = e^{t/50}$. Then (1) becomes

$$(A e^{t/50})' = e^{t/50}$$

$$\Leftrightarrow A e^{t/50} = 50 e^{t/50} + c$$

$$\Rightarrow A = 50 + c e^{-t/50}$$

Initial condition: $A(0) = 0$. Thus $c = -50$. We get

$$A(t) = 50 [1 - e^{-t/50}]$$

and $A(50) = 50(1 - e^{-1})$

Pb 6 $\quad v_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad v_2 = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \quad v_3 = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix} \quad v_4 = \begin{pmatrix} 13 \\ 14 \\ 13 \end{pmatrix}$

Solution 1: A pattern is easily extracted in the family $\{v_1, .., v_4\}$. It reveals that

$$v_2 - v_1 = v_3 - v_2 \quad \text{and} \quad v_3 - v_2 = v_4 - v_3$$

This gives 2 linear combinations $c_1 v_1 + \cdots + c_4 v_4 = 0$. Thus the maximal number of independent vector is ≤ 2. Since v_1 and v_2 are linearly independent ($v_2 \neq c v_1$), the maximal family is $\{v_1, v_2\}$

Solution 2: Form the matrix

$$A = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \end{bmatrix} \underset{A_{13}(-1)}{\overset{A_{12}(-1)}{\sim}} \begin{bmatrix} 1 & 5 & 9 & 13 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\underset{A_{12}(-1)}{\overset{A_{23}(-1)}{\sim}} \begin{bmatrix} 1 & 5 & 9 & 13 \\ 0 & -4 & -8 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\overset{M_2(-\frac{1}{4})}{\sim} \begin{bmatrix} \boxed{1} & 5 & 9 & 13 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Form the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 5 & -9 & k \\ -3 & 6 & -9 \end{bmatrix} \underset{A_{13}(3)}{\overset{A_{12}(-5)}{\sim}} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 0 & \boxed{1} & k-15 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore the system $Ax = 0$ always admits a nontrivial solution.

$$v_3 \in \text{Span}\{v_1, v_2\} \quad \text{for all } k \in \mathbb{R}$$

$$A^{\#} = \begin{bmatrix} 5 & -6 & 1 & 4 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{bmatrix} \overset{A_{31}(-1)}{\sim} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 2 & -3 & 1 & 1 \\ 4 & -3 & -1 & 5 \end{bmatrix}$$

$$\overset{A_{13}(-4)}{\underset{A_{12}(-2)}{\sim}} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 3 & -3 & 3 \\ 0 & 9 & -9 & 9 \end{bmatrix} \overset{M_2(\frac{1}{3})}{\underset{M_3(1/9)}{\sim}} \begin{bmatrix} 1 & -3 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

$$\overset{A_{23}(-1)}{\sim} \begin{bmatrix} \boxed{1} & -3 & 2 & -1 \\ 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have 2 pivots and no equation of the form $0=1$. Thus we get an ∞ number of solution. The free variable is $x_3 = t$. Then

$$x_2 - x_3 = 1 \implies x_2 = 1+t$$

$$x_1 - 3x_2 + 2x_3 = -1 \implies x_1 = 3(1+t) - 2t - 1$$

$$= 2 + t$$

Solution set

Pb 10 We have

i.e $\quad T\underbrace{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}_{v_1} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 2 \end{pmatrix} \qquad T\underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{v_2} = \begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

Then observe that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3}\left(v_2 - 2v_1\right)$$

Therefore

$$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3}\left(T(v_2) - 2T(v_1)\right)$$

$$= \frac{1}{3}\left[\begin{pmatrix} 1 \\ 0 \\ -2 \\ 2 \end{pmatrix} - 2\begin{pmatrix} -3 \\ 1 \\ 1 \\ 1 \end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{pmatrix} 7 \\ -2 \\ -4 \\ 0 \end{pmatrix}$$