## Differential equations: introduction

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Differential equations - MA 26600

## Taken from *Elementary differential equations* by Boyce and DiPrima

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#### Basic mathematical models

- Gravity example
- Mice and owl example
- Direction fields
- 2 Solving affine equations
- Olassification of differential equations



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## 2 Solving affine equations

## 3 Classification of differential equations

## Interest of differential equations

#### Differential equation:

Equation in which the derivative of a function appears.

#### Features of differential equations:

- Theoretical interest
- Always related to a physical system:
  - Fluid dynamics
  - Electrical circuits
  - Population dynamics
  - Economy, finance
- More than 300 years of study
- Still active domain of research



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# Expression of Newton's law

## Example of physical situation:

Object falling in the atmosphere near sea level

Notation:

- *t* = time variable, in seconds
- v = velocity, depends on time v = v(t), in  $m s^{-1}$
- F =force
- *a* = acceleration

Orientation: Downwards

Newton's law:

$$F = m a = m \frac{dv}{dt}$$

# Gravity example (2)

Forces acting on the object:

- Gravity: mg, where  $g = 9.81 ms^{-2}$  close to earth
- Air resistance, drag:  $-\gamma v$ , where  $\gamma$  object dependent



Total force:  $F = mg - \gamma v$ 

Resulting equation:

$$m \frac{dv}{dt} = mg - \gamma v \tag{1}$$

3

# Qualitative study

Specific values for coefficients:  $\hookrightarrow$  We take m = 10kg and  $\gamma = 2$ kg s<sup>-1</sup>

Specific equation:

$$\frac{dv}{dt} = 9.8 - \frac{v}{5}$$

(2

Note:

- One can solve equation (2)
- Qualitative study: draw conclusions from equation itself

#### Example of slope:

 $\hookrightarrow$  If v = 40, then  $\frac{dv}{dt} = 1.8$ 

# Direction field

## Meaning of the graph:

 $\hookrightarrow$  Values of  $\frac{dv}{dt}$  according to values of v



What can be seen on the graph:

- Critical value:  $v_c = 49 \text{ms}^{-1}$ , solution to  $9.8 \frac{v}{5} = 0$
- If  $v < v_c$ : positive slope
- If  $v > v_c$ : negative slope

# Qualitative study (2)

#### Equilibrium: According to the graph

- $v(t) \equiv v_c$  is solution to (2)
- All solutions converge to  $v_c$  as  $t o \infty$

## Remark:

- The facts above will be shown later on
- 2  $v_c$  is called equilibrium for system (2)

#### Generalization: For general system (1):

- Equilibrium:  $v_c = \frac{mg}{\gamma}$
- Convergence to equilibrium

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## Predator-pray model

#### Situation:

- p = p(t) = mice population
- Reproduction rate for mice: r mice/month
- Presence of owl: k mice eaten per month

Resulting equation:

 $\frac{dp}{dt} = rp - k$ 

# Direction field

## Meaning of the graph:

 $\hookrightarrow$  Values of  $\frac{dv}{dt}$  according to values of v



What can be seen on the graph:

- Critical value:  $p_c = \frac{k}{r}$ , solution to rp k = 0
- If  $p < p_c$ : negative slope
- If  $p > p_c$ : positive slope

# Qualitative study (2)

Equilibrium: According to the graph

- $p(t) \equiv p_c$  is solution to (2)
- A solution will never converge to  $p_c$  as  $t \to \infty$
- If  $p(0) > p_c$ , population increases
- If  $p(0) < p_c$ , extinction

Remark:

•  $p_c$  is an unstable equilibrium for system (2)

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## Importance of direction fields

General form of an equation:

$$\frac{dy}{dt} = f(t, y)$$

Conclusions from previous examples:

- Importance of direction fields graphs  $(t, y) \mapsto f(t, y)$
- 2 Plotting  $(t, y) \mapsto f(t, y)$  is easier than solving the equation
- It can be done with the help of a computer

# Matlab dfield8 function

#### Remote connexion to Matlab:

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## Matlab dfield8 function (2) Tip: Use Matlab 2014

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# Mice and owl equation

Specific form:

$$\frac{dp}{dt} = 0.5p - 450 \tag{3}$$

Integration of the equation: We have

$$\frac{p'}{p-900} = \frac{1}{2}$$

Integrating we obtain:

$$p(t) = 900 + c \exp\left(rac{t}{2}
ight), \quad ext{with} \quad c \in \mathbb{R}.$$

## Initial data

## Family of solutions:

- We have seen: solutions depend on parameter c
- One way to find c: specify value of p(0)
- Example: if p(0) = 850, then  $p(t) = 900 50 \exp(t/2)$

#### Graph of solutions according to initial condition:



## General solution

### Proposition 1.

Equation considered:

$$\frac{dy}{dt} = ay - b, \quad \text{and} \quad y(0) = y_0. \tag{4}$$

#### Hypothesis:

$$a, b \in \mathbb{R}, \qquad a \neq 0, \qquad y(0) \in \mathbb{R}.$$

Then the unique solution to (4) is given by:

$$y(t)=\frac{b}{a}+\left[y_0-\frac{b}{a}\right]e^{at}.$$

# Mice and owl reloaded Equation:

$$\frac{dp}{dt} = rp - k$$

Expression of solution: with initial condition  $p_0 > 0$ ,

$$p(t) = \frac{k}{r} + \left[p_0 - \frac{k}{r}\right]e^{rt}$$

Remarks:

- If  $p_0 = \frac{k}{r}$ , solution stays at equilibrium
- If p<sub>0</sub> < <sup>k</sup>/<sub>r</sub>, solution decreases until extinction
   → Negative values of p are physically meaningless
- If  $p_0 > \frac{k}{r}$ , solution grows exponentially (critics to model?)
- This could be seen on the previous graph

# Gravity reloaded Equation:

$$\frac{dv}{dt} = g - \frac{\gamma}{m}v$$

Expression of solution: with initial condition  $v_0 \in \mathbb{R}$ ,

$$v(t) = rac{mg}{\gamma} + \left[v_0 - rac{mg}{\gamma}
ight] e^{-rac{\gamma t}{m}}$$

Remarks:

- If  $v_0 = \frac{mg}{\gamma}$ , solution stays at equilibrium
- If  $v_0 \neq \frac{m_g}{\gamma}$ , convergence to equilibrium  $\hookrightarrow$  exponential convergence, rate  $\frac{\gamma}{m}$
- From v, one can retrieve position x
  - $\hookrightarrow$  find velocity v when a dropped object hits the ground

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## Olassification of differential equations

# Ordinary vs partial differential equations

Ordinary differential equation: depends on one variable only

- Gravity, v = v(t); Mice an owl, p = p(t)
- Capacitor with capacitance C, resistance R, inductance L:

$$L\frac{d^2Q}{dt} + R\frac{dQ}{dt} + \frac{Q}{C} = E$$

Partial differential equation: depends on two or more variables

Heat equation:

$$\alpha^2 \, \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

• Wave equation:

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

## Systems of differential equations

Definition: Systems of differential equations  $\hookrightarrow$  when 2 or more unknown functions are involved

Example: Lotka-Volterra predator-pray model

$$\begin{cases} \frac{dx}{dt} = ax - \alpha xy\\ \frac{dy}{dt} = -cy + \gamma xy \end{cases}$$

Remark: In many engineering situations  $\hookrightarrow$  lots of coupled differential equations

## Order of a differential equation

#### Definition: Order of a differential equation

= Order of highest derivative appearing in equation

Examples:

- Gravity, Mice-owl: first order
- Capacitor: second order
- Heat, wave: second order partial differential equations

General form of *n*-th order differential equation:

$$F(y, y', \dots, y^{(n)}) = 0$$
<sup>(5)</sup>

## Linear and nonlinear equations

#### Definition: In equation (5),

- If F is linear, differential equation is linear
- If F is not linear, differential equation is nonlinear

Examples:

- Gravity, Mice-owl, Capacitor: linear differential equations
- Heat, wave: linear partial differential equations
- Lotka-Volterra: nonlinear, because of term xy

#### Remark:

Nonlinear equations are harder to solve than linear equations

## Solutions to differential equations

Definition: Solution to equation (5) on [a, b] $\hookrightarrow$  any function  $\phi$  such that  $\phi, \phi', \dots, \phi^{(n)}$  exist and

 $F(\phi(t),\phi'(t),\ldots,\phi^{(n)}(t))=0, \quad ext{for} \quad t\in[a,b]$ 

Remark: If we have an intuition for a solution to (5)  $\hookrightarrow$  verification is easy

Example: For equation

$$y''+y=0,$$

easy to check that sin(t) and cos(t) are solutions

Issues related to differential equations

General form of equation:

 $F(y,y',\ldots,y^{(n)})=0$ 

List of problems:

- Existence to solution
- Oniqueness of solution
- Find exact solutions in simple cases
- Approximation of solution in complex cases
- Some constraints of the second second