First order differential equations

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Differential equations - MA266

Taken from *Elementary differential equations* by Boyce and DiPrima

Outline

Linear equations

- 2 Separable equations
- 3 Homogeneous equations
- 4 Modeling with first order differential equations
- 5 Differences between linear and nonlinear equations
- 6 Autonomous equations
- Exact equations and integrating factors
- 8 Numerical approximation: Euler's method

First order differential equations

General form of equation:

$$\frac{dy}{dt} = f(t, y)$$

List of problems:

- Existence of solution
- Oniqueness of solution
- Find exact solutions in simple cases
- Approximation of solution in complex cases
- Ombine analytic, graphical and numerical methods
 → to understand solutions

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General form of 1st order linear equation

General form 1:

$$\frac{dy}{dt} + p(t)y = g(t)$$

General form 2:

$$P(t)rac{dy}{dt}+Q(t)y=G(t)$$

Remark:

2 forms are equivalent if $P(t) \neq 0$

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Example with direct integration

Equation:

$$\left(4+t^2\right)\frac{dy}{dt}+2t\,y=4t$$

Equivalent form:

$$\frac{d}{dt}\left[\left(4+t^2\right)y\right]=4t$$

General solution: For a constant $c \in \mathbb{R}$,

$$y = \frac{2t^2 + c}{4 + t^2}$$

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Method of integrating factor

General equation:

$$\frac{dy}{dt} + p(t)y = g(t) \tag{1}$$

Recipe for the method:

- Consider equation (1)
- ② Multiply the equation by a function μ
- Try to choose μ such that equation (1) is reduced to:

$$\frac{d(\mu y)}{dt} = a(t) \tag{2}$$

Integrate directly equation (2)

Notation: If previous recipe works, μ is called integrating factor

Example of integrating factor Equation:

$$\frac{dy}{dt} + \frac{1}{2}y = \frac{1}{2}e^{t/3}$$
(3)

Image: A matrix

Multiplication by μ :

$$\mu(t)rac{dy}{dt}+rac{1}{2}\,\mu(t)\,y=rac{1}{2}\,\mu(t)\,e^{t/3}$$

Integrating factor: Choose μ such that $\mu' = \frac{1}{2}\mu$, i.e $\mu(t) = e^{t/2}$

Solving the equation: We have, for $c \in \mathbb{R}$

(3)
$$\iff \frac{d\left(e^{t/2}y\right)}{dt} = \frac{1}{2}e^{\frac{5t}{6}}$$

 $\iff y(t) = \frac{3}{5}e^{\frac{t}{3}} + ce^{-\frac{t}{2}}$

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Example of integrating factor (2)

Solution for a given initial data: If we know y(0) = 1, then

$$y(t) = rac{3}{5}e^{rac{t}{3}} + rac{2}{5}e^{-rac{t}{2}}$$

Direction fields and integral curves:



General case with constant coefficient



Example with exponential growth Equation:

$$\frac{dy}{dt} - 2y = 4 - t$$

General solution: for $c \in \mathbb{R}$,

$$y(t) = -\frac{7}{4} + \frac{t}{2} + c e^{2t}$$

Direction fields and integral curves:



General first order linear case

Proposition 2.

Equation considered:

$$\frac{dy}{dt} + p(t)y = g(t), \tag{5}$$

Integrating factor:

$$\mu(t) = \exp\left(\int p(r)\,dr\right).$$

Then general solution to (5) is given by:

$$y(t) = \frac{1}{\mu(t)} \left[\int_{t_0}^t \mu(s) g(s) \, ds + c \right].$$

with $t_0 \geq 0$ and $c \in \mathbb{R}$.

Example with unbounded p

Equation considered:

$$t y' + 2y = 4t^2, \qquad y(1) = 2.$$
 (6)

Equivalent form:

$$y' + \frac{2}{t}y = 4t, \qquad y(1) = 2.$$

Integrating factor:

$$\mu(t)=t^2.$$

Solution:

$$y(t) = t^2 + \frac{1}{t^2}$$

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Example with unbounded p (2)

Some integral curves:



Comments:

- First example of solution which is not defined for all $t \ge 0$
- 2 Due to singularity of $t \mapsto \frac{1}{t}$
- **(3)** Integral curves for t < 0: not part of initial value problem
- According to value of y(1), different asymptotics as $t \to 0$
- **③** Boundary between 2 behaviors: function $y(t) = t^2$

Example with no analytic solution Equation considered:

$$2y' + ty = 2,$$
 $y(0) = 1.$

Integrating factor:

$$\mu(t) = \exp\left(rac{t^2}{4}
ight).$$

Solution:

$$y(t) = \exp\left(-\frac{t^2}{4}\right) \int_0^t \exp\left(\frac{s^2}{4}\right) ds + c \, \exp\left(-\frac{t^2}{4}\right).$$

Some integral curves obtained by approximation:



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General form of separable equation

Notational change: y seen as function of x instead of t. General form of first order equation:

$$\frac{dy}{dx} = f(x, y)$$

General form of separable equation:

$$M(x)+N(y)\frac{dy}{dx}=0.$$

Heuristics to solve a separable equation:

- Write equation as: M(x) dx = -N(y) dy.
- Integrate in x on l.h.s, integrate in y on r.h.s

Example of separable equation

Equation considered:

$$rac{dy}{dx}=rac{x^2}{1-y^2} \Longleftrightarrow -x^2+(1-y^2)rac{dy}{dx}=0.$$

Chain rule:

$$\frac{df(y)}{dx} = f'(y)\frac{dy}{dx}$$

Application of chain rule:

$$(1-y^2)\frac{dy}{dx} = \frac{d}{dx}\left(y-\frac{y^3}{3}\right), \text{ and } x^2 = \frac{d}{dx}\left(\frac{x^3}{3}\right)$$

Example of separable equation (2)

Equation for integral curves: We have, for $c \in \mathbb{R}$,

$$(7) \Longleftrightarrow \frac{d}{dx} \left(-\frac{x^3}{3} + y - \frac{y^3}{3} \right) = 0 \Longleftrightarrow -x^3 + 3y - y^3 = c$$

Some integral curves obtained by approximation:



General solution for separable equations



Solvable example of separable equation

Equation considered:

$$\frac{dy}{dx} = \frac{3x^2 + 4x + 2}{2(y - 1)}$$
, and $y(0) = -1$. (9)

Integration: for a constant $c \in \mathbb{R}$,

(9)
$$\iff 2(y-1) dy = (3x^2 + 4x + 2) dx$$

 $\iff y^2 - 2y = x^3 + 2x^2 + 2x + c$

Solving the equation: if y(0) = -1, we have c = 3 and

$$y = 1 \pm \left(x^3 + 2x^2 + 2x + 4\right)^{1/2}$$

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Solvable example of separable equation (2) Determination of sign: Using y(0) = -1 again, we get

$$y = 1 - (x^3 + 2x^2 + 2x + 4)^{1/2} = 1 - ((x + 2)(x^2 + 2))^{1/2}$$

Interval of definition: $x \in (-2, \infty)$

 \hookrightarrow boundary corresponds to vertical tangent on graph below Integral curves:



Example of equation with implicit solution

Equation considered:

$$\frac{dy}{dx} = \frac{4x - x^3}{4 + y^3}.$$

General solution: for a constant $c \in \mathbb{R}$,

$$y^4 + 16y + x^4 - 8x^2 = c$$

Initial value problem: if y(0) = 1, we get

$$y^4 + 16y + x^4 - 8x^2 = 17$$

Example of equation with implicit solution (2)

Integral curves:



Interval of definition:

 \hookrightarrow boundary corresponds to vertical tangent on graph

Implicit equations with Matlab

Preliminary remark:

- Most of equations in HW will be solved explicitly
- To plot functions: use ezplot

Implicit equation:

- Of the form $H_2(y) = c$
- Use function solve

Implicit function:

- Of the form $H_2(y) = H_1(x)$
- Loop on values of x
- For each value of x, use function solve
- Easier: use ezplot

Homework hints

Problem 2.2-11:

- Equation: $x dx + y \exp(-x) dy = 0$, with y(0) = 1
- Solution: $y(x) = (2 \exp(x) 2x \exp(-x) 1)$
- Radical vanishes for $x_1 \simeq -1.7$ and $x_2 \simeq 0.77$

Problem 2.2-22:

- Equation: $(3y^2 4)dy = 3x^2dx$, with y(1) = 0
- Solution: $y^3 4y = x^3 1$
- From equation, $y' \to \infty$ when $y \to \pm \frac{2}{\sqrt{3}}$
- This corresponds to $x_1\simeq -1.276$ and $x_2\simeq 1.598$

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General form of homogeneous equation

Recall: y seen as function of x instead of t. General form of first order equation:

$$\frac{dy}{dx} = f(x, y)$$

General form of homogeneous equation:

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right).$$

Heuristics to solve homogeneous equations:

• Go back to a separable equation

Solving homogeneous equations

Equation:

$$\frac{dy}{dx} = \varphi\left(\frac{y}{x}\right). \tag{10}$$

General method:

- Set y(x) = x v(x), and express y' in terms of x, v, v'.
- **2** Replace in equation (10) \longrightarrow separable equation in v.
- Solve the separable equation in v.
- Go back to y recalling y = x v.

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Example of homogeneous equation

Equation:

$$\frac{dy}{dx} = \frac{y - 4x}{x - y}$$

Equation for *v*:

$$\frac{1-v}{v^2-4}\frac{dv}{dx} = \frac{1}{x}$$

Solution for the *v* equation: for $c \in \mathbb{R}$,

$$|v-2|^{1/4}|v+2|^{3/4} = \frac{c}{|x|}$$

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Example of homogeneous equation (2) Solution for the y equation: for $c \in \mathbb{R}$,

$$|y - 2x|^{1/4}|y + 2x|^{3/4} = c$$

Graph for the implicit equation: observe symmetry w.r.t origin



Homework hints

Problem 2.2-32:

- Equation: $y' = \frac{x^2 + 3y^2}{2xy}$
- Solution for the v equation: $v = \pm (cx 1)^{1/2}$
- Solution for the y equation: $y = \pm x(cx-1)^{1/2}$



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Homework hints (2) Problem 2.2-B:

- Equation: $y' = (y + x)^2$
- Set u = x + y
- Equation in *u*:

$$rac{u'}{u^2+1}=1, \quad ext{of the form} \quad rac{u'}{f(u)}=g(u)$$

Problem 2.2-C:

• Equation:
$$y^2y' + \frac{y^3}{x} = \frac{2}{x^2}$$

- Set $u = y^3$
- Equation in *u*: of the form

$$\frac{u'}{3} + \frac{u}{x} = \frac{2}{x^2} \quad \text{of the form} \quad u' + p(x)u = g(u)$$

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Global strategy

Interest of modeling:

Numerical predictions replace costly experiments

Basic steps:

- Translate physical principles into equations
 - Variations involved \longrightarrow 1st order equation
 - Simplifications can often be useful
- Analyze the system
 - Solve equation
 - Otherwise try to analyze behavior of system
 - Possible linearization of the system
- Omparison with experimental data

Salt concentration example

Description of experiment:

- At t = 0, Q_0 lb of salt dissolved in 100 gal of water
- Water containing $\frac{1}{4}$ lb salt/gal entering, with rate r gal/min
- Well-stirred mixture draining from tank, rate r


Salt concentration example (2)

Notation: $Q(t) \equiv$ quantity of salt at time t

Hypothesis: Variations of Q due to flows in and out,

$$\frac{dQ}{dt} =$$
rate in $-$ rate out

Equation:

$$\frac{dQ}{dt}=\frac{r}{4}-\frac{rQ}{100},\qquad Q(0)=Q_0$$

Equation, standard form:

$$\frac{dQ}{dt} + \frac{r}{100} Q = \frac{r}{4}, \qquad Q(0) = Q_0$$

Salt concentration example (3)

Integrating factor:
$$\mu(t) = e^{\frac{rt}{100}}$$

Solution:

$$Q(t) = 25 + (Q_0 - 25) e^{-rac{rt}{100}}$$

Integral curves:



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Salt concentration example (4) Expression for Q:

$$Q(t) = 25 + (Q_0 - 25) e^{-\frac{rt}{100}}$$

Question: time to reach $q \in (Q_0, 25)$?

Answer: We find

$$Q(t) = q \quad \Longleftrightarrow \quad t = rac{100}{r} \ln\left(rac{Q_0 - 25}{q - 25}
ight)$$

Application: If r = 3, $Q_0 = 50$ and q = 25.5, then:

$$t = 130.4 \text{ min}$$

Image: A matrix

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Chemical pollution example

Description of experiment:

- At t = 0, 10^7 gal of fresh water
- Water containing unwanted chemical component entering \hookrightarrow with rate $5\cdot 10^6~{\rm gal/year}$
- $\bullet\,$ Water flows out, same rate $5\cdot 10^6\,\, gal/year$
- Concentration of chemical in incoming water:

 $\gamma(t) = 2 + \sin(2t) \text{ g/gal}$



Chemical pollution example (2)

Notation: $Q(t) \equiv$ quantity of chemical comp. at time $t \rightarrow$ measured in grams

Remark: Volume is constant

Hypothesis: Variations of Q due to flows in and out,

$$\frac{dQ}{dt} =$$
rate in $-$ rate out

Equation:

$$rac{dQ}{dt} = 5 \cdot 10^6 \, \gamma(t) - 5 \cdot 10^6 \cdot rac{Q}{10^7}, \qquad Q(0) = 0$$

Image: A matrix

Chemical pollution example (3) Equation, standard form: We set $Q = 10^6 q$ and get

$$\frac{dq}{dt} + \frac{1}{2}q = 10 + 5\sin(2t), \qquad q(0) = 0$$

Integrating factor: $\mu(t) = e^{\frac{t}{2}}$

Solution:

$$q(t) = 20 - rac{40}{17}\cos(2t) + rac{10}{17}\sin(2t) - rac{300}{17}e^{-rac{t}{2}}$$

Integral curve:



Cooling cup example

Description of experiment:

• Cup of coffee cooling in a room

Notation:

- $T(t) \equiv$ temperature of cup
- $\tau \equiv$ temperature of room

Newton's law for thermic exchange:

Variations of temperature proportional to difference between T and au

Equation:

$$\frac{dT}{dt} = -k(T-\tau), \qquad T(0) = T_0.$$

Escape velocity example

Notation:

- $x \equiv$ altitude
- x = 0: surface of earth

Description of experiment:

- Body with mass m
- Initial velocity v₀, upward
- Air resistance negligible
- Newton's constant g depends on altitude



Escape velocity example (2)

Gravitational force: given by

$$w(x) = -rac{k}{(R+x)^2}$$
, and $w(0) = -mg$.

Hence:

$$w(x) = -\frac{mgR^2}{(R+x)^2}.$$

Equation: according to Newton's law,

$$m \frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2}, \qquad v(0) = v_0.$$
 (11)

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Escape velocity example (3)

Elimination of variable: We have (chain rule)

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Therefore we get a separable equation:

$$m\frac{dv}{dt} = -\frac{mgR^2}{(R+x)^2} \iff v \frac{dv}{dx} = -\frac{gR^2}{(R+x)^2}$$

Integration of equation: for $c \in \mathbb{R}$,

$$\frac{v^2}{2} = \frac{gR^2}{R+x} + c$$

Escape velocity example (4)

Solution: with $v(0) = v_0$ we get

$$v = \pm \left(v_0^2 - 2gR + \frac{2gR^2}{R+x} \right)^{1/2},$$
 (12)

and sign according to direction of velocity.

Question: maximal altitude that body reaches? \hookrightarrow We seek ξ such that v = 0 whenever $x = \xi$ in (12)

Answer: We find

$$\xi = \frac{v_0^2 R}{2gR - v_0^2}$$

Escape velocity: in order to have $\xi = \infty$ \hookrightarrow take $v_0 = (2gR)^{1/2} = 11.1$ km s⁻¹

Problem 20, section 2.3

Experiment:

- Ball with mass *m* thrown with velocity v_0
- Initial height: x_0
- Air resistance neglected

Equation:

$$v' = -g$$

Solution:

$$v = v_0 - gt$$

Maximal height: when v = 0, that is $t = \frac{v_0}{\sigma}$

Equation for height:

$$x(t) = x_0 + \int_0^t v(s) ds = x_0 + v_0 t - \frac{gt^2}{2}$$

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Problem 21, section 2.3

Experiment:

- Ball with mass m thrown with velocity v_0
- Initial height: x₀
- Air resistance: $-\gamma v$

Equation: linear 1st order equation of the form

 $\mathbf{v}' = -\mathbf{mg} - \gamma \mathbf{v}$

Solution:

$$v = -rac{mg}{\gamma} + \left(v_0 + rac{mg}{\gamma}
ight) e^{-rac{\gamma}{m}t}.$$

Maximal height: when v = 0, that is

$$t = rac{m}{\gamma} \ln \left(1 + rac{v_0 \gamma}{mg}
ight) \stackrel{\gamma ext{ small }}{\simeq} rac{v_0}{g}$$

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Problem 22, section 2.3 Experiment:

- Ball with mass m thrown with velocity v_0
- Initial height: x₀
- Air resistance: $-\gamma v^2$

Equation: whenever $v \ge 0$, separable equation of the form

$$\mathbf{v}' = -\mathbf{g} - \frac{\gamma}{m} \, \mathbf{v}^2$$

Solution: recalling $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1}(\frac{x}{a})$, we get

$$\mathbf{v} = \left(rac{mg}{\gamma}
ight)^{1/2} an\left(\mathbf{c} - \left(rac{mg}{\gamma}
ight)^{1/2} t
ight),$$

with $c = an^{-1}((rac{\gamma}{mg})^{1/2}v_0)$

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Existence and uniqueness results: why?

Interest of existence and uniqueness results:

- Mathematical interest
- ❷ Before studying a physical system represented by an equation → know if it admits solutions
- If a solution to en equation is exhibited
 - \hookrightarrow know if it is the only one

Existence and uniqueness: linear case



 \hookrightarrow Solution fails to exists only when p or g are discontinuous

Existence and uniqueness: nonlinear case

Theorem 5.

General nonlinear equation:

$$y' = f(t, y), \qquad y(0) = y_0 \in \mathbb{R}.$$
 (14)

Hypothesis:

•
$$(t_0, y_0) \in R$$
, where $R = (\alpha, \beta) \times (\gamma, \delta)$.

• f and $\frac{\partial f}{\partial y}$ continuous on R.

Conclusion:

One can find h > 0 such that there exists a unique function $y \rightarrow$ satisfying equation (14) on $(t_0 - h, t_0 + h)$.

Remark: If f is continuous, we still get existence of a solution.

Maximal interval in a linear case

Equation considered: back to equation (6), namely

$$t y' + 2y = 4t^2$$
, $y(1) = 2$.

Equivalent form:

$$y' + \frac{2}{t}y = 4t, \qquad y(1) = 2.$$

Application of Theorem 4:

- g(t) = 4t continuous on \mathbb{R}
- $p(t) = \frac{2}{t}$ continuous on $(-\infty, 0) \cup (0, \infty)$ only
- $1 \in (0,\infty)$

We thus get unique solution on $(0,\infty)$

Maximal interval in a linear case (2)

Comparison with explicit solution: We have seen that

$$y'+rac{2}{t}y=4t, \quad y(1)=2 \quad \Longrightarrow \quad y(t)=t^2+rac{1}{t^2}.$$

This is defined on $(0,\infty)$ as predicted by Theorem 4.

Changing initial condition: consider

$$y' + \frac{2}{t}y = 4t, \qquad y(-1) = 2.$$

Then:

• Solution defined on
$$(-\infty, 0)$$

• On
$$(-\infty,0)$$
 we have $y(t)=t^2+rac{1}{t^2}$.

Maximal interval in a linear case (3)

Interval of definition on integral curves:



Comments:

• Interval of definition delimited by asymptotes

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Maximal interval in a nonlinear case Equation considered: back to equation (9), namely

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}$$
, and $y(0) = -1$.

Application of Theorem 5: we have

$$f(x,y) = \frac{3x^2 + 4x + 2}{2(y-1)}, \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2 + 4x + 2}{2(y-1)^2}$$

Therefore:

• There exists rectangle R such that

$$(0,-1) \in R$$

• f and $\frac{\partial f}{\partial y}$ continuous on R

2 According to Theorem 5 there is unique solution on interval (-h, h), with h > 0

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Maximal interval in a nonlinear case (2)

Comparison with explicit solution: We have seen that

$$y = 1 - ((x+2)(x^2+2))^{1/2}$$

Interval of definition: $x \in (-2, \infty)$ \hookrightarrow much larger than predicted by Theorem 5

Changing initial condition: consider y(0) = 1, on line y = 1. Then:

- Theorem 5: nothing about possible solutions
- Oirect integration:
 - We find $y = 1 \pm (x^3 + 2x^2 + 2x)^{1/2}$
 - 2 possible solutions defined for x > 0

Maximal interval in a nonlinear case (3)

Interval of definition on integral curves:



Comments:

• Interval of definition delimited by vertical tangents

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Example with non-uniqueness Equation considered:

$$y'=y^{1/3},$$
 and $y(0)=0.$

Application of Theorem 5: $f(y) = y^{1/3}$. Hence,

- $f : \mathbb{R} \to \mathbb{R}$ continuous on \mathbb{R} , differentiable on \mathbb{R}^*
- Theorem 5: gives existence, not uniqueness

Solving the problem: Separable equation, thus

- General solution: for $c \in \mathbb{R}$, $y = \left[\frac{2}{3}(t+c)\right]^{3/2}$
- With initial condition y(0) = 0,

$$y = \left(\frac{2t}{3}\right)^{3/2}$$

Example with non-uniqueness (2) 3 solutions to the equation:

$$\phi_1(t) = \left(\frac{2t}{3}\right)^{3/2}, \quad \phi_2(t) = -\left(\frac{2t}{3}\right)^{3/2}, \quad \psi(t) = 0.$$

Family of solutions: For any $t_0 \ge 0$,

$$\chi(t) = \chi_{t_0}(t) = \begin{cases} 0 & \text{for } 0 \le t < t_0 \ \pm \left(\frac{2(t-t_0)}{3}\right)^{3/2} & \text{for } t \ge t_0 \end{cases}$$

Integral curves:



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Comparison linear/nonlinear equations

Nice properties of linear equations:

- All solutions expressed in terms of a constant *c*.
- **2** General formula for solution, involving integrations.
- Oiscontinuities/singularities deduced from properties of p and g.

Problems with nonlinear equations:

- Solutions in terms of a constant c

 → possibility of other solutions too.
- In the second second
- Often implicit solution only.
- Singularities depend on particular equation and initial condition.

Bernoulli equations (HW 2-4 27 and 28)

Equation: For $n \ge 2$,

 $y'+p(t)y=q(t)y^n.$

Interest of Bernoulli equations:

Nonlinear equations which can be turned into linear equations.

Alternative form:

$$\frac{y'}{y^n} + \frac{p(t)}{y^{n-1}} = q(t)$$

Change of variable: Set $v = \frac{1}{v^{n-1}}$. We get

 $v' - (n-1)p(t)v = -(n-1)q(t) \longrightarrow$ Linear equation

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Introduction

General form of autonomous equations:

$$\frac{dy}{dt} = f(y) \tag{15}$$

Solving autonomous equations: This is a special case of separable equation

Aim of the section:

- Information on equation (15) with graphical methods
- Applications to population growth models

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Exponential growth

Hypothesis:

Rate of change proportional to value of population

Equation: for $r \in \mathbb{R}$ and $y_0 \geq 0$,

$$\frac{dy}{dt}=ry, \qquad y(0)=y_0$$

Solution:

$$y = y_0 \exp(rt)$$

Image: A matrix

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Exponential growth (2)

Integral curves:



Limitation of model:

• Cannot be valid for large time t.

Logistic growth

Hypothesis:

- Growth rate depends on population
- Related equation: $\frac{dy}{dt} = h(y)y$

Specifications for *h*:

- $h(y) \simeq r > 0$ for small values of y
- $y \mapsto h(y)$ decreases for larger values of y
- h(y) < 0 for large values of y Possibility: h(y) = r - ay

Verhulst equation: for r, K > 0

$$\frac{dy}{dt} = f(y), \quad \text{with} \quad f(y) = r\left(1 - \frac{y}{K}\right)y$$
 (16)

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Logistic growth (2)

Vocabulary:

- r: Intrinsic growth.
- K: Saturation level.
- Solutions to f(y) = 0: critical points.

Equilibrium solutions:

- Defined as $y \equiv \ell$, where ℓ critical point
- Here 2 equilibrium: y = 0 and y = K
- If we have:
 - y(0) = 0 or y(0) = K
 - y satisfies (16),

then y stays constant

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Logistic growth (3)

Graphical interpretation 1:

- Draw $y \mapsto f(y)$.
- Here f parabola, intercepts (0,0) and (K,0).
- We have $\frac{dy}{dt} > 0$ if $y \in (0, K)$
- We have $\frac{dy}{dt} < 0$ if y > K
- Vocabulary: y-axis is called phase line



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Logistic growth (4)

Graphical interpretation 2: behavior of $t \mapsto y(t)$

- Draw line y = 0 and y = K
- Other curves:
 - ▶ Increasing if *y* < *K*
 - Decreasing if y > K
 - Flattens out as $y \to 0$ or $y \to K$
- Curves do not intersect

• Possibility of a convexity/concavity analysis (threshold $\frac{\kappa}{2}$)


Logistic growth (5)

Stable and unstable equilibrium:

- We have seen (phase diagram):
 - ▶ y increases if y < K</p>
 - y decreases if y > K

Thus K stable equilibrium

We have seen (phase diagram):

• y increases as long as y > 0 (and y < K) Thus 0 unstable equilibrium

Remark:

See also the notion of semi-stable equilibrium

Logistic growth (6)

Solving the equation: Equation (16) can be written as

$$\left[\frac{1}{y} + \frac{1/K}{1 - y/K}\right] dy = r \, dt$$

Solution is given by:

$$y = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}}$$

Equilibrium revisited: For all $y_0 > 0$ we have:

$$\lim_{t\to\infty}y(t)=K.$$

Thus K stable equilibrium.

Critical threshold example

Equation considered: for r, T > 0

$$\frac{dy}{dt} = f(y), \quad \text{with} \quad f(y) = -r\left(1 - \frac{y}{T}\right)y$$
 (17)

Critical points:

$$f(y) = 0 \iff y = 0 \text{ or } y = T$$

This corresponds to 2 equilibrium.

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Critical threshold example (2) Graphical interpretation 1:

- Here f parabola, intercepts (0,0) and (T,0).
- We have $\frac{dy}{dt} < 0$ if $y \in (0, T)$
- We have $\frac{dy}{dt} > 0$ if y > T



Conclusion for equilibrium:

- T unstable equilibrium
- 0 stable equilibrium

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Critical threshold example (3)

Graphical interpretation 2: behavior of $t \mapsto y(t)$

- Draw line y = 0 and y = T
- Other curves:
 - Increasing if y > T
 - Decreasing if y < T
 - Flattens out as $y \to 0$
- Curves do not intersect
- Possibility of a convexity/concavity analysis (threshold $\frac{T}{2}$)



Critical threshold example (4)

Solving the equation: Like for equation (16) we get

$$y(t) = \frac{y_0 T}{y_0 + (T - y_0)e^{rt}}$$

Limiting behavior: according to (18),
a) If
$$0 < y_0 < T$$
, we have $\lim_{t\to\infty} y(t) = 0$.
a) If $y_0 > T$, we have $\lim_{t\to t^*} y(t) = \infty$, where

$$t^* = \frac{1}{r} \ln \left(\frac{y_0}{y_0 - T} \right)$$

This behavior could not be inferred from graphic representation.

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HW 12, Section 2.5

Equation:

$$y'=y^2(4-y^2)\equiv f(y)$$

Graph of f: Intercepts for $x \in \{-2, 0, 2\}$



Equilibrium:

- \bullet -2 and 2 are stable
- 0 is unstable

Outline

Linear equations

- 2 Separable equations
- 3 Homogeneous equations
- 4 Modeling with first order differential equations
- 5 Differences between linear and nonlinear equations
- 6 Autonomous equations
- Exact equations and integrating factors
 - 8 Numerical approximation: Euler's method

Example of exact equation

Equation considered:

$$2x + y^2 + 2xyy' = 0 (19)$$

Image: A matrix

Remark: equation (19) neither linear nor separable

Additional function: Set $\psi(x, y) = x^2 + xy^2$. Then:

$$\frac{\partial \psi}{\partial x} = 2x + y^2$$
, and $\frac{\partial \psi}{\partial y} = xy$.

Example of exact equation (2) Expression of (19) in terms of ψ : we have

(19)
$$\iff \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0$$

Solving the equation: We assume y = y(x). Then

(19)
$$\iff \frac{d\psi}{dx}(x,y) = 0 \iff \psi(x,y) = c,$$

for a constant $c \in \mathbb{R}$.

Conclusion: equation solved under implicit form

$$x^2 + xy^2 = c.$$

General exact equation

Proposition 6.

Equation considered:

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0.$$
 (20)

Hypothesis: there exists $\psi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$rac{\partial \psi}{\partial x} = M(x,y) \quad ext{and} \quad rac{\partial \psi}{\partial y} = N(x,y).$$

Conclusion: general solution to (20) is given by:

 $\psi(x,y) = c$, with $c \in \mathbb{R}$,

provided this relation defines y = y(x) implicitely.

Criterion for exact equations

Notation: For
$$f : \mathbb{R}^2 \to \mathbb{R}$$
, set $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$

Theorem 7.

Let:

•
$$R = \{(x, y); \alpha < x < \beta, \text{ and } \gamma < y < \delta\}.$$

• M, N, M_y , N_x continuous on R.

Then there exists ψ such that:

$$\psi_x = M$$
, and $\psi_y = N$ on R ,

if and only if M and N satisfy:

$$M_{y} = N_{x}$$
 on R

Computation of function ψ

Aim: If $M_y = N_x$, find ψ such that $\psi_x = M$ and $\psi_y = N$.

Recipe in order to get ψ :

1 Write ψ as antiderivative of M with respect to x:

$$\psi(x,y) = a(x,y) + h(y)$$
, where $a(x,y) = \int M(x,y) dx$

Get an equation for h by differentiating with respect to y:

$$h'(y) = N(x, y) - a_y(x, y)$$

Finally we get:

$$\psi(x,y) = a(x,y) + h(y).$$

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Computation of ψ : example

Equation considered:

$$\underbrace{y\cos(x) + 2xe^{y}}_{M} + \underbrace{\left(\sin(x) + x^{2}e^{y} - 1\right)}_{N} y' = 0.$$
(21)

Step 1: verify that $M_y = N_x$ on \mathbb{R}^2 .

Step 2: compute ψ according to recipe. We find

$$\psi(x,y) = y\sin(x) + x^2e^y - y$$

Solution to equation (21):

$$y\sin(x)+x^2e^y-y=c.$$

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Computation of ψ : counter-example

Equation considered:

$$\underbrace{3xy+y^2}_{M} + \underbrace{\left(x^2 + xy\right)}_{N} y' = 0.$$
(22)

Step 1: verify that $M_y \neq N_x$.

Step 2: compute ψ according to recipe. We find

$$h'(y) = -\frac{x^2}{2} - xy \quad \longrightarrow \quad \text{still depends on } x!$$

Conclusion: Condition $M_y = N_x$ necessary.

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Solving an exact equation: example

Equation considered:

$$\underbrace{2x - y}_{M} + \underbrace{(2y - x)}_{N} y' = 0, \quad y(1) = 3.$$
(23)

Step 1: verify that $M_y = N_x$ on \mathbb{R}^2 .

Step 2: compute ψ according to recipe. We find

$$\psi(\mathbf{x},\mathbf{y})=\mathbf{x}^2-\mathbf{x}\mathbf{y}+\mathbf{y}^2.$$

Solution to equation (23): recalling y(1) = 3, we get

$$x^2 - xy + y^2 = 7.$$

Image: A matrix

Solving an exact equation: example (2) Expressing y in terms of x: we get

$$y=\frac{x}{2}\pm\left(7-\frac{3x^2}{4}\right)^{1/2}$$

.

Image: A matrix

Recalling y(1) = 3, we end up with:

$$y = \frac{x}{2} + \left(7 - \frac{3x^2}{4}\right)^{1/2}.$$

Interval of definition:

$$x \in \left(-2\sqrt{\frac{7}{4}}; 2\sqrt{\frac{7}{4}}\right) \simeq (-3.05; 3.05)$$

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HW 16, Section 2.6

Equation considered:

$$\underbrace{ye^{2xy} + x}_{M} + \underbrace{\left(bxe^{2xy}\right)}_{N} y' = 0, \quad b \in \mathbb{R}.$$
 (24)

Step 1: We have $M_y = N_x$ iff b = 1.

Step 2: When b = 1, compute ψ according to recipe. We find

$$\psi(\mathbf{x},\mathbf{y}) = \frac{1}{2} \left(e^{2xy} + x^2 \right).$$

Solution to equation (24):, we get

$$e^{2xy}+x^2=c.$$

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Approximations of first order equations: why?

Generic first order equation: back to equation (14), that is

$$\frac{dy}{dt} = f(t, y), \qquad y(t_0) = y_0$$

General facts about (14):

() If f is continuous, equation can be solved in neighborhood of t_0 .

Solution y cannot be computed explicitly.

Conclusion:

We need approximations in order to understand behavior of y.

Starting from direction fields

Equation considered:

$$\frac{dy}{dt} = 3 - 2t - 0.5y$$

Direction fields for (25):



Basic idea: Linking the tangent lines on the graph \hookrightarrow we get an approximation of solution.

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Questions about approximation methods

Basic issues:

- Method to link tangent lines.
- O we get an approximation of real solution?
- Solution Rate of convergence for approximation.

First steps of approximation

Equation considered: equation (14), that is

$$\frac{dy}{dt}=f(t,y), \qquad y(t_0)=y_0.$$

Approximation near t_0 :

- Solution passes through (t_0, y_0)
- Slope at (t_0, y_0) is $f(t_0, y_0)$
- Consider t₁ close to t₀

Then linear approximation of $y(t_1)$ is given by:

$$y_1 = y_0 + f(t_0, y_0) (t_1 - t_0).$$

First steps of approximation (2)

Approximation near t_1 :

- Solution passes through $(t_1, y(t_1))$
- Problem: we don't know the exact value of $y(t_1)$
- We approximate $y(t_1)$ by y_1
- Approximate slope at (t_1, y_1) is given by $f(t_1, y_1)$
- Consider t_2 close to t_1

Then linear approximation of $y(t_2)$ is given by:

$$y_2 = y_1 + f(t_1, y_1) (t_2 - t_1).$$

Euler scheme

Proposition 8.

Equation considered: equation (14), that is

$$\frac{dy}{dt}=f(t,y), \qquad y(t_0)=y_0.$$

Hypothesis: constant step in time,

$$t_{n+1}-t_n=h.$$

Notation: $f_n = f(t_n, y_n)$, $\hat{y} = \text{Euler's approximation}$.

Conclusion: Recursive formula for Euler's scheme,

$$y_{n+1} = y_n + f_n h$$

 $\hat{y}(t) = y_n + f_n (t - t_n), \text{ for } t \in [t_0 + nh, t_0 + (n+1)h)$

Example of Euler scheme

Equation considered: back to equation (25), that is

$$\frac{dy}{dt} = \overbrace{3-2t-0.5y}^{f(t,y)}, \qquad y(0) = 1$$

Exact solution: we find

$$y = \phi(t) = 14 - 4t - 13\exp\left(-\frac{t}{2}\right)$$

Euler scheme, step 1: with h = 0.2 we have

•
$$f_0 = f(0, 1) = 2.5$$

• $\hat{y}(t) = 1 + 2.5t$ for $t \in (0, 0.2)$
• $y_1 = 1.5$

Example of Euler scheme (2)

Euler scheme, step 2: with h = 0.2 we have

•
$$f_1 = f(0.2, 1.5) = 1.85$$

• $\hat{v}(t) = 1.5 \pm 1.85(t = 0.2)$ for t

$$\hat{arphi}(t) = 1.5 + 1.85(t-0.2)$$
 for $t \in (0.2, 0.4)$

•
$$y_2 = 1.87$$

Numerical results:

t	Exact	Euler with $h = 0.2$	Tangent line
0.0 0.2 0.4 0.6 0.8 1.0	1.00000 1.43711 1.75650 1.96936 2.08584 2.11510	1.00000 1.50000 1.87000 2.12300 2.27070 2.32363	y = 1 + 2.5t y = 1.13 + 1.85t y = 1.364 + 1.265t y = 1.6799 + 0.7385t y = 2.05898 + 0.26465t

Remark: about 10% error at t = 1

 \hookrightarrow Approximation not accurate enough, smaller *h* needed.

Example of Euler scheme (3)

Numerical results with varying *h*:

t	Exact	h = 0.1	h = 0.05	h = 0.025	h = 0.01
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
1.0	2.1151	2.2164	2.1651	2.1399	2.1250
2.0	1.2176	1.3397	1.2780	1.2476	1.2295
3.0	-0.9007	-0.7903	-0.8459	-0.8734	-0.8898
4.0	-3.7594	-3.6707	-3.7152	-3.7373	-3.7506
5.0	-7.0671	-7.0003	-7.0337	-7.0504	-7.0604

Comments:

- Error decreases with time step.
- Error could possibly be of order *h*.

Example of Euler scheme (4)

Graphical comparison for h = 0.2:



Remark: $\hat{y} \ge y$ \hookrightarrow Due to the fact that y concave \implies tangent above graph Euler scheme for fast increasing solution

Equation considered:

$$\frac{dy}{dt} = 4 - t + 2y, \qquad y(0) = 1$$
 (26)

Exact solution: we find

$$y = \phi(t) = -\frac{7}{4} - \frac{t}{2} + \frac{11}{4} \exp(2t)$$

Thus exponential growth for y.

Euler scheme for fast increasing solution (2)

Numerical results with varying *h*:

t	Exact	h = 0.1	h = 0.05	h = 0.025	h = 0.01
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
1.0	19.06990	15.77728	17.25062	18.10997	18.67278
2.0	149.3949	104.6784	123.7130	135.5440	143.5835
3.0	1109.179	652.5349	837.0745	959.2580	1045.395
4.0	8197.884	4042.122	5633.351	6755.175	7575.577
5.0	60573.53	25026.95	37897.43	47555.35	54881.32

Comments:

- Error still decreases with h
- Worse performance than for (25).

Explanation of difference:

- For (25) all solutions converge to φ(t) = 14 − 14t
 → successive errors are not propagating
- For (26) solutions diverge exponentially
 - \hookrightarrow strong propagation of successive errors