The Laplace transform

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Differential equations - MA 266

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Outline

1. Definition of Laplace transform
2. Solution of initial value problems
3. Step functions
4. Differential equations with discontinuous forcing functions
5. Impulse functions
6. The convolution integral

Interest of Laplace transform

Laplace:

- 1749-1827, lived in France
- Mostly mathematician
- Called the French Newton
- Contributions in
  - Mathematical physics
  - Analysis, partial differential equations
  - Celestial mechanics
  - Probability (central limit theorem)

General interest of Laplace transform:

In many branches of mathematics (analysis - geometry - probability)

Interest for differential equations:

Deal with impulsive (discontinuous) forcing terms.

Outline

Improper integrals

Definition:

$$\int_a^{\infty} f(t)\,dt = \lim_{A\to\infty} \int_a^{A} f(t)\,dt. \tag{1}$$

Vocabulary:

- If limit exists in 1: convergent integral.
- Otherwise: divergent integral.

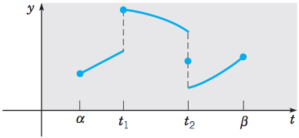
Examples:

- $\int_0^{\infty} e^{ct} dt$ convergent iff $c < 0$.
- $\int_1^{\infty} t^{-p}$ convergent iff $p > 1$.

Piecewise continuous function

Definition: f piecewise continuous on $I = [\alpha, \beta]$
$\hookrightarrow$ if there exists $\alpha = t_1 < \cdots < t_n = \beta$ such that

- f continuous on each (t_i, t_{i+1}).
- f admits left and right limits at each t_i.



Integral of piecewise continuous function:

$$\int_{\alpha}^{\beta} f(t)\, dt = \sum_{i=1}^{n-1} \int_{t_i}^{t_{i+1}} f(t)\, dt.$$

Laplace transform

Definition: Let $f : \mathbb{R}_+ \to \mathbb{R}$. Then

$$\mathcal{L}f(s) = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Possible strategy to solve a differential equation:

1. Transform diff. equation into algebraic problem in s variable.
2. Solve algebraic problem and find F.
3. Invert Laplace transform and find f.

Existence of Laplace transform

Theorem 1.

Hypothesis:

- f piecewise continuous on $[0, A]$ for each $A > 0$.
- $|f(t)| \leq K e^{at}$ for $K \geq 0$ and $a \in \mathbb{R}$.

Conclusion:
$\mathcal{L}f(s)$ exists for $s > a$.

Vocabulary: f satisfying $|f(t)| \leq K e^{at}$
$\hookrightarrow$ Called function of exponential order.

Table of Laplace transforms

Function f	Laplace transform F	Domain of F
$\mathbf{1}$	$\frac{1}{s}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$\mathbf{1}_{[0,1)}(t) + k\,\mathbf{1}_{(t=1)}$	$\frac{1-e^{-s}}{s}$	$s > 0$
$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p,\ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$s > \|a\|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$	$s > \|a\|$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$	$s > a$

Table of Laplace transforms (2)

Function f	Laplace transform F	Domain of F
$t^n e^{at}$, $n \in \mathbb{N}$	$\frac{n!}{(s-a)^n}$	$s > a$
$u_c(t)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$	
$e^{ct} f(t)$	$F(s-c)$	
$f(ct)$, $c > 0$	$\frac{1}{c} F(\frac{s}{c})$	
$\int_0^t f(t-\tau) g(\tau)$	$F(s) G(s)$	
$\delta(t-c)$	e^{-cs}	
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	
$(-t)^n f(s)$	$F^{(n)}(s)$	

Linearity of Laplace transform

Example of function f:

$$f(t) = 5\,e^{-2t} - 3\,\sin(4t).$$

Laplace transform by linearity: we find

$$\begin{aligned} \mathcal{L}f(s) &= 5\left[\mathcal{L}(e^{-2t})\right](s) - 3\left[\mathcal{L}(\sin(4t)\right](s) \\ &= \frac{5}{s+2} - \frac{12}{s^2+16}. \end{aligned}$$

Outline

Relation between $\mathcal{L}f$ and $\mathcal{L}f'$

Theorem 2.

Hypothesis:

1. f continuous, f' piecewise continuous on $[0, A]$
   $\hookrightarrow$ for each $A > 0$.
2. $|f(t)| \leq Ke^{at}$ for $K, a \geq 0$.

Conclusion: $\mathcal{L}f'$ exists and

$$\mathcal{L}f'(s) = s\mathcal{L}f(s) - f(0)$$

Proof: By integration by parts

$$\int_0^A e^{-st} f'(t)\, dt = \left[e^{-st} f(t)\right]_0^A + s \int_0^A e^{-st} f(t)\, dt$$

Laplace transform of higher order derivatives

Theorem 3.

Hypothesis:

1. $f, \ldots, f^{(n-1)}$ cont., $f^{(n)}$ piecewise cont. on $[0, A]$
   $\hookrightarrow$ for each $A > 0$.
2. $|f(t)|, \ldots, |f^{(n-1)}| \leq K e^{at}$ for $K, a \geq 0$.

Conclusion: $\mathcal{L} f^{(n)}$ exists and

$$\mathcal{L} f^{(n)}(s) = s^n \mathcal{L} f(s) - s^{n-1} f(0) - \cdots - s\, f^{(n-2)}(0) - f^{(n-1)}(0)$$

Application to differential equations

Equation:

$$y'' - y' - 2y = 0, \qquad y(0) = 1,\ y'(0) = 0$$

Solution by usual methods:

$$y = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$$

Application to differential equations (2)

Expression for Laplace transform: Set $Y \equiv \mathcal{L}(y)$. Then:

$$\left(s^2 - s - 2\right) Y(s) + (1-s)y(0) - y'(0) = 0$$

Plugging initial condition:

$$\left(s^2 - s - 2\right) Y(s) + (1-s) = 0$$

and thus

$$Y(s) = \frac{s-1}{s^2 - s - 2} = \frac{1/3}{s-2} + \frac{2/3}{s+1}$$

Inverting Laplace transform: we find

$$y = \frac{2}{3}e^{-t} + \frac{1}{3}e^{2t}$$

Generalization

Proposition 4.

Equation:

$$ay'' + by' + cy = f(t)$$

Hypothesis:

- f and y satisfy assumptions of Theorem 3.

Conclusion: setting $Y = \mathcal{L}(y)$ and $F = \mathcal{L}(f)$ we have

$$Y(s) = \frac{(as+b)y(0) + ay'(0)}{as^2 + bs + c} + \frac{F(s)}{as^2 + bs + c}$$

Features of Laplace transform

1. Made to solve equations we cannot solve with previous methods
2. Differential equation $\longrightarrow$ algebraic equation
3. Task of computing c_1, c_2 (more or less) avoided
4. Homog. and non homog. cases treated in the same way
5. Second and higher order cases treated in the same way
6. Characteristic polynomial appear in denominator
7. Main difficulty: inverting Laplace transform $\longrightarrow$ see table
8. Physical applications: springs, electrical circuits
   $\hookrightarrow$ Impulse (or step) functions needed

Non homogeneous example

Equation:

$$y'' + y = \sin(2t), \qquad y(0) = 2,\ y'(0) = 1$$

Equation with Laplace transform: setting $Y = \mathcal{L}(y)$,

$$s^2 Y(s) - sy(0) - y'(0) + Y(s) = \frac{2}{s^2 + 4}$$

Expression for Y: with initial values,

$$Y(s) = \frac{2s^3 + s^2 + 8s + 6}{(s^2+1)(s^2+4)} \overset{\text{computation}}{=} \frac{2s}{s^2+1} + \frac{5/3}{s^2+1} - \frac{2/3}{s^2+4}$$

Expression for y: Inverting Laplace transform,

$$y = 2\cos(t) + \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)$$

Outline

Heaviside

Heaviside:

- 1850-1925, lived in England
- Electrical Engineer
- Self taught
- Contributions in
  - Differential equations
  - Functional calculus
  - Electromagnetism
  - Practical telephone transmission

A quote by Heaviside:

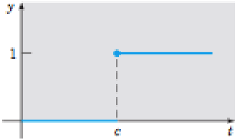
- Mathematics is an experimental science, and definitions do not come first, but later on

Heaviside function u_c

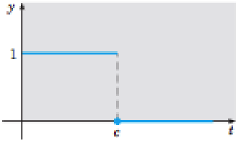
Definition: we set

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

Graph of u_c:



Negative step: for $y = 1 - u_c$ the graph is



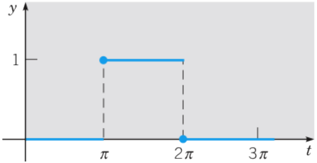
Sum of 2 Heaviside

Example: Consider

$$h(t) = u_\pi(t) - u_{2\pi}(t).$$

Then

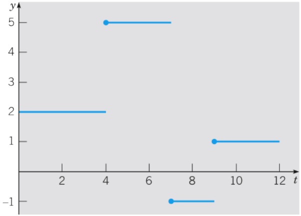
$$h(t) = \begin{cases} 0, & 0 \le t < \pi \\ 1, & \pi \le t < 2\pi \\ 0 & 2\pi \le t < \infty \end{cases}$$



Piecewise constant function as sum of Heaviside

Piecewise constant function: Consider

$$f(t) = \begin{cases} 2, & 0 \le t < 4 \\ 5, & 4 \le t < 7 \\ -1 & 7 \le t < 9 \\ 1 & 9 \le t < \infty \end{cases}$$



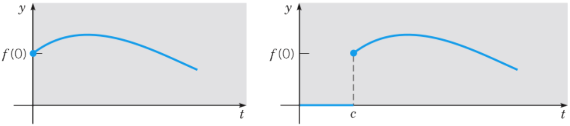
Expression in terms of Heaviside functions:

$$f = 2 + 3u_4 - 6u_7 + 2u_9$$

Shifted function

Shifted function: For a function f we set

$$g(t) = \begin{cases} 0, & t < c \\ f(t-c), & t \geq c \end{cases}$$



Expression in terms of Heaviside:

$$g(t) = u_c(t)\, f(t-c)$$

Laplace transform of a shifted function

Theorem 5.

Hypothesis:

1. $F = \mathcal{L}(f)$ exists for $s > a \geq 0$
2. $c \geq 0$

Conclusion:

$$\mathcal{L}\left(u_c(t) f(t-c)\right) = e^{-cs} F(s)$$

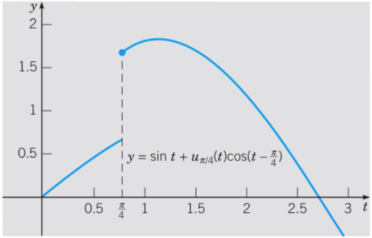
Corollary: we have

$$\mathcal{L}\left(u_c(t) f(t-c)\right) = \frac{e^{-cs}}{s}$$

Computation with shift

Function f: Consider

$$f(t) = \begin{cases} \sin(t), & 0 \le t < \frac{\pi}{4} \\ \sin(t) + \cos\left(t - \frac{\pi}{4}\right), & t \ge \frac{\pi}{4} \end{cases}$$



Computation with shift (2)

Other expression for f:

$$f = \sin + g, \quad \text{where} \quad g(t) = u_{\frac{\pi}{4}}(t)\,\cos\left(t - \frac{\pi}{4}\right)$$

Laplace transform of f:

$$\begin{aligned}\mathcal{L}(f(t)) &= \mathcal{L}(\sin(t)) + \mathcal{L}\left(u_{\frac{\pi}{4}}(t)\,\cos\left(t - \frac{\pi}{4}\right)\right)\\ &= \frac{1}{s^2+1} + e^{-\frac{\pi s}{4}}\frac{s}{s^2+1}\end{aligned}$$

Example of inverse Laplace transform

Function F:

$$F(s) = \frac{1 - e^{-2s}}{s^2} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}$$

Inverse Laplace transform: we find

$$\begin{aligned} f(t) = \mathcal{L}^{-1}(F(s)) &= \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right) \\ &= t - u_2(t)\,(t-2) \end{aligned}$$

Other expression for f:

$$f(t) = \begin{cases} t, & 0 \le t < 2 \\ 2, & t \ge 2 \end{cases}$$

Multiplication by exponential and Laplace

Theorem 6.

Hypothesis:

1. $F = \mathcal{L}(f)$ exists for $s > a \geq 0$
2. $c \in \mathbb{R}$

Conclusion:

$$\mathcal{L}\left(e^{ct} f(t)\right) = F(s - c)$$

Example of inverse Laplace transform

Function G:

$$G(s) = \frac{1}{s^2 - 4s + 5}$$

Other expression for G:

$$G(s) = \frac{1}{(s-2)^2 + 1}$$

Inverse Laplace transform:

$$\mathcal{L}^{-1}\left(G(s)\right) = e^{2t}\sin(t)$$

Outline

Example of equation with discontinuity

Differential equation:

$$2y'' + y' + 2y = g, \quad \text{with} \quad g = u_5 - u_{20},\ y(0) = 0,\ y'(0) = 0$$

Equation for Laplace transform:

$$Y(s) = \left(e^{-5s} - e^{-20s}\right) H(s), \quad \text{with} \quad H(s) = \frac{1}{s\,(2s^2 + s + 2)}$$

Decomposition of H:

$$H(s) = \frac{1/2}{s} - \frac{s + 1/2}{2s^2 + s + 2}$$
$$= \frac{1/2}{s} - \frac{1}{2}\frac{s + 1/4}{(s + 1/4)^2 + (\sqrt{15}/4)^2} + \frac{1}{\sqrt{15}}\frac{\sqrt{15}/4}{(s + 1/4)^2 + (\sqrt{15}/4)^2}$$

Example of equation with discontinuity (2)

Inverting Laplace transform: we find

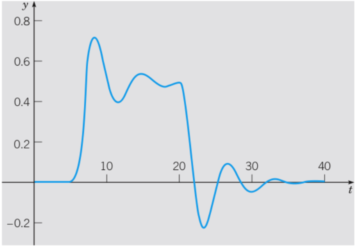
$$h(t) = \mathcal{L}^{-1}(H) = \frac{1}{2} - \frac{1}{2} e^{-\frac{t}{4}} \cos\left(\frac{\sqrt{15}t}{4}\right) + \frac{1}{\sqrt{15}} e^{-\frac{t}{4}} \sin\left(\frac{\sqrt{15}t}{4}\right)$$

Solution of equation:

$$y = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

Example of equation with discontinuity (3)

Graph of y: 3 regimes for $t \in [0,5)$, $t \in [5,20)$ and $t > 20$



Details on 3 regimes:

1. For $t \in [0,5)$: solution of $2y'' + y' + 2y = 0 \longrightarrow y \equiv 0$
2. For $t \in [5,20)$: sol. of $2y'' + y' + 2y = 1 \longrightarrow y \equiv \frac{1}{2} +$ transient
3. For $t > 20$: sol. of $2y'' + y' + 2y = 0 \longrightarrow y \equiv$ Damped vib.
4. Discontinuities of $g \longrightarrow$ Discontinuities for y''

Ramp example

Forcing term:

$$g(t) = \begin{cases} 0, & 0 \le t < 5 \\ \frac{t-5}{5}, & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$$

Other expression for g:

$$g(t) = \frac{1}{5}\left[u_5(t)(t-5) - u_{10}(t)(t-10)\right].$$

Equation:

$$y'' + 4y = g(t), \quad y(0) = 0, \; y'(0) = 0$$

Ramp example (2)

Equation for Laplace transform:

$$Y(s) = \frac{(e^{-5s} - e^{-10s})H(s)}{5}, \quad \text{where} \quad H(s) = \frac{1}{s^2(s^2+4)}$$

First expression for y: if $h = \mathcal{L}^{-1}(H)$ we have

$$y = \frac{1}{5}\left[h(t-5)u_5(t) - h(t-10)u_{10}(t)\right]$$

Decomposition of H:

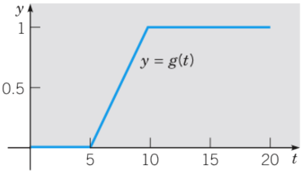
$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

Computation of h:

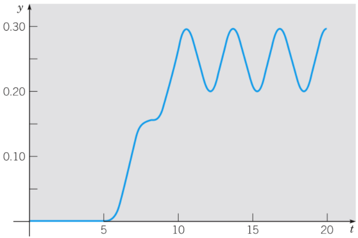
$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

Ramp example (3)

Graph of g:



Graph of y:



Outline

Impulse model

Equation:

$$ay'' + by' + cy = g$$

Desired external force: For τ small, we want a function g such that

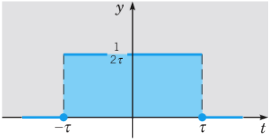
$$g(t) = \begin{cases} \text{Large value}, & t \in (t_0 - \tau, t_0 + \tau) \\ 0, & t \notin (t_0 - \tau, t_0 + \tau) \end{cases}$$

Strength of g:

$$I(\tau) = \int_{t_0-\tau}^{t_0+\tau} g(t)\, dt = \int_{-\infty}^{\infty} g(t)\, dt.$$

Indicator functions

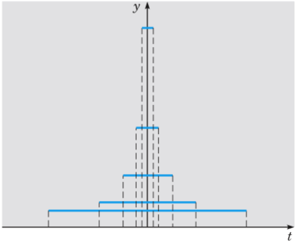
An indicator centered at 0: We consider



$$g(t) = d_\tau(t) = \begin{cases} \frac{1}{2\tau}, & t \in (-\tau, \tau) \\ 0, & t \notin (-\tau, \tau) \end{cases}$$

Idealized version of impulse: limit of d_τ as $\tau \searrow 0$

$\hookrightarrow$ Called Dirac delta function



Definition of δ

Formal definition: for $t_0 \in \mathbb{R}$

$$\delta(t - t_0) = 0, \quad t \neq t_0, \quad \text{and} \quad \int_{\mathbb{R}} \delta(t - t_0)\, dt = 1$$

Laplace transform of indicators: we have

$$\mathcal{L}\left(d_\tau(t - t_0)\right) = e^{-t_0 s} \mathcal{L}\left(d_\tau(t)\right) = e^{-t_0 s} \frac{\sinh(\tau s)}{\tau s}$$

Definition of δ (2)

Laplace transform of δ:
we set $\mathcal{L}(\delta(t-t_0)) = \lim_{\tau\to 0^+} \mathcal{L}(d_\tau(t-t_0))$, which gives

$$\begin{aligned} \mathcal{L}\left(\delta(t-t_0)\right) &= e^{-t_0 s} \\ \mathcal{L}\left(\delta(t)\right) &= \mathbf{1} \end{aligned}$$

Dirac function as a distribution: For a continuous f

$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

Equation with Dirac forcing

Equation:

$$2y'' + y' + 2y = \delta(t-5), \qquad y(0) = 0,\ y'(0) = 0$$

Equation for Y:

$$(2s^2 + s + 2)Y = e^{-5s}$$

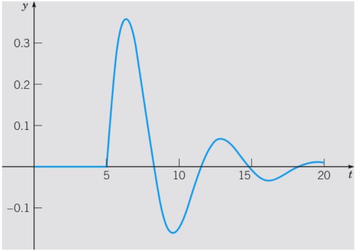
Thus

$$Y(s) = \frac{e^{-5s}}{2} H(s), \quad \text{with} \quad H(s) = \frac{1}{\left(s + \frac{1}{4}\right)^2 + \frac{15}{16}}$$

Equation with Dirac forcing (2)

Solution y:

$$y(t) = \frac{1}{2}\, u_5(t) \,\exp\left(-\frac{t-5}{4}\right)\, \sin\left(\frac{\sqrt{15}}{4}(t-5)\right)$$



Remark:

- Dirac function is an idealized impulse
- Simplifies computations in Laplace mode

Outline

Convolution product

Discrete convolution: For filter $(a)_{n\geq 0}$ and sequence $(v)_{n\geq 0}$

$$w_n = [a * v]_n = \sum_{j=0}^{n} a_{n-j}\, v_j$$

Convolution integral: For f, g functions,

$$h(t) \equiv [f * g](t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$$

Uses of convolution: When system output depends on the past

- Population dynamics
- Signal transmission

Rules for convolutions

Rules to follow:

$$f * g = g * f \qquad \text{Commutative law}$$

$$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \text{Distributive law}$$

$$f * (g * h) = (f * g) * h \qquad \text{Associative law}$$

$$f * 0 = 0 \qquad \text{Absorbing state}$$

Rules **not** to follow:

- $f * 1 \neq f$ in general.
- $f * f$ not positive in general.

Laplace transform and convolution

Theorem 7.

Consider:

- f, g functions with Laplace transform F and G
- $h(t) = [f * g](t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

Then

$$H(s) = F(s)G(s)$$

Proof:

Changing order of integration for integration in the plane.

Equation expressed with convolution

Equation:

$$y'' + 4y = g(t), \quad y(0) = 3, \; y'(0) = 1$$

Expression for Laplace transform:

$$\begin{aligned} Y(s) &= \frac{3s - 1}{s^2 + 4} + \frac{G(s)}{s^2 + 4} \\ &= 3\frac{s}{s^2 + 4} - \frac{1}{2}\frac{2}{s^2 + 4} + \frac{G(s)}{s^2 + 4} \end{aligned}$$

Solution of equation:

$$y = 3\cos(2t) - \frac{1}{2}\sin(2t) + \frac{1}{2}\int_0^t \sin\left(2(t - \tau)\right)\, g(\tau)\, d\tau$$