Integrating factor

Eq. $t^2 y' + 3ty = 6t$ $\qquad y(1)=0$

$$\Leftrightarrow \quad y' + \frac{3}{t} y = \frac{6}{t}$$

Integrating factor: $\mu = t^3$. Then

Eq $\Leftrightarrow (t^3 y)' = 6t^2 \Leftrightarrow t^3 y = 2t^3 + c \Leftrightarrow y = 2 + \frac{c}{t^3}$

With initial condition, $c = -2$. Thus

$$y(t) = 2 - \frac{2}{t^3} \quad , \; y'(1) = 6$$

Separable equation

$$y' = \frac{e^{-x} - e^{x}}{3+4y}$$

$$\Leftrightarrow (3+4y)\,dy = (e^{-x} - e^{x})\,dx$$

$$\Leftrightarrow 3y + 2y^2 = (-e^{-x} - e^{x}) + c$$

$$\Leftrightarrow 3y + 2y^2 + e^{-x} + e^{x} = c$$

If $x = 0$ then $y = 1$

$$\Rightarrow c = 7$$

Unique solution:

$$3y + 2y^2 + e^{-x} + e^{x} = 7$$

Homogeneous equation. For $x > 0$,

$$y' = \frac{3y^2/x^2 + 1}{2y/x}. \quad \text{Let} \quad y = xv.$$

We get

$$x v' = \frac{v^2+1}{2v}$$

which is a separable equation. Integrating we obtain

$$\ln(v^2+1) = \ln(x) + c_1$$

$$\Leftrightarrow \quad v^2 = c_2 x - 1$$

$$\Leftrightarrow \quad y^2 = c_2 x^3 - x^2$$

If $x = 1$ then $y = 2$

$$\Rightarrow \quad 4 = c_2 - 1, \quad c_2 = 5$$

Solution: $y^2 = 5x^3 - x^2$

Tank

Let Q = quantity of salt. Then

(i) $Q_{in} = 3$

(ii) $Q_{out} = 2 \dfrac{Q(t)}{V(t)}$, with $V(t) = 200 + t$

Then $\dfrac{dQ}{dt} = Q_{in} - Q_{out} = 3 - \dfrac{2Q}{200+t}$

Initial condition: $Q(0) = 100$

Solution

$$Q' + \frac{2}{200+t} Q = 3$$

Integrating factor: μ s.t.

$$\frac{\mu'}{\mu} = \frac{2}{200+t}, \text{ take } \mu(t) = (200+t)^2$$

Then

$$\left[(200+t)^2 Q\right]' = 3(200+t)^2$$

$$\Leftrightarrow \quad (200+t)^2 Q = (200+t)^3 + c$$

$$\Leftrightarrow \quad Q = 200 + t + \frac{c}{200+t}$$

Determine c with initial condition

Equilibrium for autonomous equations

$y' = f(y)$, with $f(y) = y^2(4-y^2)$

Then

(i) $f(y) = 0 \Leftrightarrow y \in \{-2, 0, 2\}$

(ii) $f(y) \geq 0$ iff $y \in [-2, 2]$

Thus

- -2 unstable
- 0 semi-stable
- 2 stable

Exact equation.

Eq. $\underbrace{(2xy^2+2y)}_{M}+\underbrace{(2x^2y+2x)}_{N}y'$

One verifies that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. Hence the equation is exact.

Next $\varphi(x,y)=\int M\,dx = x^2y^2+2xy+h(y)$

and $\frac{\partial \varphi}{\partial y}=N \Leftrightarrow 2x^2y+2x+h'(y)=2x^2y+2x$

$\Leftrightarrow h'(y)=0.$

The solution is thus of the form

$x^2y^2+2xy=c.$

If $y(1)=1$, we get $c=3$. Hence

$x^2y^2+2xy=3$

Euler's method

Eq. $y' = \overbrace{3+t-y}^{f(t,y)}$, $y(0)=1$, $h=0.5$

Recall that for Euler's method we have

$$y_{k+1} = y_k + f(t_k, y_k)\, h$$

Here $y_0 = 1$, and we wish to find y_2. We get

$$y_1 = y_0 + f(0, y_0)\, h = 1 + 2 \times 0.5 = 2$$

$$y_2 = y_1 + f(0.5, 2) \times 0.5 = 2 + (3 + 0.5 - 2) \times 0.5$$

Hence

$$y_2 = 2.75$$

Undamped vibration

$$y'' + 4y = 0 \qquad y(0) = 2 \qquad y'(0) = 4$$

The unique solution is given by

$$y = 2\cos(2t) + 2\sin(2t)$$

Then

$$y = R\cos(2t - \delta)$$

with

$$R = (c_1^2 + c_2^2)^{\frac{1}{2}} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\delta = \frac{c_2}{c_1} = 1 \implies \delta = \frac{\pi}{4}$$

Therefore

$$y = 2\sqrt{2}\cos\left(2t - \frac{\pi}{4}\right)$$

Undetermined coefficients

$$y^{(4)} - 2y^{(3)} - 3y^{(2)} = 4t^2 - 3 + e^{3t} + e^{t} - 6\sin(t)$$

For the homogeneous equation, the characteristic polynomial is given by

$$P(r) = r^4 - 2r^3 - 3r^2 = r^2(r^2 - 2r - 3) = r^2(r-3)(r+1)$$

The general solution of the homogeneous problem is thus

$$y = c_1 t + c_2 + c_3 e^{3t} + c_4 e^{-t}$$

The particular solution is obtained as follows, taking into account the fact that 0 is double root and 3 is a root:

$$Y = t^2(d_1 t^2 + d_2 t + d_3) + d_4 t e^{3t} + d_5 e^{t}$$

$$+ d_6 \cos(t) + d_7 \sin(t)$$

Variation of parameters

Eq. $y'' + 2y' + y = \overbrace{\frac{e^{-t}}{t}}^{\equiv g(t)}$

Homogeneous eq: $y'' + 2y' + y = 0$

$\Rightarrow \quad y_1 = e^{-t} \qquad y_2 = t e^{-t}$

and $W[y_1, y_2](t) = \begin{vmatrix} e^{-t} & t e^{-t} \\ -e^{-t} & (-t+1)e^{-t} \end{vmatrix} = e^{-2t}$

The particular solution is given by

$$Y = -y_1 \int \frac{y_2\, g}{W[y_1, y_2]} + y_2 \int \frac{y_1\, g}{W[y_1, y_2]}$$

$$= -e^{-t} \int \frac{t e^{-t}\, t^{-1} e^{-t}}{e^{-2t}}\, dt$$

$$+ t e^{-t} \int \frac{e^{-t}\, t^{-1} e^{-t}}{e^{-2t}}\, dt$$

$$= -t e^{-t} + t \ln(t) e^{-t}$$

Rmk: $-t e^{-t}$ solves the homogeneous equation.

Thus we end up with

$$Y = t \ln(t) e^{-t}$$

Interval of defition

Eq. $(4-t^2)y^{(3)} + 2ty = 3\ln(t)$

$y(1) = -3 \qquad y'(1) = \pi \qquad y''(1) = 0$

This is a linear non homogeneous 3^{rd} order equation which can be written as

$$y^{(3)} + \overbrace{\frac{2t}{4-t^2}}^{p(t)} y = \overbrace{\frac{3\ln(t)}{4-t^2}}^{g(t)}$$

Then

- p is continuous on $(-\infty,-2) \cup (-2,2) \cup (2,\infty)$
- g is continuous on $(0,\infty)$
- $t_0 = 1$, which belongs to $(0,2)$

We get $I = (0,2)$

Reduction of order

$$(x-1)y'' - xy' + y = 0 \qquad y_1 = e^x$$

Set $y_2 = v\, e^x$. Then

$$(x-1)v'' + (x-2)v' = 0$$

Set $w = v'$. Then

$$\frac{w'}{w} = \frac{1}{x-1} - 1$$

We get

$$w = c_2 (x-1) e^{-x}$$

$$v = x e^{-x}$$

$$y_2 = x$$

Steady State

$$y'' + 4y' + 5y = 2\cos(t) - \sin(t)$$

Characteristic eq: $r^2 + 4r + 5 = 0$. Roots: $r = -2 \pm i$

Since $\Re(r) < 0$, the steady state is the particular solution Y.

A priori form: $Y = a\cos t + b\sin t$

We find $\begin{cases} a+b = \frac{1}{2} \\ -a+b = -\frac{1}{4} \end{cases} \Rightarrow a = \frac{1}{8} \qquad b = \frac{3}{8}$

Steady state: $Y = \frac{1}{8}\cos(t) + \frac{3}{8}\sin(t)$

Laplace Transform

$$h(t) = \begin{cases} 0 & 0 \le t < 3 \\ (t-1)e^{2t} & t \geq 3 \end{cases}$$

We have $h(t) = u_3(t)\, g(t-3)$,

where $g(t) = (t+2)\, e^{2(t+3)} = e^6 t e^{2t} + 2e^6 e^{2t}$

Hence $H(s) = e^{-3s}\, G(s)$

and $G(s) = e^6 \times \frac{1}{(s-2)^2} + 2e^6 \times \frac{1}{s-2}$

$$= \frac{e^6}{(s-2)^2}\left[1 + 2(s-2)\right]$$

$$= e^6 \frac{2s-1}{(s-2)^2}$$

We get $H(s) = e^{-3s+6} \frac{2s-1}{(s-2)^2}$

Inverse Laplace transform

$$F(s) = \frac{s^2-4s+12}{(s+1)(s^2-6s+10)}$$

For $s^2-6s+10$ we have $\Delta = -4 < 0$.

Hence the a priori decomposition of F is

$$F(s) = \frac{a}{s+1} + \frac{bs+c}{s^2-6s+10}$$

Then $a = (s+1)F(s)|_{s=-1} = 1$

$$a+b = \lim_{s \to \infty} sF(s) = 1 \quad \Rightarrow b = 0$$

$$a + \frac{c}{10} = F(0) = \frac{12}{10} \quad \Rightarrow c = 2$$

We thus get

$$F(s) = \frac{1}{s+1} + \frac{2}{(s-3)^2+1}$$

$$\Rightarrow \quad f(t) = \mathcal{L}^{-1}(F) = e^{-t} + 2 \cdot e^{3t} \sin(t)$$

Equation with impulse

$$y'' + 9y = \delta(t-2) \qquad y(0)=0 \quad y'(0)=0$$

Equation for $Y = \mathcal{L}(y)$:

$$(s^2+9)Y = e^{-2s}$$

Hence $Y = e^{-2s} \times \frac{1}{s^2+9} = \frac{1}{3} e^{-2s} \frac{3}{s^2+9}$

and $y = \frac{1}{3} u_2(t) \sin(3(t-2))$

Convolution integral. We have

$$H(s) = G(s-1)$$

where

$$G(s) = \mathcal{L}(e^{-t})\,\mathcal{L}(\cos(2t)) = \frac{s}{(s+1)(s^2+4)}.$$

Thus

$$H(s) = \frac{s-1}{s\,(s^2-2s+5)}$$

System with real eigenvalues

Eq $x' = \overbrace{\begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix}}^{A} x \qquad x(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Then $P_A(r) = (r+2)(r-4)+5 = r^2 - 2r - 3$

Eigenvalues: $r_1 = -1 \qquad r_2 = 3$

Eigenvectors: . $\xi^{(1)}$ satisfies $-\xi_1 + \xi_2 = 0$.

We choose $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

. $\xi^{(2)}$ satisfies $-5\xi_1 + \xi_2 = 0$

We choose $\xi^{(2)} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

General solution:

$$y = c_1 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Initial data

$$\begin{pmatrix} 1 & 1 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow c_1 = \frac{1}{2} \qquad c_2 = \frac{1}{2}$$

Hence

$$y = \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

Phase portrait

Eq. $x' = Ax$, with $A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$

We have $P_A(r) = (r+2)(r-2)+5 = r^2+1$

Eigenvalues: $r_1 = i$ $\quad r_2 = -i$

Hence 0 is a center

$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow$ counterclockwise motion

System with complex eigenvalues

Eq. $x' = Ax \qquad A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} \qquad x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

We have $P_A(r) = (r-1)(r+3) + 5 = r^2 + 2r + 2$

Eigenvalues: $r_1 = -1 + i \qquad r_2 = -1 - i$

Eigenvector for r_1: such that $(2-i)\xi_1 - 5\xi_2 = 0$

We take $\xi^{(1)} = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$. Thus

$$y^{(1)} = e^{-t} \begin{pmatrix} 5e^{it} \\ (2-i)e^{it} \end{pmatrix}$$

$$= e^{-t} \begin{pmatrix} 5\cos t + i\, 5 \sin t \\ (2\cos t + \sin t) + i\,(2\sin t - \cos t) \end{pmatrix}$$

General solution:

$$x = c_1 e^{-t} \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 5 \sin t \\ 2\sin t - \cos t \end{pmatrix}$$

Initial data:

$$\begin{pmatrix} 5 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \quad c_1 = \frac{1}{5} \qquad c_2 = -\frac{3}{5}$$

Non homogeneous system

$$x' = \underbrace{\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 0 \\ e^t \end{pmatrix}}_{g}$$

We have $\det(A-\lambda I) = (2-\lambda)(-2-\lambda)+3 = \lambda^2-4+3 = \lambda^2-1$

Hence $r_1 = 1$, $r_2 = -1$

Eigenvectors: $\xi^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad \xi^{(2)} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

Thus $T = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$ $\quad T^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix}$ $\quad T^{-1}g = \frac{e^t}{2}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Set $x = Ty$. Then y solves

$$\begin{cases} y_1' - y_1 = -\frac{e^t}{2} \\ y_2' + y_2 = \frac{e^t}{2} \end{cases} \Rightarrow \begin{matrix} y_1 = -\frac{t}{2} e^t \quad (+c_1 e^t) \\ y_2 = \frac{1}{4} e^t \quad (+c_2 e^{-t}) \end{matrix}$$

We obtain a particular solution of the form

$$X(t) = T \begin{pmatrix} -\frac{t}{2} e^t \\ \frac{1}{4} e^t \end{pmatrix} = \frac{1}{2} e^t \, T \begin{pmatrix} -t \\ \frac{1}{2} \end{pmatrix}$$

$$= \frac{1}{2} e^t \begin{pmatrix} -t + \frac{1}{2} \\ -t + \frac{3}{2} \end{pmatrix}$$