Second order linear equations

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Differential equations - MA 266

Taken from *Elementary differential equations* by Boyce and DiPrima

Outline

Second order differential equations

General form of equation:

$$\frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right)$$

Importance of second order equations:

1. Instructive methods of resolution
2. Crucial for modeling in physics:
   - Fluid mechanics
   - Heat transfer
   - Wave motion
   - Electromagnetism

Outline

General form of 2nd order linear equation

General form 1:

$$y'' + p(t)y' + q(t)y = g(t)$$

General form 2:

$$P(t)y'' + Q(t)y' + R(t)y = G(t)$$

Remark:

2 forms are equivalent if $P(t) \neq 0$

Initial condition:

- Given by $y(t_0) = y_0$ and $y'(t_0) = y'_0$
- Two conditions necessary because two integrations performed

Homogeneous linear equations

Homogeneous equations: When $g \equiv 0$, that is

$$y'' + p(t)y' + q(t)y = 0$$

Remark:
Nonhomogeneous solutions can be deduced from homogeneous ones

Homogeneous equations with constant coefficients:

$$ay'' + by' + cy = 0,$$

for $a, b, c \in \mathbb{R}$.

Simple example

Equation:

$$y'' - y = 0. \tag{1}$$

Initial condition:

$$y(0) = 2, \quad \text{and} \quad y'(0) = -1.$$

Two simple functions satisfying (1):

$$y = \exp(t), \quad \text{and} \quad y = \exp(-t).$$

Using linear form of (1): for $c_1, c_2 \in \mathbb{R}$,

$$y = c_1 \exp(t) + c_2 \exp(-t).$$

is solution to the equation.

Simple example (2)

First conclusion:
We obtain an infinite family of solutions indexed by c_1, c_2.

Initial value problem: with $y(0) = 2$ and $y'(0) = -1$ we find

$$\begin{cases} c_1 + c_2 & = 2 \\ c_1 - c_2 & = -1 \end{cases}$$

Solution: $c_1 = \frac{1}{2}$ and $c_2 = \frac{3}{2}$.

Solution to initial value problem:

$$y = \frac{1}{2}\exp(t) + \frac{3}{2}\exp(-t).$$

Generalization

Proposition 1.

Equation considered: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0. \tag{2}$$

Characteristic equation:

$$ar^2 + br + c = 0.$$

Hypothesis:
Characteristic equation has 2 distinct real roots r_1, r_2.

Conclusion: general solution to (2) given by:

$$y = c_1 \exp(r_1 t) + c_2 \exp(r_2 t). \tag{3}$$

Generalization: initial value

Initial value problem: under assumptions of Proposition 1,

$$ay'' + by' + cy = 0, \quad y(t_0) = y_0, \quad y'(t_0) = y'_0. \tag{4}$$

Solution to (4): given by

$$y = c_1 \exp(r_1 t) + c_2 \exp(r_2 t),$$

with

$$c_1 = \frac{y'_0 - y_0 r_2}{r_1 - r_2} \exp(-r_1 t_0), \qquad c_2 = \frac{y'_0 - y_0 r_1}{r_2 - r_1} \exp(-r_2 t_0)$$

Example 1

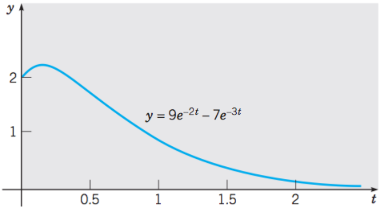
Equation considered:

$$y'' + 5y' + 6y = 0 \quad y(0) = 2, \quad y'(0) = 3. \tag{5}$$

Solution: given by

$$y = 9\exp(-2t) - 7\exp(-3t)$$

Graph of solution:



Example 2

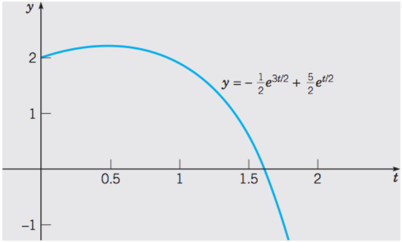
Equation considered:

$$4y'' - 8y' + 3y = 0 \quad y(0) = 2, \quad y'(0) = \frac{1}{2}.$$

Solution: given by

$$y = -\frac{1}{2}\exp\left(\frac{3t}{2}\right) + \frac{5}{2}\exp\left(\frac{t}{2}\right)$$

Graph of solution:



Asymptotic behavior of solutions

3 cases: under assumptions of Proposition 1,

1. If both $r_1, r_2 < 0$, then $\lim_{t\to\infty} y(t) = 0$
2. If $r_1 > 0$ or $r_2 > 0$, exponential growth for y
3. If $r_1 < 0$ and $r_2 = 0$, then $\lim_{t\to\infty} y(t) = \ell \in \mathbb{R}$

Outline

Definition of an operator

Definition 2.

Let

- $I = (\alpha, \beta)$, that is
$$I = \{t \in \mathbb{R}; -\infty \leq \alpha < t < \beta \leq \infty\}.$$
- $\phi : I \to \mathbb{R}$ twice differentiable.
- $p, q : I \to \mathbb{R}$

We define $L[\phi] : I \to \mathbb{R}$ by:

$$L[\phi] = \phi'' + p\phi' + q\phi$$

Homogeneous equation in terms of L

Equation considered: Under conditions of Definition 2,

$$L[y] = 0 \quad \Longleftrightarrow \quad y'' + py' + qy = 0$$

Initial conditions: for $t_0 \in I$,

$$y(t_0) = y_0, \quad \text{and} \quad y'(t_0) = y_0'.$$

Existence and uniqueness theorem

Theorem 3.

General linear equation:

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y'_0. \tag{6}$$

Hypothesis:

- $t_0 \in I$, where $I = (\alpha, \beta)$.
- p, q and g continuous on I.

Conclusion:
There exists a unique function y satisfying equation (6) on I.

Existence and uniqueness theorem (2)

Important conclusions of the theorem:

1. There exists a solution to (6).
2. There is only one solution.
3. The solution y is defined and twice differentiable on I.

Back to equation (5):

- We had existence part.
- Uniqueness is harder to see.

Major difference with first order equations:

- No general formula for solution to (6).

Example of maximal interval

Equation considered:

$$\left(t^2 - 3t\right) y'' + t\, y' - (t+3)y = 0, \quad y(1) = 2, \quad y'(1) = 1.$$

Equivalent form:

$$y'' + \frac{1}{t-3} y' + \frac{t+3}{t(t-3)}\, y = 0, \quad y(1) = 2, \quad y'(1) = 1.$$

Application of Theorem 3:

- $g(t) = 0$ continuous on $\mathbb{R}$
- $p(t) = \frac{1}{t-3}$ continuous on $(-\infty, 3) \cup (3, \infty)$
- $q(t) = \frac{t+3}{t(t-3)}$ continuous on $(-\infty, 0) \cup (0, 3) \cup (3, \infty)$
- $1 \in (0, 3)$

We thus get unique solution on $(0, 3)$

A trivial example of equation

Equation considered:

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 0$$

Hypothesis:

- p and q continuous on I
- $t_0 \in I$

Application of Theorem 3:

1. $y \equiv 0$ solves equation.
2. According to Theorem 3 it is the unique solution.

Principle of superposition

Theorem 4.

Equation considered:

$$y'' + p(t)y' + q(t)y = 0. \tag{7}$$

Hypothesis:

- y_1 and y_2 are 2 solutions to equation (7).
- c_1 and c_2 are 2 constants.

Conclusion:

$y = c_1 y_1 + c_2 y_2$ also solves (7).

Additional question:

Are all the solutions of the form $y = c_1 y_1 + c_2 y_2$?

Proof of Theorem 4

Step 1: prove that

$$L[c_1 y_1 + c_2 y_2] = c_1 \, L[y_1] + c_2 \, L[y_2].$$

Step 2: We obtain

$$L[y_1] = 0, \quad L[y_2] = 0 \quad \Longrightarrow \quad L[c_1 y_1 + c_2 y_2] = 0.$$

Wronskian

Definition 5.

Consider:

- Equation $y'' + p(t)y' + q(t)y = 0$.
- Two solutions y_1, y_2 on interval I.
- $t_0 \in I$.

The Wronskian for y_1, y_2 at t_0 is:

$$W = W[y_1, y_2](t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}.$$

Wronskian and determination of solutions

Theorem 6.

Equation: back to (6) that is

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Hypothesis:

- Existence of two solutions y_1, y_2.
- Initial condition $y(t_0) = y_0$ and $y'(t_0) = y'_0$ assigned.

Conclusion: One can find c_1, c_2 such that

$$y = c_1 y_1 + c_2 y_2$$

satisfies (6) with initial condition iff

$$W[y_1, y_2](t_0) \neq 0$$

Complement to Theorem 6

Expression for c_1, c_2: Under assumptions of Theorem 6 we have

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad \text{and} \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

Justification: c_1, c_2 are solution to the system

$$\begin{cases} c_1 y_1(t_0) + c_2 y_2(t_0) & = y_0 \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) & = y_0' \end{cases}$$

Example 1

Equation considered: back to (5), that is

$$y'' + 5y' + 6y = 0.$$

2 solutions: given by

$$y_1 = \exp(-2t), \quad \text{and} \quad y_2 = \exp(-3t)$$

Expression of Wronskian: for $t \in \mathbb{R}$,

$$W[y_1, y_2](t) = \exp(-5t).$$

Solving the equation: $W[y_1, y_2](t) \neq 0$ for all $t \in \mathbb{R}$
$\Longrightarrow$ initial value problem can be solved at any $t \in \mathbb{R}$.

Wronskian and uniqueness of solutions

Theorem 7.

Equation: back to (6) that is

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Hypothesis:

- Existence of two solutions y_1, y_2.

Conclusion: The general solution

$$y = c_1 y_1 + c_2 y_2, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}$$

includes all solutions to (6) iff:

$$\text{there exists } t_0 \in I \text{ such that } W[y_1, y_2](t_0) \neq 0.$$

Example: equations with constant coefficients

Equation considered: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0. \tag{8}$$

Characteristic equation:

$$ar^2 + br + c = 0.$$

Hypothesis:
Characteristic equation has 2 real roots r_1, r_2.

2 solutions:

$$y_1 = \exp(r_1 t), \quad \text{and} \quad y_2 = \exp(r_2 t).$$

Example: equations with constant coefficients (2)

Wronskian:

$$W[y_1, y_2](t) = (r_2 - r_1) \exp\left((r_1 + r_2)t\right)$$

Conclusion: The general solution

$$y = c_1 y_1 + c_2 y_2, \quad \text{with} \quad c_1, c_2 \in \mathbb{R}$$

includes all solutions to (8) iff $r_1 \neq r_2$.

Notation:
In this context the functions y_1, y_2 are called fundamental solutions

Summary

Homogeneous linear second order equation:

$$L[y] = y'' + p(t)y' + q(t)y = 0.$$

Recipe to solve the equation:

1. Find 2 solutions y_1 and y_2
2. Find a point t_0 such that $W[y_1, y_2](t_0) \neq 0$
3. Then y_1 and y_2 are fundamental solutions

Outline

General situation

Linear equation with constant coefficients: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0. \tag{9}$$

Characteristic equation:

$$ar^2 + br + c = 0.$$

Situation considered up to now:
2 real roots, that is $b^2 - 4ac \geq 0$.

Situation considered in this section:
2 complex roots, that is $b^2 - 4ac < 0$.

Complex roots case: general solution

Proposition 8.

Equation considered: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0. \tag{10}$$

Characteristic equation:

$$ar^2 + br + c = 0.$$

Hypothesis: Characteristic equation has 2 complex roots
$\hookrightarrow r_1 = \lambda + \imath\mu$ and $r_2 = \lambda - \imath\mu$.

Conclusion: general solution to (10) given by:

$$y = c_1 \exp(r_1 t) + c_2 \exp(r_2 t). \tag{11}$$

Complex exponential

Aim: understand meaning of expression

$$y_1(t) = \exp\left((\lambda + \imath\mu)t\right).$$

Example: if $\lambda = -1$, $\mu = 2$ and $t = 3$, $y_1(t) = e^{-3+6\imath}$.

Definition 9.

For $t \in \mathbb{R}$ we set:

$$e^{\imath t} = \cos(t) + \imath \sin(t).$$

Properties of complex exponential

1. For $t \in \mathbb{R}$,
$$\exp(-\imath t) = \cos(t) - \imath \sin(t) = \overline{\exp(\imath t)}.$$
2. For $z = \alpha + \imath\beta \in \mathbb{C}$,
$$\exp(z) = \exp(\alpha)\cos(\beta) + \imath \exp(\alpha)\sin(\beta).$$
3. For $z_1, z_2 \in \mathbb{C}$,
$$\exp(z_1 + z_2) = \exp(z_1)\, \exp(z_2).$$
4. For $z \in \mathbb{C}$,
$$\frac{d}{dt}\exp(zt) = z\exp(zt).$$

Example of differential equation

Equation:

$$y'' + y' + 9.25y = 0. \tag{12}$$

Characteristic equation:

$$r^2 + r + 9.25 = 0.$$

Roots of characteristic equation:

$$r_1 = -\frac{1}{2} + 3\imath, \qquad r_2 = -\frac{1}{2} - 3\imath$$

Example of differential equation (2)

Fundamental solutions for (12):

$$\begin{aligned} y_1(t) &= e^{\left(-\frac{1}{2}+3\imath\right)t} = e^{-\frac{t}{2}}\left[\cos(3t) + \imath \sin(3t)\right] \\ y_2(t) &= e^{\left(-\frac{1}{2}-3\imath\right)t} = e^{-\frac{t}{2}}\left[\cos(3t) - \imath \sin(3t)\right] \end{aligned}$$

Wronskian:

$$W[y_1, y_2](t) = -6\imath e^{-t} \neq 0.$$

Thus all solutions of (12) are of the form:

$$y = c_1 y_1 + c_2 y_2.$$

Example of differential equation (3)

Real valued fundamental solutions:

$$u = e^{-\frac{t}{2}} \cos(3t), \qquad v = e^{-\frac{t}{2}} \sin(3t).$$

Wronskian for u, v:

$$W[u, v](t) = 3e^{-t} \neq 0.$$

Thus all solutions of (12) are of the form:

$$y = c_1 u + c_2 v.$$

Example of differential equation (4)

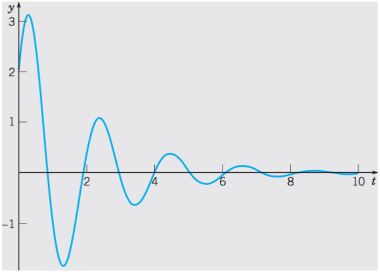
Initial value problem: equation (12) with

$$y(0) = 2, \quad \text{and} \quad y'(0) = 8.$$

Solution:

$$y = e^{-\frac{t}{2}} \left[2\cos(3t) + 3\sin(3t) \right].$$

Graph: decaying oscillations



Complex roots case: real valued solutions

Proposition 10.

Equation considered: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0. \tag{13}$$

Hypothesis: Characteristic equation has 2 complex roots
$\hookrightarrow r_1 = \lambda + \imath\mu$ and $r_2 = \lambda - \imath\mu$.

Conclusion: fundamental solutions to (13) given by:

$$y_1 = \exp(\lambda t)\cos(\mu t), \qquad y_2 = \exp(\lambda t)\sin(\mu t). \tag{14}$$

Example of application

Equation:

$$16y'' - 8y' + 145y = 0. \tag{15}$$

Roots of characteristic equation:
We have $\Delta = -9216 = -(96)^2$, thus

$$r_1 = \frac{1}{4} + 3\imath, \qquad r_2 = \frac{1}{4} - 3\imath$$

Real valued fundamental solutions:

$$y_1 = e^{\frac{t}{4}} \cos(3t), \qquad y_2 = e^{\frac{t}{4}} \sin(3t).$$

Example of application (2)

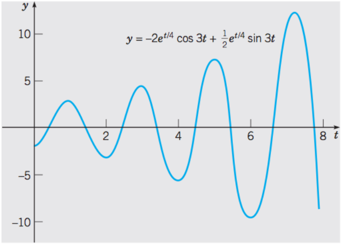
Initial value problem: equation (15) with

$$y(0) = -2, \quad \text{and} \quad y'(0) = 1.$$

Solution:

$$y = e^{\frac{t}{4}} \left[-2\cos(3t) + \frac{1}{2}\sin(3t) \right].$$

Graph: increasing oscillations



Outline

General situation

Linear equation with constant coefficients: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0. \tag{16}$$

Characteristic equation:

$$ax^2 + bx + c = 0.$$

Situation considered up to now:

1. 2 distinct real roots, that is $b^2 - 4ac > 0$.
2. 2 distinct complex roots, that is $b^2 - 4ac < 0$.

Situation considered in this section:
Only one root, that is $b^2 - 4ac = 0$.

D'Alembert

Some facts about d'Alembert:

- Abandoned after birth
- Mathematician
- Contribution in fluid dynamics
- Philosopher
- Participation in 1st Encyclopedia

Problem with double root

Expression for the root: if $b^2 - 4ac = 0$ then

$$ax^2 + bx + c = a\,(x - r)^2\,, \quad \text{with} \quad r = -\frac{b}{2a}.$$

Consequence on equation (16): only one fundamental solution,

$$y_1(t) = \exp\,(rt)\,.$$

D'Alembert's method in order to get 2 fundamental solutions:

1. Look for solutions under the form $y(t) = v(t)\exp(rt)$.
2. We will see: $v(t)$ of the form $v(t) = c_1 t + c_2$.

Example of d'Alembert's method

Equation:

$$y'' + 4y' + 4y = 0. \tag{17}$$

Double root of characteristic equation: $r = -2$.

Applying d'Alembert's method:

1. Look for solutions under the form $y(t) = v(t)\exp(rt)$.
2. We find: $v'' = 0$, thus $v(t) = c_1 t + c_2$.
3. Fundamental solutions: $y_1 = \exp(-2t)$ and $y_2 = t\exp(-2t)$.

Example of d'Alembert's method (2)

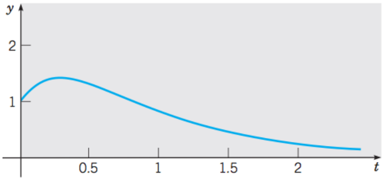
Wronskian:

$$W[y_1, y_2](t) = e^{-4t} \neq 0.$$

Thus all solutions of (20) are of the form:

$$y = c_1 y_1 + c_2 y_2.$$

Graph of a typical solution:



Double root case: generalization

Proposition 11.

Equation considered: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0. \tag{18}$$

Hypothesis: Characteristic equation has 1 double root
$\hookrightarrow r = -\frac{b}{2a}$.

Conclusion: fundamental solutions to (18) given by:

$$y_1 = \exp(rt), \qquad y_2 = t\exp(rt). \tag{19}$$

Example of application

Equation:

$$y'' - y' + 0.25y = 0. \tag{20}$$

Roots of characteristic equation:

$$r = \frac{1}{2}$$

Fundamental solutions:

$$y_1 = e^{\frac{t}{2}}, \qquad y_2 = te^{\frac{t}{2}}.$$

Example of application (2)

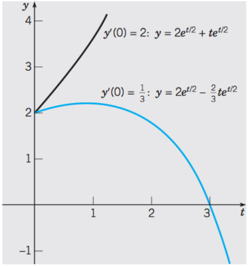
Initial value problem: equation (20) with

$$y(0) = 2, \quad \text{and} \quad y'(0) = \frac{1}{3}.$$

Solution:

$$y = \left(2 - \frac{2}{3}t\right) e^{\frac{t}{2}}.$$

Graph:



Example of application (3)

Modification of initial value: equation (20) with

$$y(0) = 2, \quad \text{and} \quad y'(0) = 2.$$

Solution:

$$y = (2 + t)\, e^{\frac{t}{2}}.$$

Question:
Separation between increasing and decreasing behavior of y
$\hookrightarrow$ according to value of $y'(0)$.

Equations with constant coefficients: summary

Equation: for $a, b, c \in \mathbb{R}$,

$$ay'' + by' + cy = 0.$$

3 cases:

1. 2 real roots r_1, r_2: fundamental solutions given by

$$y_1(t) = \exp(r_1 t), \quad \text{and} \quad y_2(t) = \exp(r_2 t)$$

2. 1 double root r: fundamental solutions given by

$$y_1(t) = \exp(rt), \quad \text{and} \quad y_2(t) = t \exp(rt)$$

3. 2 complex roots $r_1 = \lambda + \imath\mu$, $r_2 = \lambda - \imath\mu$: fund. sol. given by

$$y_1(t) = \exp(\lambda t) \cos(\mu t), \quad \text{and} \quad y_2(t) = \exp(\lambda t) \sin(\mu t)$$

Reduction of order method

Equation: General linear 2nd order,

$$y'' + p(t)y' + q(t)y = 0. \tag{21}$$

Hypothesis: we know 1 solution y_1 to equation (21).

Method: find 2nd solution y_2 given by

$$y_2 = v\, y_1.$$

Equation for $w \equiv v'$: we find

$$y_1 w' + (2y_1' + py_1)\, w = 0.$$

This is a first order equation.

Example of application

Equation: for $t > 0$,

$$2t^2y'' + 3ty' - y = 0. \tag{22}$$

An elementary solution:

$$y_1(t) = \frac{1}{t}.$$

Reduction of order: find 2nd solution y_2 given by

$$y_2 = v\, y_1.$$

Equation for $w \equiv v'$: we find

$$2tw' - w = 0.$$

Example of application (2)

Solving equation for w: we find

$$w(t) = ct^{1/2} \quad \Longrightarrow \quad v(t) = c_1 t^{3/2} + c_2$$

Fundamental solutions for (22): we obtain

$$y_1(t) = \frac{1}{t}, \quad \text{and} \quad y_2(t) = t^{1/2}$$

Wronskian:

$$W[y_1, y_2](t) = \frac{3}{2t^{3/2}}.$$

Thus y_1, y_2 fundamental solutions for $t > 0$.

Outline

Setting

General linear equation:

$$y'' + p(t)y' + q(t)y = g(t). \tag{23}$$

Corresponding homogeneous equation:

$$y'' + p(t)y' + q(t)y = 0. \tag{24}$$

Aim:
Deduce solutions to (23) from solutions to (24).

From nonhomogeneous to homogeneous

Theorem 12.

Consider:

- Y_1, Y_2 solutions to nonhomogeneous equation (23).
- y_1, y_2 fund. solts. to homogeneous equation (24).

Then

1. $Y_1 - Y_2$ solves homogeneous equation (24).
2. $Y_1 - Y_2$ can be written, for $c_1, c_2 \in \mathbb{R}$, as:

$$Y_1 - Y_2 = c_1 y_1 + c_2 y_2.$$

General form of solutions: nonhomogeneous case

Theorem 13.

Equation: nonhomogeneous (23), that is

$$y'' + p(t)y' + q(t)y = g(t).$$

Consider:

1. y_1, y_2 fund. solts. to homogeneous equation (24).
2. Specific solution Y to nonhomogeneous equation (23).

Conclusion: all solutions to nonhomogeneous equation (23)
$\hookrightarrow$ can be written as:

$$y = Y + c_1 y_1 + c_2 y_2.$$

Summary

Nonhomogeneous linear second order equation:

$$L[y] = y'' + p(t)y' + q(t)y = g(t).$$

Recipe to solve the equation:

1. Find 2 fundamental solutions y_1 and y_2 to homogeneous equation.
2. Find a particular solution Y to nonhomogeneous equation
3. Then general solution has the form

$$y = Y + c_1 y_1 + c_2 y_2.$$

Method of undetermined coefficients

Nonhomogeneous linear equation with constant coefficients:

$$L[y] = ay'' + by' + cy = g(t).$$

Aim: Find a particular solution Y to the equation.

Table of possible guess: restricted to a limited number of cases,

Function g	Guess
$\alpha \exp(at)$	$A \exp(at)$
$\alpha \sin(\gamma t) + \beta \cos(\gamma t)$	$A \sin(\gamma t) + B \cos(\gamma t)$
$\alpha_n t^n + \cdots + \alpha_0$	$A_n t^n + \cdots + A_0$
$(\alpha_n t^n + \cdots + \alpha_0) \exp(at)$	$(A_n t^n + \cdots + A_0) \exp(at)$
$(\alpha \sin(\gamma t) + \beta \cos(\gamma t)) \exp(at)$	$(A \sin(\gamma t) + B \cos(\gamma t)) \exp(at)$

Example of application

Equation:

$$y'' - 3y' - 4y = 2\sin(t) \tag{25}$$

Guess for particular solution:

$$Y(t) = A\sin(t) + B\cos(t)$$

Equation for A, B: plugging into (25) we get

$$-5A + 3B = 2, \quad \text{and} \quad -3A - 5B = 0.$$

Particular solution:

$$Y(t) = -\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t)$$

Example of application (2)

Homogeneous equation:

$$y'' - 3y' - 4y = 0$$

Solution of homogeneous equation:

$$y = c_1 e^{-t} + c_2 e^{4t}.$$

General solution of nonhomogeneous equation (25):

$$y = c_1 e^{-t} + c_2 e^{4t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t).$$

Second example of application

Equation:

$$y'' - 3y' - 4y = -8e^t \cos(2t) \tag{26}$$

Guess for particular solution:

$$Y(t) = Ae^t \cos(2t) + Be^t \sin(2t)$$

Equation for A, B: plugging into (26) we get

$$10A + 2B = 8, \quad \text{and} \quad 2A - 10B = 0.$$

Particular solution:

$$Y(t) = \frac{10}{13}e^t \cos(2t) + \frac{2}{13}e^t \sin(2t)$$

Second example of application (2)

Homogeneous equation:

$$y'' - 3y' - 4y = 0$$

Solution of homogeneous equation:

$$y = c_1 e^{-t} + c_2 e^{4t}.$$

General solution of nonhomogeneous equation (26):

$$y = c_1 e^{-t} + c_2 e^{4t} + \frac{10}{13}\cos(2t) + \frac{2}{13}\sin(2t).$$

Sums of particular solutions

Proposition 14.

Consider the 2 equations:

$$ay'' + by' + cy = g_1(t) \tag{27}$$
$$ay'' + by' + cy = g_2(t) \tag{28}$$

Let

- Y_1 particular solution of equation (27).
- Y_2 particular solution of equation (28).

Now consider the following equation:

$$ay'' + by' + cy = g_1(t) + g_2(t). \tag{29}$$

Then $Y = Y_1 + Y_2$ particular solution of (29).

Example of application

Equation:

$$y'' - 3y' - 4y = 2\sin(t) - 8e^{t}\cos(2t) \tag{30}$$

Splitting the equation: into (25) and (26).

Recalling previous results:
particular solutions to (25) and (26) are given by

$$\begin{aligned} Y_1(t) &= -\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t) \\ Y_2(t) &= -\frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}e^{t}\sin(2t) \end{aligned}$$

Particular solution for (30):

$$Y(t) = -\frac{5}{17}\sin(t) + \frac{3}{17}\cos(t) - \frac{10}{13}e^{t}\cos(2t) + \frac{2}{13}e^{t}\sin(2t).$$

Elaboration of the guess

Equation:

$$L[y] = y'' - 3y' - 4y = 2e^{-t} \tag{31}$$

Initial guess for particular solution: $Y(t) = A\,e^{-t}$.

Problem: e^{-t} solution of homogeneous equation $\Longrightarrow L[Y] = 0$.

Second guess: $Y(t) = A\,te^{-t}$

Particular solution:

$$Y(t) = -\frac{2}{5}\,te^{-t}.$$

Generalization:

1. When initial guess is solution to homogeneous equation
   $\hookrightarrow$ multiply initial guess by t
2. Sometimes initial guess has to be multiplied by t^2

Outline

Introduction

Equation:

$$ay'' + by' + c = g(t).$$

Hypothesis:

1. We have fundamental solutions y_1, y_2 to homogeneous equation.
2. g has not a simple form allowing a guess for particular solution.

Method: find solution under the form

$$y = u_1(t)\, y_1 + u_2(t)\, y_2.$$

Comments:

- Advantage: very general method.
- Problem: involves nontrivial integration steps.

Generalization

General equation:

$$y'' + p(t)y' + q(t)y = g(t) \tag{32}$$

Hypothesis:

- p, q, g continuous functions on interval I.
- We know y_1, y_2 fundamental solutions to

$$y'' + p(t)y' + q(t)y = 0,$$

  with non vanishing Wronskian.

Aim: find general solution to (32)

Generalization (2)

Theorem 15.

Under previous assumptions, general solution to (32) is:

$$y = c_1 y_1 + c_2 y_2 + Y,$$

where Y is given as:

$$Y = -y_1 \int \frac{y_2(s)g(s)}{W[y_1, y_2](s)}\, ds + y_2 \int \frac{y_1(s)g(s)}{W[y_1, y_2](s)}\, ds.$$

Main difficulties of application:

- Find y_1, y_2 for non constant coefficients p, q.
- Integration step.

Variation of parameters: example

Equation:

$$y'' + 4y = 3\csc(t) \tag{33}$$

Solution to homogeneous equation:

$$c_1 \cos(2t) + c_2 \sin(2t).$$

Variation of parameters: example (2)

Wronskian: $W[y_1, y_2](t) = 2$

Integrals:

$$\int \frac{y_2(s)\, g(s)}{W[y_1, y_2](s)}\, ds = \frac{3}{2} \int \frac{\sin(2t)}{\sin(t)}\, dt = 3\sin(t) + c_1$$

and

$$\begin{aligned} \int \frac{y_1(s)\, g(s)}{W[y_1, y_2](s)}\, ds &= \frac{3}{2} \int \frac{\cos(2t)}{\sin(t)}\, dt \\ &= \frac{3}{2} \ln\left(|\csc(t) - \cot(t)|\right) + 3\cos(t) + c_2 \end{aligned}$$

Variation of parameters: example (3)

Particular solution of equation (33):

$$Y = -3\sin(t)\cos(2t) + \frac{3}{2}\ln\left(|\csc(t) - \cot(t)|\right)\sin(2t) + 3\cos(t)\sin(2t)$$

General solution to equation (33):

$$y = c_1\cos(2t) + c_2\sin(2t) - 3\sin(t)\cos(2t) + \frac{3}{2}\ln\left(|\csc(t) - \cot(t)|\right)\sin(2t) + 3\cos(t)\sin(2t)$$

Some trigonometric formulae

Functions secant, cosecant:

$$\sec(t) = \frac{1}{\cos(t)}, \qquad \csc(t) = \frac{1}{\sin(t)}$$

Integral of csc:

$$\int \csc(t)\, dt = \ln\left(|\csc(t) - \cot(t)|\right) + c$$

Double angles:

$$\begin{aligned} \cos(2t) &= \cos^2(t) - \sin^2(t) = 1 - 2\sin^2(t) \\ \sin(2t) &= 2\sin(t)\cos(t) \end{aligned}$$

HW 3.6 exercice 10

Equation:

$$y'' - 2y' + y = \frac{e^t}{1+t^2}$$

Fundamental solutions: $y_1 = e^t$ and $y_2 = te^t$.

Wronskian: $W[y_1, y_2](t) = e^{2t}$

Integrals:

$$\begin{aligned}
\int \frac{y_2(s)\, g(s)}{W[y_1, y_2](s)}\, ds &= \frac{1}{2} \ln\left(1 + t^2\right) + c_1 \\
\int \frac{y_1(s)\, g(s)}{W[y_1, y_2](s)}\, ds &= \arctan(t) + c_2
\end{aligned}$$

HW 3.6 exercice 10 (2)

Particular solution:

$$Y(t) = -\frac{e^t}{2}\ln\left(1+t^2\right) + te^t\arctan(t)$$

General solution:

$$y(t) = c_1 e^t + c_2 te^t - \frac{e^t}{2}\ln\left(1+t^2\right) + te^t\arctan(t)$$

Outline

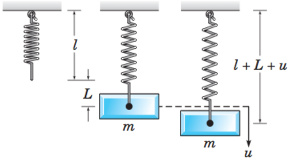
Mass on a spring

Situation:

- Mass m hanging at rest on end of vertical spring.
- Original length of spring: ℓ.
- Elongation of spring due to mass: L.

Forces:

- Gravitational: mg.
- Spring: $F_s = -kL$.
- At equilibrium: $mg = kL \longrightarrow$ can be used to compute k.



Mass on a spring dynamics

Situation 2: we have either

- Force applied to spring.
- Initial displacement.

Forces: if $u \equiv$ displacement from equilibrium position,

1. Gravitational: mg.
2. Spring: $F_s = -k(L + u)$.
3. Resistive or damping: $F_d = -\gamma u'$.
4. Applied external force F, often periodic.

Remark:

Expressions for F_s and F_d are approximate.

Equation for dynamics

Newton's law:

$$mu'' = mg - k(L + u) - \gamma u' + F$$

Simplification: since $mg = kL$, we get

$$mu'' + \gamma u' + ku = F. \tag{34}$$

Solution: With initial condition $u(0), u'(0)$ and continuous F
$\hookrightarrow$ according to Theorem 3 there exists a unique solution to (34).

Numerical example

Situation:

- A mass weighing 4lb stretches a spring 2in.
- Additional 6in displacement given, then released.
- Viscous resistance of medium is 6lb when velocity is 3 ft/s.

Model: equation (34)
$\hookrightarrow$ determination of m, γ, k

Numerical example (2)

Constants: we have

$$
\begin{aligned}
m &= \frac{w}{g} = \frac{4 \text{ lb}}{32 \text{ ft s}^{-2}} = \frac{1}{8} \, \frac{\text{lb s}^2}{\text{ft}} \\
\gamma &= \frac{6\text{lb}}{3 \text{ ft s}^{-1}} = 2 \, \frac{\text{lb s}}{\text{ft}} \\
k &= \frac{mg}{L} = \frac{w}{L} = \frac{4 \text{ lb}}{2 \text{ in}} = \frac{4 \text{ lb}}{1/6 \text{ ft}} = 24 \, \frac{\text{lb}}{\text{ft}}
\end{aligned}
$$

Initial condition:

$$
u(0) = 6 \text{ in} = \frac{1}{2} \text{ ft}.
$$

Undamped free vibration

Particular situation: $F = 0$ and $\gamma = 0$.

Resulting equation:

$$mu'' + ku = 0.$$

General solution:

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \quad \text{with} \quad \omega_0 = \left(\frac{k}{m}\right)^{1/2}$$

Other expression for solution:

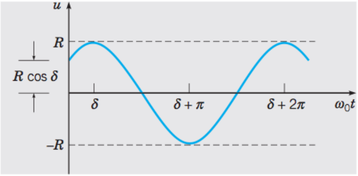
$$u = R\cos\left(\omega_0 t - \delta\right), \quad \text{where} \quad R = \left(c_1^2 + c_2^2\right)^{1/2}, \quad \tan(\delta) = \frac{c_2}{c_1}.$$

Undamped free vibration (2)

Vocabulary: we call

- ω_0: natural frequency (does not depend on initial condition).
- R: amplitude of motion (does depend on initial condition).
- δ: phase.

Period of motion: $T = 2\pi \left(\frac{m}{k}\right)^{1/2}$
$\hookrightarrow$ Larger mass vibrates more slowly.



Numerical example

Situation:

- A mass weighing 10lb stretches a spring 2in.
- Additional 2in displacement given.
- Mass released with upward velocity 1ft/s

Numerical example (2)

Constants: we have

$$
\begin{aligned}
m &= \frac{w}{g} = \frac{10 \text{ lb}}{32 \text{ ft s}^{-2}} = \frac{5}{16} \frac{\text{lb s}^2}{\text{ft}} \\
k &= \frac{mg}{L} = \frac{w}{L} = \frac{10 \text{ lb}}{2 \text{ in}} = \frac{10 \text{ lb}}{1/6 \text{ ft}} = 60 \, \frac{\text{lb}}{\text{ft}}
\end{aligned}
$$

Initial condition:

$$
u(0) = 2 \text{ in} = \frac{1}{6} \text{ ft}, \qquad u'(0) = -1 \text{ ft s}^{-1}.
$$

Numerical example (3)

Equation:

$$u'' + 192u = 0.$$

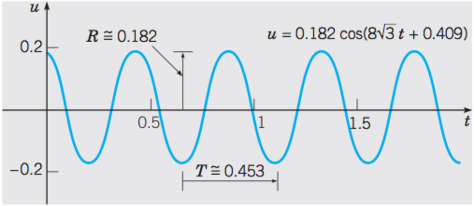
Solution: taking initial condition into account,

$$u = \frac{1}{6}\cos\left(8\sqrt{3}t\right) - \frac{1}{8\sqrt{3}}\sin\left(8\sqrt{3}t\right).$$

Numerical example (4)

Quantities of interest:

- Natural frequency: $\omega_0 = \sqrt{192} = 13.85$ rad s^{-1}.
- Period: $T = \frac{2\pi}{\omega_0} = 0.45$ s.
- Amplitude: $R = (c_1^2 + c_2^2)^{1/2} = 0.182$ ft.
- Phase: $\delta = -\arctan(\frac{6}{8\sqrt{3}}) = -0.41$ rad.



Damped free vibrations

Equation:

$$mu'' + \gamma u' + ku = 0.$$

Roots:

$$\begin{aligned} r_1,\ r_2 &= \frac{\gamma}{2m}\left[-1 \pm \left(1 - \frac{4km}{\gamma^2}\right)^{1/2}\right] \\ &= -\frac{\gamma}{2m} \pm \frac{(\gamma^2 - 4km)^{1/2}}{2m}. \end{aligned}$$

Remark:
$\mathcal{R}(r_1),\ \mathcal{R}(r_2) < 0$, thus exponentially decreasing amplitude

Damped free vibrations (2)

3 cases:

1. If $\gamma^2 - 4km > 0$, then:

$$u = c_1 \exp\left(r_1 t\right) + c_2 \exp\left(r_2 t\right).$$

2. If $\gamma^2 - 4km = 0$, then:

$$u = [c_1 + c_2 t] \exp\left(-\frac{\gamma t}{2m}\right).$$

3. If $\gamma^2 - 4km < 0$, then:

$$u = [c_1 \cos(\mu t) + c_2 \sin(\mu t)] \exp\left(-\frac{\gamma t}{2m}\right), \tag{35}$$

where $\mu = \frac{(4km - \gamma^2)^{1/2}}{2m} > 0$.

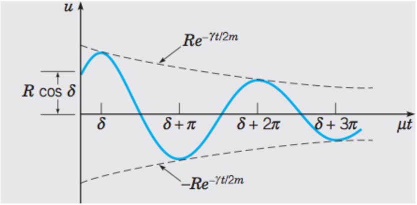
Small damping case

Case under consideration:

If γ small, we have $\gamma^2 - 4km < 0 \Longrightarrow$ motion governed by (35).

Expression for u:

$$u = R \exp\left(-\frac{\gamma t}{2m}\right) \cos(\mu t - \delta).$$



Small damping (2)

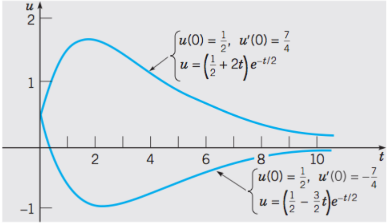
Quasi-frequency: when $\gamma^2 - 4km < 0$, given by μ. We have

$$\frac{\mu}{\omega_0} = \frac{(4km - \gamma^2)^{1/2}}{2m} \left(\frac{m}{k}\right)^{1/2} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2} \simeq 1 - \frac{\gamma^2}{8km}.$$

Conclusion:
Small damping $\Longrightarrow$ smaller frequency for oscillations.

Critical and over damping

Critically damped case: when $\gamma^2 - 4km = 0$
$\hookrightarrow$ mass passes through equilibrium at most once.



Overdamped case: when $\gamma^2 - 4km > 0$
$\hookrightarrow$ mass passes through equilibrium at most once.

Outline

Introduction

Situation:

External force applied to a spring mass.

Equation:

$$mu'' + \gamma u' + ku = F. \tag{36}$$

Example of forced vibration

Equation:

$$u'' + u' + 1.25u = 3\cos(t) \tag{37}$$

Guess for particular solution:

$$U(t) = A\cos(t) + B\sin(t)$$

Equation for A, B: plugging into (37) we get

$$0.25A + B = 3, \quad \text{and} \quad -A + 0.25B = 0.$$

Particular solution:

$$U(t) = \frac{12}{17}\cos(t) + \frac{48}{17}\sin(t)$$

Example of forced vibration (2)

Homogeneous equation:

$$u'' + u' + 1.25u = 0$$

Solution of homogeneous equation:

$$u = c_1 e^{-\frac{t}{2}} \cos(t) + c_2 e^{-\frac{t}{2}} \sin(t).$$

General solution of nonhomogeneous equation (37):

$$u = c_1 e^{-\frac{t}{2}} \cos(t) + c_2 e^{-\frac{t}{2}} \sin(t) + \frac{12}{17} \cos(t) + \frac{48}{17} \sin(t).$$

Example of forced vibration (3)

Initial conditions: we assume

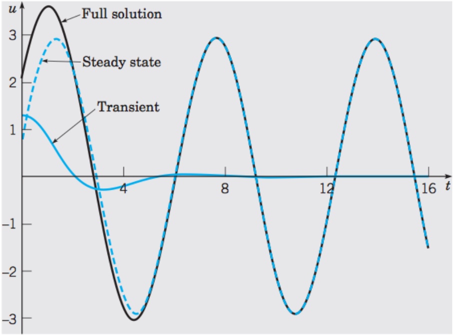
$$u(0) = 2 \quad \text{and} \quad u'(0) = 3.$$

Expression for u:

$$u = \overbrace{\frac{22}{17} e^{-\frac{t}{2}} \cos(t) + \frac{14}{17} e^{-\frac{t}{2}} \sin(t)}^{\text{Transient part}} + \overbrace{\frac{12}{17} \cos(t) + \frac{48}{17} \sin(t)}^{\text{Steady state}}.$$

Example of forced vibration (4)

Observation on graph:
Solution to (37) gets quickly close to steady state.



Generalization

Equation with sinusoidal forcing:

$$m u'' + \gamma u' + k u = F_0 \cos(\omega t). \tag{38}$$

Solution of homogeneous equation:

$$u_c = c_1 e^{-rt} \cos(\mu t) + c_2 e^{-rt} \sin(\mu t).$$

Transience of u_c: we have

$$\lim_{t \to \infty} u_c(t) = 0, \quad \text{exponentially fast.}$$

Generalization (2)

Particular solution: of the form

$$U(t) = A\cos(\omega t) + B\sin(\omega t) = R\cos(\omega t - \delta).$$

General solution to (38): given by

$$u = \overbrace{c_1 e^{-rt}\cos(\mu t) + c_2 e^{-rt}\sin(\mu t)}^{\text{Transient part}} + \overbrace{R\cos(\omega t - \delta)}^{\text{Steady state}}.$$

Resonance phenomenon

Equation with sinusoidal forcing: back to (38), that is

$$mu'' + \gamma u' + ku = F_0 \cos(\omega t).$$

Quantity of interest:

$$\varphi(\omega) \equiv \frac{Rk}{F_0} = \frac{\text{Amplitude}}{\text{Displacement due to } F_0 \text{ at equilibrium}}$$

Expression for φ:

$$\varphi(\omega) = \left[\left(1 - \frac{\omega^2}{\omega_0^2} \right)^2 + \Gamma \frac{\omega^2}{\omega_0^2} \right]^{-1/2}, \quad \text{where} \quad \Gamma = \frac{\gamma^2}{mk}.$$

Resonance phenomenon (2)

Proposition 16.

Let us assume

- Γ small (small damping case)

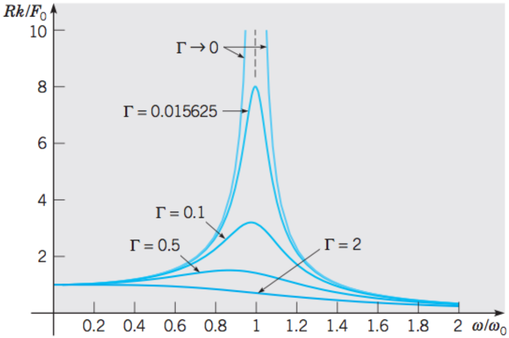
Then we have

$$\omega_{\max} \simeq \omega_0, \quad \text{and} \quad \varphi(\omega_{\max}) \simeq \frac{1}{\Gamma^{1/2}} = \frac{(mk)^{1/2}}{\gamma}.$$

Resonance phenomenon (3)

Interpretation of Proposition 16:

- Amplitude can get high if γ small and ω close to ω_0.
- This has to be taken into account in real situations.



Proof of Proposition 16

Differentiating φ: we find

$$\varphi'(\omega) = 2\left(1 - \frac{\omega^2}{\omega_0^2} - \frac{\Gamma}{2}\right)\left[\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + \Gamma\,\frac{\omega^2}{\omega_0^2}\right]^{-3/2}$$

Optimizing φ: if $\Gamma < 2$ we have

$$\omega_{\max} = \omega_0\left(1 - \frac{\Gamma}{2}\right)^{1/2},$$

and recalling $\Gamma = \frac{\gamma^2}{mk}$,

$$\varphi(\omega_{\max}) = \left[\Gamma\left(1 - \frac{\Gamma}{4}\right)\right]^{-1/2} = \left(\frac{mk}{\gamma^2}\right)^{1/2}\left(1 - \frac{\gamma^2}{4mk}\right)^{-1/2}.$$

Proof of Proposition 16 (2)

Behavior of φ: if $\Gamma < 2$ we have

1. $\varphi(0) = 1 \longrightarrow$ corresponds to equilibrium.
2. $\omega_{\max} = \omega_0 \left(1 - \frac{\Gamma}{2}\right)^{1/2}$ and $\varphi(\omega_{\max}) = \left[\Gamma\left(1 - \frac{\Gamma}{4}\right)\right]^{-1/2}$.
3. $\lim_{\omega\to\infty} \varphi(\omega) = 0$.

Behavior of φ, small damping case: if Γ small we have

$$\omega_{\max} \simeq \omega_0, \quad \text{and} \quad \varphi(\omega_{\max}) \simeq \frac{1}{\Gamma^{1/2}} = \frac{(mk)^{1/2}}{\gamma}.$$

Forced vibration without damping

Proposition 17.

Equation with sinusoidal forcing and no damping:

$$mu'' + ku = F_0 \cos(\omega t). \tag{39}$$

Interesting initial condition: mass at rest, that is

$$u(0) = 0, \quad \text{and} \quad u'(0) = 0.$$

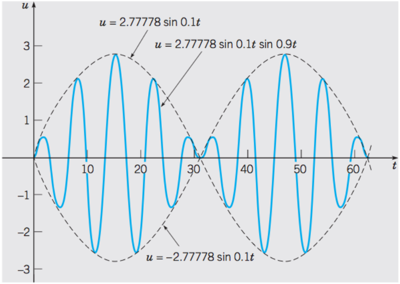
Then we get a useful expression for u:

$$u = \overbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)}^{\text{Slowly oscillating term}} \overbrace{\sin\left(\frac{(\omega_0 + \omega)t}{2}\right)}^{\text{Fast oscillating term}}. \tag{40}$$

Interpretation of Proposition 17

Amplitude modulation: according to formula (41) we have

1. A fast oscillating motion.
2. A periodic slow variation of amplitude.



Remark: This type of wave is observed in reality
 $\hookrightarrow$ guitar almost tuned.

Proof of Proposition 17

Equation with sinusoidal forcing and no damping:
Recall equation (39)

$$mu'' + ku = F_0 \cos(\omega t).$$

Solution of homogeneous equation:

$$u_c = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t), \quad \text{where} \quad \omega_0 = \left(\frac{k}{m}\right)^{1/2}.$$

Particular solution, case $\omega \neq \omega_0$:

$$U(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Proof of Proposition 17 (2)

General solution to (39): given by

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

Interesting initial condition: mass at rest, that is

$$u(0) = 0, \quad \text{and} \quad u'(0) = 0.$$

Solution to initial value problem:

$$u = \frac{F_0}{m(\omega_0^2 - \omega^2)} \left[\cos(\omega t) - \cos(\omega_0 t)\right].$$

Proof of Proposition 17 (3)

Elementary trigonometric formula:

$$\cos(a+b) - \cos(a-b) = -2\sin(a)\sin(b).$$

Another expression for u:

$$u = \overbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)}^{\text{Slowly oscillating term}} \overbrace{\sin\left(\frac{(\omega_0 + \omega)t}{2}\right)}^{\text{Fast oscillating term}}. \qquad (41)$$

Resonance case

Equation in resonance case:

$$mu'' + ku = F_0 \cos(\omega_0 t). \tag{42}$$

General solution to (42): given by

$$u = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0}{2m\omega_0} t \sin(\omega_0 t).$$

Remarks: (1) Unbounded response is not physically realistic.
(2) Response remains bounded when damping is considered.