Systems of first order linear equations

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Differential equations - MA 266

Taken from *Elementary differential equations* by Boyce and DiPrima

Outline

Outline

Preliminary remarks

Applications:

- Mechanics
- Electrical networks

Framework for this course:

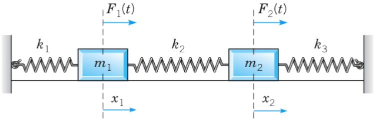
- Linear equations with constant coefficients
- Restriction to 2d systems
- Fundamental tool: linear algebra

Notation:

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = (x_1(t), x_2(t))^T$$

Spring example

Physical setting: Interacting springs



Equation:

$$
\begin{aligned}
m_1 \frac{d^2 x_1}{dt^2} &= k_2(x_2 - x_1) - k_1 x_1 + F_1(t) \\
m_2 \frac{d^2 x_2}{dt^2} &= -k_2(x_2 - x_1) - k_3 x_2 + F_2(t)
\end{aligned}
$$

Second order equation as first order system

Equation:

$$u'' + 0.125u' + u = 0$$

Change of variable: set

$$x_1 = u, \qquad x_2 = u'$$

New equation:

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 - 0.125x_2 \end{aligned}$$

Outline

Transpose, conjugate, adjoint

Generic matrix in $\mathbb{R}^{2\times 2}$:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{or} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{or} \quad \mathbf{A} = (a_{ij})$$

Transposition: interchange columns and lines

$$\mathbf{A}^T = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}$$

Conjugate and adjoint:

$$\bar{\mathbf{A}} = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{pmatrix}, \qquad \mathbf{A}^* = \bar{\mathbf{A}}^T = \begin{pmatrix} \bar{a}_{11} & \bar{a}_{21} \\ \bar{a}_{12} & \bar{a}_{22} \end{pmatrix}$$

Elementary operations on matrices

Addition:

$$\mathbf{A} + \mathbf{B} = (a_{ij}) + (b_{ij}) = (a_{ij} + b_{ij})$$

Addition: for $\alpha \in \mathbb{R}$

$$\alpha\mathbf{A} = \alpha(a_{ij}) = (\alpha a_{ij})$$

Multiplication:

$$\mathbf{C} = \mathbf{AB} \quad \Longrightarrow \quad c_{ij} = \sum_k a_{ik} b_{kj}$$

Rules for multiplications

Rules to follow:

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} \qquad \text{Distributive law}$$
$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C} \qquad \text{Associative law}$$
$$\mathbf{A}\,0 = 0\,\mathbf{A} = 0 \qquad \text{Absorbing state}$$
$$\mathbf{A}\,\mathbf{Id} = \mathbf{Id}\,\mathbf{A} = \mathbf{A} \qquad \text{Identity element}$$

Rule **not** to follow:

- $\mathbf{AB} \neq \mathbf{BA}$ in general.

Example

Matrices:

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$$

Products:

$$\mathbf{AB} = \begin{pmatrix} 0 & 3 \\ 2 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{BA} = \begin{pmatrix} 2 & -2 \\ 1 & -4 \end{pmatrix}$$

Operations on vectors

Inner product: also called scalar product

$$\mathbf{x}^T\mathbf{y} = x_1y_1 + x_2y_2, \quad \text{for} \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^2$$

Norm: also called magnitude,

$$\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}} = \left(x_1^2 + x_2^2\right)^{1/2}$$

Inverse

Definition: for $\mathbf{A}$ such that $\det(\mathbf{A}) \neq 0$,

$$\mathbf{A}^{-1} \quad \text{is the matrix such that} \quad \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\,\mathbf{A}^{-1} = \mathbf{Id}$$

Computation in dimension 2:

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \Longrightarrow \quad \mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Computation in dimension ≥ 2:

- Use Gaussian elimination

Functions

Vector and matrix functions:

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$$

Derivatives:

$$\mathbf{x}' = \begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix}, \quad \text{and} \quad \mathbf{A}' = \begin{bmatrix} a_{11}'(t) & a_{12}'(t) \\ a_{21}'(t) & a_{22}'(t) \end{bmatrix}$$

Rules:

- $(\mathbf{A} + \mathbf{B})' = \mathbf{A}' + \mathbf{B}'$
- $(\mathbf{AB})' = \mathbf{A}'\mathbf{B} + \mathbf{A}\,\mathbf{B}'$ (remember: non commutative product)
- $(\mathbf{CA})' = \mathbf{C}\,\mathbf{A}'$ if $\mathbf{C}$ is constant

Outline

Systems of equations

System of equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 & = b_1 \\ a_{21}x_1 + a_{22}x_2 & = b_2 \end{cases} \quad \text{or} \quad \mathbf{A}\,\mathbf{x} = \mathbf{b}$$

Solution, case **A** nonsingular: if $\det(\mathbf{A}) \neq 0$,

$$x_1 = \frac{1}{\det(\mathbf{A})} \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} \quad \text{or} \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Solution, case **A** singular: if $\det(\mathbf{A}) = 0$, we have 2 cases

1. Infinite number of solutions
2. No solution

Singular homogeneous example

Matrix:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

System:

$$\mathbf{A}\,\mathbf{x} = \mathbf{0}$$

Solving the system:

- $\det(\mathbf{A}) = 0$, thus only 1 condition is enough
- Solution: $x_1 - x_2 = 0$

General solution: for $\alpha \in \mathbb{R}$,

$$\mathbf{x} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Singular non homogeneous example

System: For $\mathbf{b} \in \mathbb{R}^2$,

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \mathbf{x} = \mathbf{b}$$

Augmented matrix:

$$\left(\begin{array}{cc|c} -1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array}\right) \xrightarrow{R_2 := R_2 + R_1} \left(\begin{array}{cc|c} -1 & 1 & b_1 \\ 0 & 0 & b_1 + b_2 \end{array}\right)$$

Singular non homogeneous example (2)

Solution, case 1:

$\hookrightarrow$ If $b_1 + b_2 \neq 0$, there is no solution

Solution, case 1: If (particular example) $\mathbf{b} = (-2, 2)^T$, 4 steps

1. System reduces to $x_1 - x_2 = 2$
2. Particular solution: $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$
3. Solution of homogeneous system: for $\alpha \in \mathbb{R}$,
$$\mathbf{x} = \begin{pmatrix} \alpha \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
4. General solution:
$$\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Linear dependence

Definition:
$x^{(1)}$ and $x^{(2)}$ linearly dependent if there exist $(c_1, c_2) \neq \mathbf{0}$ such that

$$c_1 x^{(1)} + c_2 x^{(2)} = 0$$

Criterion: $x^{(1)}$ and $x^{(2)}$ linearly dependent iff

$$\det(\mathbf{A}) = 0, \quad \text{where} \quad \mathbf{A} = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} \\ x_2^{(1)} & x_2^{(2)} \end{pmatrix}$$

Eigenvalues, eigenvectors

Problem: find $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}^2$ such that

$$(\mathbf{A} - \lambda\, \mathbf{Id})\mathbf{x} = \mathbf{0}$$

Vocabulary: if problem above admits a solution,

- λ is an eigenvalue for $\mathbf{A}$
- $\mathbf{x}$ is an eigenvector for $\mathbf{A}$

Characteristic polynomial: eigenvalues are roots of the polynomial

$$P_{\mathbf{A}}(\lambda) = \det(\mathbf{A} - \lambda\, \mathbf{Id})$$

Example

Matrix:

$$\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

Characteristic polynomial:

$$\det(\mathbf{A} - \lambda \, \mathbf{Id}) = \lambda^2 - \lambda - 2$$

Eigenvalues and eigenvectors:

$$\lambda_1 = 2, \quad \mathbf{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda_1 = -1, \quad \mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

Remark: If $\mathbf{x}$ eigenvector, $\alpha \, \mathbf{x}$ is eigenvector too for any $\alpha \in \mathbb{R}$.

Linearly independent eigenvectors

Distinct eigenvalues case:

- Hypothesis: λ_1, λ_2 eigenvalues, with $\lambda_1 \neq \lambda_2$
- $x^{(1)}, x^{(2)}$ corresponding eigenvectors
- Then $x^{(1)}, x^{(2)}$ are linearly independent

Repeated eigenvalue case:

- Hypothesis: λ_0 double eigenvalue, i.e $P_{\mathbf{A}}(\lambda) = c\,(\lambda - \lambda_0)^2$
- Two subcases:
  1. Nice case: there exist $x^{(1)}, x^{(2)}$ linearly independent eigenvectors
  2. More difficult: only 1 eigenvector $x^{(1)}$

Outline

Aim

Recalling 1d case: $x' = ax$, with $a \in \mathbb{R}$ and $a \neq 0$

- Equilibrium: $x = 0$
- If $a < 0$, then 0 is a stable equilibrium
- If $a > 0$, then 0 is an unstable equilibrium
- Important information given by direction fields

2-dimensional case: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{A} \in \mathbb{R}^{2,2}$ and $\det(\mathbf{A}) \neq 0$

- Equilibrium: $\mathbf{x} = 0$
- Main issue: equilibrium stability
- Important information given by a plot of $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$ in $x_1 x_2$-plane

Toy example

Equation:

$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} \mathbf{x}$$

Decoupled equation:

$$x_1' = 2x_1, \qquad x_2' = -3x_2$$

Solution:

$$x_1 = c_1 e^{2t}, \qquad x_2 = c_2 e^{-3t}$$

Toy example (2)

Vector form:

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}$$

Fundamental solutions:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-3t}.$$

Wronskian:

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{vmatrix} = e^{-t} \neq 0.$$

Conclusion: $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent

Generalization

System: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{A} \in \mathbb{R}^{2,2}$ and $\det(\mathbf{A}) \neq 0$

Strategy:

- Form of the solution: $\mathbf{x} = \boldsymbol{\xi}\, e^{rt}$
- r eigenvalue of $\mathbf{A}$
- $\boldsymbol{\xi}$ eigenvector corresponding to r

3 cases: as for second order differential equations,

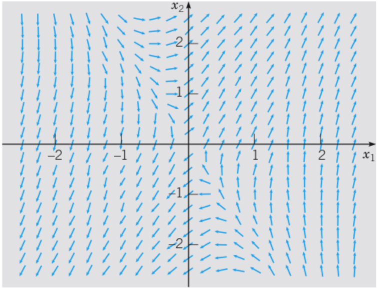
1. 2 real valued eigenvalues
2. 2 complex valued eigenvalues
3. 1 repeated eigenvalue

Example with real eigenvalues

Equation:

$$\mathbf{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \mathbf{x}$$

Direction field: Attracting 1st and 3rd quadrants



Example with real eigenvalues (2)

Eigenvalue decomposition:

$$r_1 = 3, \;\; \xi^{(1)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \qquad r_2 = -1, \;\; \xi^{(2)} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Fundamental solutions:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

Wronskian:

$$W[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}](t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Conclusion: $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ are linearly independent

Example with real eigenvalues (3)

General solution:

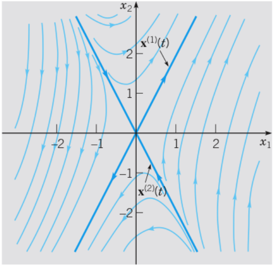
$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

Asymptotic behavior:

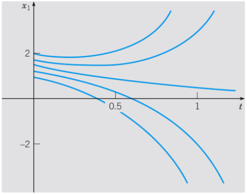
- Dominant term: $c_1 (1,2)^T e^{3t}$ if $c_1 \neq 0$
- We have $\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = 2$ if $c_1 \neq 0$
- Origin is a saddle point: solutions escape from 0 as $t \to \infty$

Example with real eigenvalues (4)

Graph in the x_1x_2 plane:



Graph of $t \mapsto x_1(t)$ for several initial conditions:



Example with negative eigenvalues

Equation:

$$\mathbf{x}' = \begin{pmatrix} -3 & \sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix} \mathbf{x}$$

Characteristic polynomial:

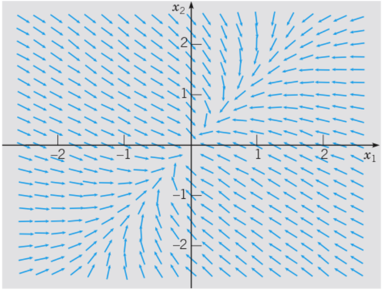
$$P_{\mathbf{A}}(\lambda) = (\lambda + 1)(\lambda + 4)$$

Fundamental solutions:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{-t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix} e^{-4t}.$$

Example with negative eigenvalues (2)

Direction field: Origin is a node

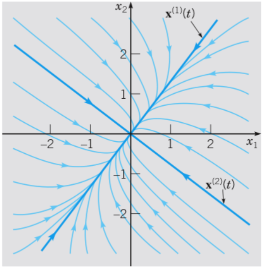


Comments:

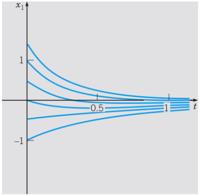
- All solutions approach 0
- In this case, origin is called a node

Example with negative eigenvalues (3)

Graph in the x_1x_2 plane: direction $x_2 = \sqrt{2}x_1$ dominant



Graph of $t \mapsto x_1(t)$ for several initial conditions:



Program for next week

In class:

- Possibly Lesson 35 (Section 7.9)
- Main focus: review problems

Homework:

- Lesson 31 due on $4/20$?
- Luxury HW: 32-33, Ordinary HW: 34-35
- Extra credits on Midterm 2:
  1. Based on a luxury HW: improved redaction and presentation
  2. Possible bonus: $+10$ on Midterm 2
  3. If grade on Midterm $2 = 100$
     $\hookrightarrow$ bonus transferred to Midterm 1 or HW
- Project 3: dropped

Outline

Framework

System: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{A} \in \mathbb{R}^{2,2}$ and $\det(\mathbf{A}) \neq 0$

Strategy:

- Form of the solution: $\mathbf{x} = \xi\, e^{rt}$
- r eigenvalue of $\mathbf{A}$
- ξ eigenvector corresponding to r

3 cases: as for second order differential equations,

1. 2 real valued eigenvalues
2. 2 complex valued eigenvalues, $r = \lambda \pm \imath\mu$
3. 1 repeated eigenvalue

Example with complex eigenvalues

Equation:

$$\mathbf{x}' = \begin{pmatrix} -\frac{1}{2} & 1 \\ -1 & -\frac{1}{2} \end{pmatrix} \mathbf{x}$$

Eigenvalue decomposition:

$$r_1 = -\frac{1}{2} + \imath, \quad \xi^{(1)} = \begin{pmatrix} 1 \\ \imath \end{pmatrix}; \qquad r_2 = -\frac{1}{2} + \imath, \quad \xi^{(2)} = \begin{pmatrix} 1 \\ -\imath \end{pmatrix}$$

Fundamental solutions:

$$\mathbf{x}^{(1)}(t) = \begin{pmatrix} 1 \\ \imath \end{pmatrix} e^{(-\frac{1}{2}+\imath)t}, \qquad \mathbf{x}^{(2)}(t) = \begin{pmatrix} 1 \\ -\imath \end{pmatrix} e^{(-\frac{1}{2}-\imath)t}.$$

Example with complex eigenvalues (2)

Real valued fundamental solutions:

$$\begin{aligned} \mathbf{u}(t) &= \Re \mathbf{x}^{(1)}(t) = \begin{pmatrix} \cos(t) \\ -\sin(t) \end{pmatrix} e^{-\frac{1}{2}t} \\ \mathbf{v}(t) &= \Im \mathbf{x}^{(1)}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} e^{-\frac{1}{2}t}. \end{aligned}$$

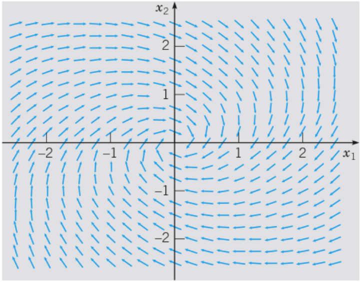
Wronskian:

$$W[\mathbf{u}, \mathbf{v}](t) = \begin{vmatrix} \cos(t)e^{-\frac{1}{2}t} & \sin(t)e^{-\frac{1}{2}t} \\ -\sin(t)e^{-\frac{1}{2}t} & \cos(t)e^{-\frac{1}{2}t} \end{vmatrix} = e^{-t} \neq 0.$$

Conclusion: **u** and **v** are linearly independent

Example with complex eigenvalues (3)

Direction field: Origin is an asymptotically stable spiral point

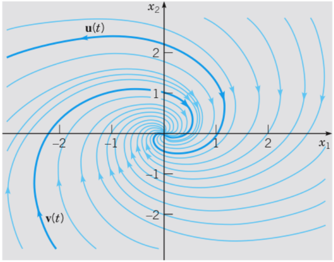


Comments:

- All solutions approach 0
- Infinite number of loops on their way to 0

Example with complex eigenvalues (4)

Graph in the x_1x_2 plane:



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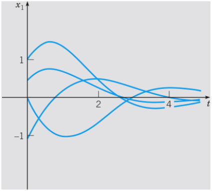


Vocabulary

5 cases for the behavior of $\mathbf{x}$:

1. Real eigenvalues, one of them positive
   $\hookrightarrow$ $\mathbf{0}$ is a saddle point
2. Real eigenvalues, both negative
   $\hookrightarrow$ $\mathbf{0}$ is a node
3. Complex eigenvalues, with positive real part
   $\hookrightarrow$ $\mathbf{0}$ is an unstable spiral point
4. Complex eigenvalues, with negative real part
   $\hookrightarrow$ $\mathbf{0}$ is an asymptotically stable spiral point
5. Complex eigenvalues, vanishing real part
   $\hookrightarrow$ $\mathbf{0}$ is a center (periodic solutions)

Outline

Example of matrix with repeated root

Matrix:

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

Characteristic polynomial:

$$P_{\mathbf{A}}(r) = \det(\mathbf{A} - r\,\mathbf{Id}) = (r-2)^2$$

Eigenvalues and eigenvectors:

$$r = 2, \qquad \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Remark: $r = 2$ is a double eigenvalue, with 1 eigenvector only.

General recipe

System: $\mathbf{x}' = \mathbf{A}\mathbf{x}$, with $\mathbf{A} \in \mathbb{R}^{2,2}$ and $\det(\mathbf{A}) \neq 0$

Situation:

- $\mathbf{A}$ has a double eigenvalue r
- Unique eigenvector $\boldsymbol{\xi}$ (up to constant factor)

First fundamental solution:

- $\mathbf{x}^{(1)} = \boldsymbol{\xi}\, e^{rt}$

Recipe to find the second fundamental solution:

1. Find $\boldsymbol{\eta}$ such that $(\mathbf{A} - r\mathbf{Id})\boldsymbol{\eta} = \boldsymbol{\xi}$
2. Then $\mathbf{x}^{(2)} = \boldsymbol{\xi}\, te^{rt} + \boldsymbol{\eta}\, e^{rt}$

Remark: This is an elaboration of d'Alembert method

Example

Equation:

$$\mathbf{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \mathbf{x}$$

Eigenvalues and eigenvector:

$$r = 2, \qquad \xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

First fundamental solution:

$$\mathbf{x}^{(1)} = e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example (2)

Applying the recipe to find a second fundamental solution:

1. Find $\boldsymbol{\eta}$:

$$(\mathbf{A} - 2\mathbf{Id})\boldsymbol{\eta} = \boldsymbol{\xi} \quad \Longleftrightarrow \quad \eta_1 + \eta_2 = -1$$

We choose: $\boldsymbol{\eta} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

2. We obtain:

$$\mathbf{x}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$$

General solution:

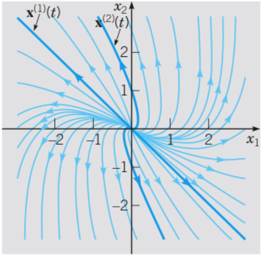
$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} te^{2t} + c_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{2t}$$

Example (3)

Asymptotic behavior: As $t \to \infty$

- $\mathbf{x}(t) \to \infty$
- $\lim_{t\to\infty} \frac{x_2(t)}{x_1(t)} = -1$, thus slope $\simeq -1$
- $\mathbf{x}(t)$ does not approach the asymptote

Graph in the $x_1 x_2$ plane:



Outline

Setting

Equation: For $\mathbf{g} = \mathbf{g}(t)$ we set

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{g} \tag{1}$$

Hypothesis:

- $\mathbf{A}$ admits eigenvalues r_1 and r_2
- Eigenvectors $\xi^{(1)}$ and $\xi^{(2)}$

Eigenvector matrix: We set

$$\mathbf{T} = \left[\boldsymbol{\xi}^{(1)}\ \boldsymbol{\xi}^{(2)}\right]$$

General method

Proposition 1.

Under the previous assumptions, set

$$\mathbf{x} = \mathbf{T}\,\mathbf{y}$$

Then $\mathbf{y}$ solves

$$\mathbf{y}' = \mathbf{D}\,y + \mathbf{h},$$

where

$$\mathbf{D} = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad \text{and} \quad \mathbf{h} = \mathbf{T}^{-1}\mathbf{g}$$

Advantage:
This system is as simple as the toy example

Example (1)

System:

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{g}$$

Eigenvalues:

$$r_1 = -3, \quad \text{and} \quad r_2 = -1$$

Eigenvectors:

$$\xi^{(1)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \xi^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Eigenvector matrix:

$$\mathbf{T} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{T}^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Example (2)

Diagonal system: Set $\mathbf{x} = \mathbf{T}\,\mathbf{y}$. Then

$$\mathbf{y}' = \mathbf{D}\mathbf{y} + \mathbf{h}$$

with

$$\mathbf{D} = \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{h} = \begin{pmatrix} e^{-t} - \frac{3}{2}t \\ e^{-t} + \frac{3}{2}t \end{pmatrix}$$

System for coordinates: We are back to the toy example

$$\begin{cases} y_1' + 3y_1 & = e^{-t} - \frac{3}{2}t \\ y_1' + y_1 & = e^{-t} + \frac{3}{2}t \end{cases}$$

Example (3)

Solving for $\mathbf{y}$: With integrating factor method we get

$$\begin{cases} y_1 &= -\frac{t}{2} + \frac{1}{6} + \frac{1}{2}e^{-t} + c_1 e^{-3t} \\ y_2 &= \frac{3t}{2} - \frac{3}{2} + te^{-t} + c_2 e^{-t} \end{cases}$$

Solving for $\mathbf{x}$: We write $\mathbf{x} = \mathbf{T}\mathbf{y}$, which yields

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} + \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$