Review problems

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Probability - MA 416

Mostly taken from *A first course in probability* by S. Ross

Outline

1. Central limit theorem

2. Joint random variables

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1. Central limit theorem

2. Joint random variables

Problem 8.6 (1)

Situation: We have

- A die is continually rolled until the total sum of all rolls exceeds 300
- Rolls are independent, each uniformly distributed on $\{1, 2, 3, 4, 5, 6\}$

Question:
Approximate the probability that at least 80 rolls are necessary.

Problem 8.6 (2)

Model: Set

$$X_i \equiv \text{result of } i\text{-th roll}$$

The event "at least 80 rolls are necessary" is $\left\{\sum_{i=1}^{79} X_i \leq 300\right\}$, so we wish to find

$$\mathbf{P}\left(\sum_{i=1}^{79} X_i \leq 300\right)$$

Law of X_i: We have

- X_i's i.i.d., each uniform on $\{1, 2, 3, 4, 5, 6\}$
- $\mathbf{E}[X_i] = \frac{7}{2}$, and $\mathbf{Var}(X_i) = \sigma^2 = \frac{35}{12}$

Problem 8.6 (3)

Application of CLT: Write

$$\begin{aligned}
\mathbf{P}\left(\sum_{i=1}^{79} X_i \leq 300\right) &= \mathbf{P}\left(\bar{X}_{79} \leq \frac{300}{79}\right) \\
&= \mathbf{P}\left(\sqrt{79}\,\frac{\bar{X}_{79} - \frac{7}{2}}{\sigma} \leq \frac{300 - 79 \cdot \frac{7}{2}}{\sigma\sqrt{79}}\right) \\
&\overset{CLT}{\simeq} \Phi\left(\frac{23.5}{\sqrt{79 \cdot \frac{35}{12}}}\right) \\
&= \Phi(1.55) \simeq .94
\end{aligned}$$

Problem 8.10 (1)

Situation: We consider

- $W \sim \mathcal{N}(400, 40^2)$: weight (in 1000 lbs) a bridge span can withstand without structural damage
- Each car has weight X_i (in 1000 lbs) with

$$\text{Mean} = 3, \text{ and Standard deviation} = 0.3$$

Question:

Approximately how many cars must be on the bridge for the probability of structural damage to exceed .1?

Problem 8.10 (2)

Model: Set

$$X_i \equiv \text{weight of } i\text{-th car}, \qquad W \equiv \text{bridge capacity}$$

We wish to find n minimal such that

$$\mathbf{P}\left(\sum_{i=1}^{n} X_i > W\right) > .1$$

Law of X_i and W: We have

- X_i's i.i.d., independent of W
- $\mathbf{E}[X_i] \equiv \mu = 3, \quad \text{and} \quad \sqrt{\mathbf{Var}(X_i)} \equiv \sigma = .3$
- $W \sim \mathcal{N}(400, 40^2)$

Problem 8.10 (3)

Application of CLT: Set

$$S_n = \sum_{i=1}^{n} X_i - W$$

By CLT, S_n is approx. normal with

$$\mathbf{E}[S_n] = 3n - 400, \qquad \mathbf{Var}(S_n) = 0.09n + 1600$$

Problem 8.10 (4)

Application of CLT -ctd: Write

$$\begin{aligned}
\mathbf{P}\left(\sum_{i=1}^{n} X_i > W\right) &= \mathbf{P}\left(S_n > 0\right) \\
&= \mathbf{P}\left(\frac{S_n - (3n - 400)}{\sqrt{0.09n + 1600}} > \frac{400 - 3n}{\sqrt{0.09n + 1600}}\right) \\
&\overset{CLT}{\simeq} 1 - \Phi\left(\frac{400 - 3n}{\sqrt{0.09n + 1600}}\right)
\end{aligned}$$

Problem 8.10 (4)

Reading the Gaussian table: Since

$$\mathbf{P}(Z > 1.28) \simeq .1,$$

we are looking for n minimal such that

$$\frac{400 - 3n}{\sqrt{0.09n + 1600}} < 1.28$$

Approximation: Since $0.09n \ll 1600$ for relevant n, we use $\sqrt{0.09n + 1600} \simeq 40$, giving

$$400 - 3n < 51.2$$

This yields

$$n \geq 117$$

Outline

1 Central limit theorem

2 Joint random variables

Problem 6.23 (1)

Situation: X and Y have joint density function

$$f(x,y) = 12\,x\,y\,(1-x), \quad 0 < x < 1, \quad 0 < y < 1$$

Questions:

1. Are X and Y independent?
2. Find $\mathbf{E}[X]$ and $\mathbf{E}[Y]$
3. Find $\mathbf{Var}(X)$ and $\mathbf{Var}(Y)$

Problem 6.23 (2)

Marginal densities:

$$
\begin{aligned}
f_X(x) &= \int_0^1 12\,xy(1-x)\,dy = 12\,x(1-x)\int_0^1 y\,dy = 6\,x(1-x) \\
f_Y(y) &= \int_0^1 12\,xy(1-x)\,dx = 12\,y\int_0^1 x(1-x)\,dx = 2\,y
\end{aligned}
$$

Independence (part 1): Since

$$
f(x,y) = 12\,xy(1-x) = \underbrace{6x(1-x)}_{f_X(x)} \cdot \underbrace{2y}_{f_Y(y)}
$$

X and Y are independent.

Problem 6.23 (3)

Expectations (part 2):

$$\mathbf{E}[X] = \int_0^1 x \cdot 6x(1-x)\, dx = 6 \int_0^1 \left(x^2 - x^3\right) dx = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

$$\mathbf{E}[Y] = \int_0^1 y \cdot 2y\, dy = 2 \int_0^1 y^2\, dy = \frac{2}{3}$$

Problem 6.23 (4)

Second moments (part 3):

$$\mathbf{E}[X^2] = \int_0^1 x^2 \cdot 6x(1-x)\, dx = 6\left(\frac{1}{4} - \frac{1}{5}\right) = \frac{3}{10}$$

$$\mathbf{E}[Y^2] = \int_0^1 y^2 \cdot 2y\, dy = 2 \cdot \frac{1}{4} = \frac{1}{2}$$

Variances:

$$\mathbf{Var}(X) = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$$

$$\mathbf{Var}(Y) = \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$$

Problem 6.24 (1)

Situation: Consider

- Independent trials, each with outcome $i \in \{0, 1, \dots, k\}$
- $\mathbf{P}(\text{Outcome}_k = i) = p_i$, with $\sum_{i=0}^{k} p_i = 1$
- N = number of trials needed to get outcome $\neq 0$
- X = that first non-zero outcome

Questions:

1. Find $\mathbf{P}(N = n)$, $n \geq 1$
2. Find $\mathbf{P}(X = j)$, $j = 1, \dots, k$
3. Show $\mathbf{P}(N = n, X = j) = \mathbf{P}(N = n)\,\mathbf{P}(X = j)$
4. Is it intuitive that N and X are independent of each other?

Problem 6.24 (2)

Distribution of N (part 1): $\{N = n\}$ means the first $n-1$ trials gave outcome 0 and the n-th did not, so

$$\mathbf{P}\{N = n\} = p_0^{n-1}(1 - p_0), \quad n \geq 1$$

Distribution of X (part 2): By total probability

$$\begin{aligned} \mathbf{P}\{X = j\} &= \sum_{n=1}^{\infty} \mathbf{P}\{N = n, X = j\} = \sum_{n=1}^{\infty} p_0^{n-1}\, p_j \\ &= p_j \sum_{n=1}^{\infty} p_0^{n-1} = \frac{p_j}{1 - p_0} \end{aligned}$$

Problem 6.24 (3)

Independence (part 3): Directly

$$\mathbf{P}\{N = n,\, X = j\} = p_0^{n-1}\, p_j$$

and

$$\mathbf{P}\{N = n\}\,\mathbf{P}\{X = j\} = p_0^{n-1}(1 - p_0) \cdot \frac{p_j}{1 - p_0} = p_0^{n-1}\, p_j$$

so N and X are independent.

Intuition (part 4):

- N encodes only how long we waited (how many 0's): it carries no information about which non-zero value appeared
- X encodes only which non-zero outcome appeared: it carries no information about the waiting time