## Answer Key

## Spring 2018, MA/STAT 519 Mid term Exam

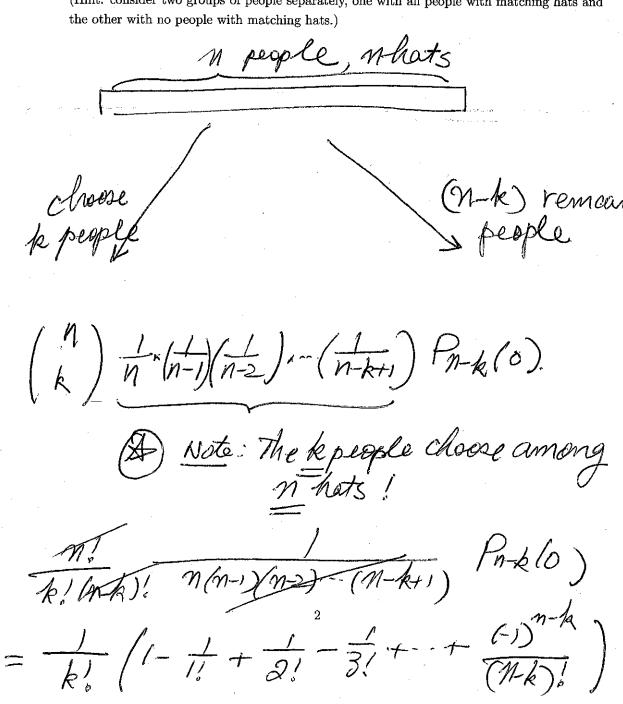
1. Consider the matching-hat-problem for n people with n hats. All the people throw their hats into a container and then each of them picks one at random. Let  $P_n(0)$  be the probability that no people get their own hats back. In class, it was derived that

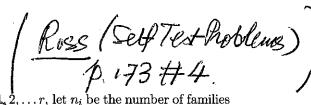
$$P_n(0) = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}.$$

Now let  $P_n(k)$  be the probability of exactly k  $(k \leq n)$  people get their own hats back. From that

$$P_n(k) = \frac{1}{k!} \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^{(n-k)}}{(n-k)!} \right).$$

(Hint: consider two groups of people separately, one with all people with matching hats and the other with no people with matching hats.)





2. A certain community consists of m families. For i = 1, 2, ..., r, let  $n_i$  be the number of families with i children so that  $\sum_{i=1}^{r} n_i = m$  and  $\sum_{i=1}^{r} i n_i$  gives the total number of children in the community.

If one of the families is randomly chosen, let X be the number of children in that family. Alternatively, if one of the children in the whole community is randomly chosen, let Y be the total number of children in the family of that child.

Which of E(X) and E(Y) is bigger? Prove your statement rigorously, i.e. mathematically. (Hint: simplify, expand, and compare terms.)

$$EX = \sum_{i=1}^{r} \tau P(X=i) = \sum_{i=1}^{r} \frac{i m_i}{m}$$

$$=\left(\frac{\sum_{i=1}^{N}in_{i}}{m}\right)$$

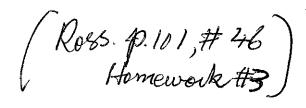
$$\pm Y = \sum_{i=1}^{Y} i P(Y=i) = \sum_{i=1}^{Y} \frac{i m_i}{\sum_{i=1}^{Y} i n_j}$$

Because children from Cargor families are more likely to be picked

$$=\frac{\sum_{i=1}^{r}iM_{i}}{\left(\sum_{j=1}^{r}in_{j}\right)}$$

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$$\frac{\sum_{i} i m_{i}}{\sum_{i} m_{i}} \stackrel{?}{=} \frac{\sum_{i} m_{i}}{\sum_{i} m_{i$$



- 3. In any given year, a male automobile policyholder will make a claim with probability  $p_M$ and a female automobile policyholder will make a claim with probability  $p_F$ . Let  $\alpha$  be the fraction of male policyholders (and  $(1-\alpha)$  be the fraction of female policyholders). Now a policyholder is randomly chosen and fixed for the following consideration. Let  $A_i$  denote the event that this person makes a claim during the i-th year.
  - (a) Suppose  $p_M \neq p_F$ . Prove that  $P(A_2|A_1) > P(A_1)$ . Are  $A_1$  and  $A_2$  independent?
  - (b) Suppose  $p_M > p_F$ . Let M be the event that the policyholder is a male. Prove that  $P(M|A_1) > P(M).$

P(AZNA, /M) P(M) + P(AZNA, /F) P(F)
P(A, /M) P(M) + P(A, /F) P(F)

PM X+ P= (1-X) ? PM X+ PF(FX)

1=> PM x+ PF(1-x) > (PM x+ PF(1-x))

=> Pm (x-x2)+ PF (1-x)-(1-x3)-2PmPFX(1-x)

> pm (xfras) + P= (1-a)(x) - 2PmPFX(+a)

(PN-PF) >0

A, Az are not independent as P(Az/Ai) FP(Ai)

You can use this blank page. > P(M) P(M/A) P(A,/M)P(M) Pm > P(A) PM > PMX+ PF(+X) PM (1-00) > PF(1-00) PM > PF

4. Let X be a positive-integer-valued random variables, i.e. it takes values in  $\{1, 2, 3, \ldots\}$ . X is said to enjoy the memoryless property if for any positive integers m, n, it holds that

$$P(X > m + n | X > m) = P(X > n).$$

Prove that X enjoys the memoryless property if and only if X is a geometric random variable, i.e. there is a p (with  $0 ) such that <math>P(X = i) = p(1 - p)^{i-1}$ .

Note: you need to prove two separate statements: geometric implies memoryless and memoryless implies geometric. For simplicity, just ignore the "degenerate" case p=0 or p=1.

 $P(X>m+n|X>m) = \frac{P(X>m+n,X>m)}{P(X>m)}$ 

 $=\frac{P(X>m+n)}{P(X>m)}$ 

Hence memoryles property is equivalent to: P(X>m+n)=P(X>n)P(X>m)

(1) geometric > memoryless.

P(X>n) = P(M) 1st n trials give 0)

 $(ov, P(X>n) = \sum_{i=n+1}^{\infty} P(X=i) - \sum_{i=n+1}^{\infty} pg^{i}$ 

= pgn = gn = gn)

Hence, Similarly, P(X>m)=gm You can use this blank page. P(X > m + n) = P(X > m + n)Honote gurn gn. gn Hence menongless property-holds. (2) Memonfler -> geome sic P(X>mm)>P(X>m)P(X>n) n=1+ (n-1) In particular, P(X>n) P(X>1) P(X>n-1) = P(X>1) P(X> n-2) Let p= P(X=1) =P(X>15P(X>n-3) Then P(X>1)=1-P(X=1) = P(X>1) P(X>1)  $= P(X>1)^n$ P(X=n) = P(X>n-1) - P(X>n) $= (1-p)^{n-1} - (7-p)^n = (1-p)^{n-1}(1-1+p)$   $= (1-p)^{n-1}p$ 

