MA/STAT 532 Spring 2024
Stochastic Processes

Midterm Exam

- You can use a calculator.
- A 2 pages long handwritten cheat sheet is allowed. It should only contain formulae and theorems (no example, no solved problem).
- You have 50 minutes.
- Show your work.
- In order to get full credits, you need to give correct and simplified answers and explain in a comprehensible way how you arrive at them.
- GOOD LUCK!

Name:

Problem 1. We consider a random walk X on $\mathbb{Z}^2$. Namely $X_0 = (0,0)$, and for $n \geq 1$ we have

$$X_n = \sum_{k=1}^{n} Z_k ,$$

where $\{Z_k;\, k \geq 1\}$ is a sequence of independent and identically distributed random variables with

$$\mathbf{P}\left(Z_k = (-1,0)\right) = \mathbf{P}\left(Z_k = (1,0)\right) = \mathbf{P}\left(Z_k = (0,-1)\right) = \mathbf{P}\left(Z_k = (0,1)\right) = \frac{1}{4} .$$

Since both X_n and Z_k take values in $\mathbb{Z}^2$, we will write

$$X_n = \left(X_n^1, X_n^2\right), \quad \text{and} \quad Z_k = \left(Z_k^1, Z_k^2\right).$$

1.1. For a given $k \geq 1$, prove that Z_k^1 and Z_k^2 are not independent.

Solution:

1.2. For a given $k \geq 1$, we set

$$Y_k^1 = Z_k^1 + Z_k^2, \quad \text{and} \quad Y_k^2 = Z_k^1 - Z_k^2.$$

Prove that (Y_k^1, Y_k^2) is a couple of independent random variables such that for $j = 1, 2$ we have

$$\mathbf{P}\left(Y_k^j = -1\right) = \mathbf{P}\left(Y_k^j = 1\right) = \frac{1}{2}.$$

Solution:

1.3. Recall that $X_n = \sum_{k=1}^{n} Z_k$. We now set

$$U_n = X_n^1 + X_n^2, \quad \text{and} \quad V_n = X_n^1 - X_n^2.$$

Show that $U = \{U_n;\, n \geq 1\}$ and $V = \{V_n;\, n \geq 1\}$ are two independent symmetric random walks starting at 0.

Solution:

1.4. Let T_1 be the random time defined by

$$T_1 = \inf\{n \geq 1;\, U_n = 1\}.$$

Prove that the probability generating function G_1 of T_1 has the expression

$$G_1(s) = \frac{1 - (1 - s^2)^{1/2}}{s}.$$

Please don't use directly the result from class, it is expected that you prove the above formula carefully. However, you can resort to the following identity: if $T_2 = \inf\{n \geq 1;\, U_n = 2\}$, and if we set

$$f_1(n) = \mathbf{P}(T_1 = n), \quad f_2(n) = \mathbf{P}(T_2 = n), \quad F_1(s) = \sum_{n=1}^{\infty} f_1(n) s^n, \quad F_2(s) = \sum_{n=1}^{\infty} f_2(n) s^n,$$

then we have

$$F_2(s) = (F_1(s))^2.$$

Solution:

1.5. Let D_1 be the line

$$D_1 = \inf\left\{(x, y) \in \mathbb{R}^2;\ x + y = 1\right\}.$$

We wish to get some information about the random time

$$\hat{T}_1 = \inf\left\{n \geq 1;\ X_n \in D_1\right\}.$$

Prove that $\hat{T}_1 = T_1$, and deduce an expression for the probability generating function $\hat{G}_1$ of the random variable $\hat{T}_1$.

Solution:

1.6. Show that $\hat{T}_1$ is a finite random variable, namely

$$\mathbf{P}\left(\hat{T}_1 = \infty\right) = 0.$$

Solution:

Problem 2. Let $\{U_n;\, n \geq 1\}$ be a sequence representing successive dice rolls. That is the U_n's are independent uniform random variables in $\{1, \ldots, 6\}$. Prove that the following processes are Markov chains and specify their transition matrix. Be sure to specify the state space S for each case.

2.1. $X_n \equiv$ largest roll U_j shown up to n-th roll.

Solution:

2.2. $Y_n \equiv$ Number of sixes in the first n rolls.

Solution: