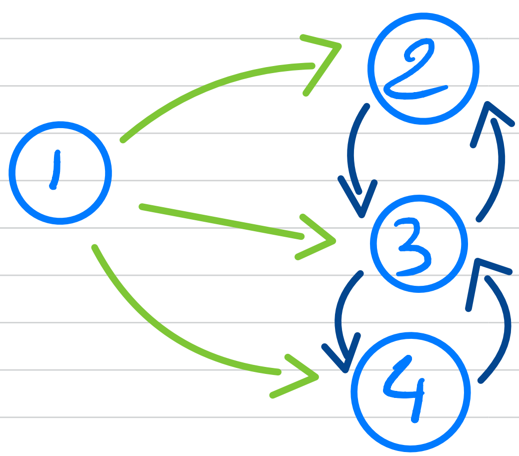
Chain on $S = \{1, 2, 3, 4\}$

Monday: what else can we say with π?

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



classes

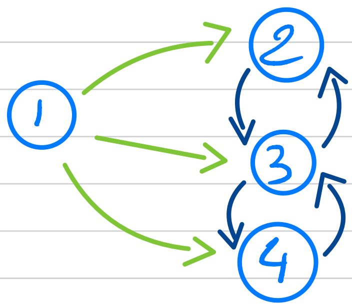
$C_1 = \{1\}$ not closed

$C_2 = \{2, 3, 4\}$ closed

Classification of states

1 is transient 2, 3, 4 are persistent

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



Question: can we directly apply Thm 35 ?

Hyp for Thm 35

- Transition P
- $\times$ irreducible $\longrightarrow$ not satisfied
- stat distribution π

From decomposition thm: Here we have

$T = \{1\}$, $C_1 = \{2,3,4\}$ closed

Strategy: Look at the "subchain" on $\{2,3,4\}$ with transition

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

Study of the subchain

This chain is irreducible.

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

Question: do we have an invariant ~~measure~~ distribution

$$\pi = [\pi_2, \pi_3, \pi_4]$$

s.t. $\pi Q = \pi$, i.e

$$\begin{cases} \pi_2 = \frac{1}{2}\pi_3 \\ \pi_3 = \pi_2 + \pi_4 \\ \pi_4 = \frac{1}{2}\pi_3 \end{cases}$$

Solution of the form

$$\pi = [a, 2a, a]$$

If π is a distribution we get

$$4a = 1 \iff a = \frac{1}{4}$$

Unique invariant probability distribution for Q

$$\pi = [\,1/4 \quad 1/2 \quad 1/4\,]$$

Application of Thm 35

$$\boxed{\begin{aligned} E[T_2 \mid X_0 = 2] &= \frac{1}{\pi_2} = 4 \\ E[T_3 \mid X_0 = 3] &= \frac{1}{\pi_3} = 2 \\ E[T_4 \mid X_0 = 4] &= 4 \end{aligned}}$$

Back to Markov chain X

First way: We have seen that 1 is transient

$$P = \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$\Rightarrow \quad E[T_1 \mid X_0 = 1] = \infty$

$\Rightarrow$ We guess $\pi_1 = 0$

Second way: Solve the linear system

$\pi P = \pi$ with $\pi = [\pi_1 \ \pi_2 \ \pi_3 \ \pi_4]$

Extra equation for π_1: $\pi_1 = \frac{1}{4}\pi_1 \Rightarrow \pi_1 = 0$

Other equation now unchanged

$\Rightarrow$ inv measure $\pi = [0, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$

Example (2)

Related classes:
$C_1 = \{1\}$, $C_2 = \{2, 3, 4\}$
$\hookrightarrow$ C_1 closed C_2 non closed.

Partial conclusion: C_1 transient, at least one recurrent state in C_2.

Invariant measure:
Solve the system $\pi = \pi P$ and $\langle \pi, \mathbf{1} \rangle = 1$. We find

$$\pi = (0, 1/4, 1/2, 1/4) .$$

Conclusion: All states in C_2 are non-null persistent

Example (3)

Remark:

- It is almost always easier to solve the system

$$\pi = \pi\, p \quad \text{and} \quad \langle \pi, \mathbf{1} \rangle = 1$$

than to compute $\mathbf{E}_i[T_i]$

- However, in the current case a direct computation is possible

$T_3 = \inf\{n \geq 1;\ X_n = 3\}$

Computation on the subchain

$$Q = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

value of T_3

$$P(T_3 = 2 \mid X_0 = 3) \Rightarrow E[T_3 \mid X_0 = 3] = 2$$

Analysis for T_2

(i) T_2 can only take even values

(ii) $P(T_2 > 2k+2 \mid X_0 = 2)$

$P(A \cap B \mid C)$
$= P(A \mid B \cap C)\, P(B \mid C)$

$$= P(X_{2k+2} = 4, X_{2k} = 4, \ldots, X_2 = 4 \mid X_0 = 2)$$

$$= P(X_{2k+2} = 4 \mid X_{2k} = 4, \ldots, X_2 = 4, X_0 = 2)$$

$$P(X_{2k} = 4, \ldots, X_2 = 4 \mid X_0 = 2)$$

Markov

$$= P(X_{2k+2} = 4 \mid X_{2k} = 4) \quad P(T_2 > 2k \mid X_0 = 2)$$

Summary We have found

$$P(T_2 > 2k+2 \mid X_0 = 2)$$

$$= P(X_{2k+2} = 4 \mid X_{2k} = 4) \quad P(T_2 > 2k \mid X_0 = 2)$$

$$= \underbrace{P_{43}}_{1} \underbrace{P_{34}}_{1/2} \quad P(T_2 > 2k \mid X_0 = 2)$$

$$\Rightarrow \quad P(T_2 > 2k+2 \mid X_0 = 2) = \frac{1}{2} P(T_2 > 2k \mid X_0 = 2)$$

(geometric induction)

Conclusion we find $\boxed{T_2 \sim 2 \times \mathcal{G}(\tfrac{1}{2})}$

(check)

$$\Rightarrow \quad E[T_2 \mid X_0 = i] = 2 \times 2 = 4$$

Example (4)

Direct analysis: We find

- $\mathbf{E}_1[T_1] = \infty$ since 1 is transient
- $\mathbf{E}_3[T_3] = 2$ since $T_3 = 2$ under $\mathbf{P}_3$.
- In order to compute $\mathbf{E}_2[T_2]$:

$$\begin{aligned}
\mathbf{E}_2\left[\mathbf{1}_{(T_2>2k+2)}\right] &= \mathbf{E}_2\left[\mathbf{1}_{(T_2>2k)}\,\mathbf{1}_{(T_2>2k+2)}\right] \\
&= \mathbf{E}_2\left\{\mathbf{1}_{(T_2>2k)}\,\mathbf{E}_{X_{2k}}\left[\mathbf{1}_{(T_2(A^{2k})>2)}\right]\right\} \\
&= \mathbf{E}_2\left\{\mathbf{1}_{(T_2>2k)}\,\mathbf{E}_4\left[\mathbf{1}_{(T_2(A^{2k})>2)}\right]\right\} \\
&= \mathbf{E}_2\left[\mathbf{1}_{(T_2>2k)}\,p_{4,3}\,p_{3,4}\right] = \frac{1}{2}\mathbf{E}_2\left[\mathbf{1}_{(T_2>2k)}\right]
\end{aligned}$$

We deduce $\mathbf{P}_2(T_2 > 2k) = 1/2^k$ and $\mathbf{E}_2[T_2] = 4 = \mathbf{E}_4[T_4]$.