Some problems from Grimmet-Stirzacker

6-1-6. **Strong Markov property.** Let X be a Markov chain on S, and let T be a random variable taking values in $\{0, 1, 2, \dots\}$ with the property that the indicator function $I_{\{T=n\}}$, of the event that $T = n$, is a function of the variables $X_1, X_2, \dots, X_n$. Such a random variable T is called a *stopping time*, and the above definition requires that it is decidable whether or not $T = n$ with a knowledge only of the past and present, $X_0, X_1, \dots, X_n$, and with no further information about the future.

Show that

$$\mathbb{P}\big(X_{T+m} = j \,\big|\, X_k = x_k \text{ for } 0 \le k < T,\ X_T = i\big) = \mathbb{P}(X_{T+m} = j \mid X_T = i)$$

Stopping time One way to express that T is a stopping time is to write

$$\mathbb{1}_{(T=n)} = \varphi_n(X_0, \dots, X_n),$$

where φ_n is a given function.

φ_n as indicator we have

$$\varphi_n : S^{n+1} \longrightarrow \{0,1\}$$

In fact

$$\varphi_n(x_0, \dots, x_n) = \mathbb{1}(x_0 \in A_0) \times \dots \times \mathbb{1}(x_n \in A_n)$$

for sets $A_0, \dots, A_n \subset S$

Conditional probability we wish to evaluate

$$Q_m \equiv \mathbb{P}(X_{T+m} = j \mid X_0 = i_0, \dots, X_T = i)$$

$$= \frac{\mathbb{E}[\mathbb{1}(X_{T+m} = j)\, \mathbb{1}(X_T = i) \cdots \mathbb{1}(X_0 = i_0)]}{\mathbb{P}(X_T = i, \dots, X_0 = i_0)}$$

$$= \frac{N_m}{D}$$

Term N_m we have

$$N_m = \mathbb{E}[\mathbb{1}(X_{T+m} = j)\, \mathbb{1}(X_T = i) \cdots \mathbb{1}(X_0 = i_0)]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}[\mathbb{1}(T = n)\, \mathbb{1}(X_{T+m} = j)\, \mathbb{1}(X_T = i) \cdots \mathbb{1}(X_0 = i_0)]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}[\varphi_n(X_0, \dots, X_n)\, \mathbb{1}(X_{n+m} = j)\, \mathbb{1}(X_n = i) \cdots \mathbb{1}(X_0 = i_0)]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}[\mathbb{1}(X_0 \in A_0) \cdots \mathbb{1}(X_n \in A_n) \cdot \mathbb{1}(X_0 = i_0) \cdots \mathbb{1}(X_n = i)$$

$$\times\ \mathbb{1}(X_{n+m} = j)]$$

Simplification for N_m we have found

$$N_m$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\left[\mathbb{1}_{(X_0 \in A_0)} \cdots \mathbb{1}_{(X_n \in A_n)} \cdot \mathbb{1}_{(X_0 = i_0)} \cdots \mathbb{1}_{(X_n = i)} \times \mathbb{1}_{(X_{n+m} = j)} \right]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\left[\mathbb{1}_{(X_0 \in (A_0 \cap \{i_0\}))} \cdots \mathbb{1}_{(X_n \in A_n \cap \{i\})} \mathbb{1}_{(X_{n+m} = j)} \right]$$

Markov

$$= \left(\sum_{n=0}^{\infty} \mathbb{E}\left[\mathbb{1}_{(X_0 \in (A_0 \cap \{i_0\}))} \cdots \mathbb{1}_{(X_n \in A_n \cap \{i\})} \right] \right) \times p_{ij}(m)$$

$$= \mathbb{E}\left[\mathbb{1}_{(X_0 = i_0)} \cdots \mathbb{1}_{(X_T = i)} \right] p_{ij}(m)$$

$$= D \quad p_{ij}(m)$$

Conclusion we have seen

$$\mathbb{P}(X_{T+m} = j \mid X_0 = i_0, \ldots, X_T = i)$$

$$= \frac{N_m}{D} = \frac{D \; p_{ij}(m)}{D} = p_{ij}(m)$$

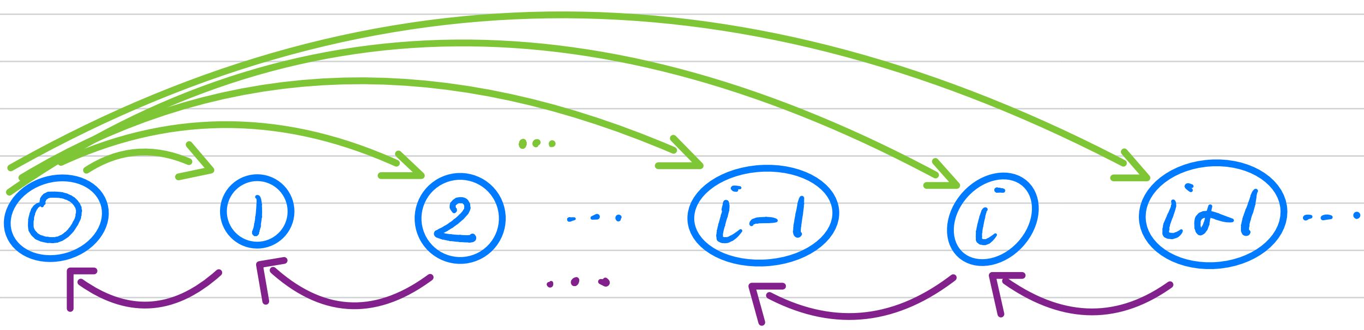
↳ Markov property for $\{X_{T+m} ; m \geq 0\}$

$\boxed{6\text{-}3\text{-}1.}$ Let X be a Markov chain on $\{0, 1, 2, \dots\}$ with transition matrix given by $p_{0j} = a_j$ for $j \geq 0$, $p_{ii} = r$ and $p_{i,i-1} = 1 - r$ for $i \geq 1$. Classify the states of the chain, and find their mean recurrence times.

Rmk We focus on non-degenerate cases, that is

$r \in (0,1)$, $a_j > 0$ for all j

Graph of X of the form



This chain is irreducible. Thus

(i) If 1 state is transient, all states are transient

(ii) If 1 stat is null persistent, all states are null persistent

Decomposition for T_0 starting from $X_0 = 0$ we have

$$T_0 = \mathbb{1}_{(X_1=0)} + \sum_{i=1}^{\infty} (1 + T_{i,0}) \mathbb{1}_{(X_1=i)}$$

where

$T_{i,0}(X_1)$ = Time to go from i to 0 starting from $X_1 = i$

Decomposition for $E[T_0]$ we get

$$E[T_0] = P(X_1 = 0) \qquad (1)$$

$$+ \sum_{i=1}^{\infty} E[(1 + T_{i,0}(X_1)) \mid X_1 = i] \; P(X_1 = i)$$

$$= r + \sum_{i=1}^{\infty} (1 + E[T_{i,0} \mid X_0 = i]) \, a_i$$

Decomposition for $T_{i,0}$ If $X_0 = i$, we have

$$T_{i,0} = T_{i,i-1} + \cdots + T_{1,0}$$

where $T_{j,j-1}$ i.i.d $G(1-r)$

Formula for $E[T_{i,0} \mid X_0 = i]$ we get

$$E[T_{i,0} \mid X_0 = i] = \sum_{j=1}^{i} E[T_{j,j-1} \mid X_0 = j]$$

$$= \sum_{j=1}^{i} \frac{1}{1-r}$$

$$= \frac{i}{1-r}$$

Formula for $E[T_0]$ Going back to (1) we obtain

$$E[T_0] = r + \sum_{i=1}^{\infty} \left(1 + \frac{i}{1-r}\right) a_i$$

$$= \underbrace{\left(r + \sum_{i=1}^{\infty} a_i\right)}_{=1} + \frac{1}{1-r} \sum_{i=1}^{\infty} i\, a_i$$

Thus

$$\boxed{E[T_0] = 1 + \frac{1}{1-r} \sum_{i=1}^{\infty} i\, a_i}$$

This is finite iff $\sum_{i=1}^{\infty} i\, a_i < \infty$