

Outline

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2 Modes of convergence

- 2.1 Reviewing the modes of convergence
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- 2.3 Results for almost sure convergence
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Probability space

Probability space: $(\Omega, \mathcal{F}, \mathbf{P})$ with

- Ω set
- \mathcal{F} a σ -algebra or σ -field
- \mathbf{P} probability measure

Prob space : $(\Omega, \mathcal{F}, \mathbb{P})$

\mathcal{F} : σ -algebra, i.e. \mathcal{F} is a collection of subsets of Ω s.t.

(i) $\Omega \in \mathcal{F}$, $\emptyset \in \mathcal{F}$ or \bar{A}

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(iii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$

↳ In fact if $\{A_i; i \geq 1\}$ we want

$$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}, \quad \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$$

About \mathbb{P} : \mathbb{P} is a probability measure,

$$\mathbb{P}: \mathcal{F} \xrightarrow{\tau-e} [0,1] \quad \text{s.t.}$$

$$(i) \quad \mathbb{P}(\emptyset) = 0 \quad \mathbb{P}(\Omega) = 1$$

(ii) If $\{A_i; i \geq 1\}$ are disjoint,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

$$(iii) \quad \mathbb{P}(A^c) = 1 - \mathbb{P}(A) \quad \forall A \in \mathcal{F}$$

Rmk If $A \in \mathcal{F}$, we say that A is measurable

Example 1 Tossing 2 dice . Then

$$\Omega = \{1, \dots, 6\} \times \{1, \dots, 6\} = \{1, \dots, 6\}^2$$

Since Ω is finite (or countable),
we can take

$$\begin{aligned} \mathcal{F} &= \mathcal{P}(\Omega) \equiv \{ \text{all subsets of } \Omega \} \\ &= \{ \emptyset, \{(1,1)\}, \{(1,2)\} \dots \{(1,1), (1,2)\} \dots \} \end{aligned}$$

Question : $|\Omega| \equiv \# \text{ elements in } \Omega = 36$
 $|\mathcal{F}| = 2^{36}$

On $(\Omega, \mathcal{F}) = (\{1, \dots, 6\}^2, \mathcal{P}(\Omega))$ we
define the uniform probability
by specifying

$$\mathbb{P}(\{(i, j)\}) = \frac{1}{36} \quad \begin{array}{l} \text{(uniform since} \\ \text{it does not depend} \\ \text{on } (i, j)) \end{array}$$

$(i, j) \in \{1, \dots, 6\}^2$

+ properties (i) - (ii) - (iii)

Example 2 $\Omega = [0,1]$

$\mathcal{F} =$ Borel σ -algebra, i.e. we declare
that any interval $(a,b) \subset [0,1]$
belongs to \mathcal{F}

+ properties (i) - (ii) - (iii)

Uniform prob measure: If $A = (a,b)$
Then

$$P(A) = b - a$$

Then use (i) - (ii) - (iii)

Rmk It is important to consider only countable unions. Otherwise one could write

$$P(\{t\}) = P([t, t]) = 0$$

Thus

$$\begin{aligned} 1 &= P([0, 1]) = P(\bigcup_{t \in [0, 1]} \{t\}) \\ &\stackrel{\text{WRONG}}{=} \sum_{t \in [0, 1]} P(\{t\}) = 0 \end{aligned}$$

disjoint

Complete probability space

Hypothesis: We assume that \mathbf{P} is **complete**, i.e

$$\begin{aligned} A \in \mathcal{F} \text{ such that } \mathbf{P}(A) = 0, \text{ and } B \subset A \\ \implies \\ B \in \mathcal{F} \text{ and } \mathbf{P}(B) = 0. \end{aligned}$$

Remark: A probability can always be completed

Simple examples (1)

Tossing 2 dice:

- $\Omega = \{1, 2, 3, 4, 5, 6\}^2$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- $\mathbf{P}(A) = \frac{|A|}{36}$

Uniform distribution on $[0, 1]$:

- $\Omega = [0, 1]$
- $\mathcal{F} = \mathcal{B}([0, 1])$
- $\mathbf{P} = \lambda$, Lebesgue measure