Outline

Probability space

Probability space: $(\Omega, \mathcal{F}, \mathbf{P})$ with

- Ω set
- $\mathcal{F}$ a σ-algebra or σ- field
- $\mathbf{P}$ probability measure

Prob space : $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathcal{F}$: σ-algebra, i.e $\mathcal{F}$ is a collection of subsets of Ω s.t.

(i) $\Omega \in \mathcal{F}$, $\emptyset \in \Omega$ or $\bar{A}$

(ii) If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(iii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$

In fact if $\{A_i ; i \geq 1\}$ we want

$\bigcap_{i=1}^{\infty} A_i \in \mathcal{F}$, $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$

About $\mathbb{P}$: $\mathbb{P}$ is a probability measure, i.e

$$\mathbb{P} : \mathcal{F} \longrightarrow [0,1] \quad \text{s.t.}$$

(i) $\mathbb{P}(\emptyset) = 0$ $\quad$ $\mathbb{P}(\Omega) = 1$

(ii) If $\{A_i \, ; \, i \geq 1\}$ are disjoint,

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

(iii) $\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$ $\quad \forall A \in \mathcal{F}$

Rmk If $A \in \mathcal{F}$, we say that A is measurable

Example 1 Tossing 2 dice. Then

$$\Omega = \{1, \dots, 6\} \times \{1, \dots, 6\} = \{1, \dots, 6\}^2$$

Since Ω is finite (or countable), we can take

$$\mathcal{F} = \mathcal{P}(\Omega) = \{\text{all subsets of } \Omega\}$$

$$= \{\emptyset, \{(1,1)\}, \{(1,2)\} \dots \{(1,1), (1,2)\} \dots\}$$

Question: $|\Omega| = \#\text{ elements in } \Omega = 36$

$|\mathcal{F}| = 2^{36}$

On $(\Omega, \mathcal{F}) = (\{1,\dots,6\}^2, \mathcal{P}(\Omega))$ we define the <u>uniform probability</u> by specifying

$$\mathbb{P}(\{\underbrace{(i,j)}_{(i,j)\in\{1,\dots,6\}^2}\}) = \frac{1}{36}$$

(uniform since it does not depend on (i,j))

+ properties (i) - (ii) - (iii)

Example 2 $\Omega = [0,1]$

$\mathcal{F}$ = Borel σ-algebra, i.e. we declare that any interval $(a,b) \subset [0,1]$ belongs to $\mathcal{F}$

+ properties (i) - (ii) - (iii)

Uniform prob measure: If $A = (a,b)$ then

$$\boxed{\mathbb{P}(A) = b - a}$$

Then use (i) - (ii) - (iii)

Rmk It is important to consider only countable unions. Otherwise one could write

$$\mathbb{P}(\{t\}) = \mathbb{P}([t,t]) = 0$$

Thus

$$1 = \mathbb{P}([0,1]) = \mathbb{P}\Big(\overbrace{\bigcup_{t\in[0,1]} \{t\}}^{\text{disjoint}}\Big)$$

WRONG

$$= \sum_{t\in[0,1]} \overbrace{\mathbb{P}(\{t\})}^{0} = 0$$

Complete probability space

Hypothesis: We assume that $\mathbf{P}$ is complete, i.e

$$A \in \mathcal{F} \text{ such that } \mathbf{P}(A) = 0, \text{ and } B \subset A$$
$$\Longrightarrow$$
$$B \in \mathcal{F} \text{ and } \mathbf{P}(B) = 0.$$

Remark: A probability can always be completed

Simple examples (1)

Tossing 2 dice:

- $\Omega = \{1, 2, 3, 4, 5, 6\}^2$
- $\mathcal{F} = \mathcal{P}(\Omega)$
- $\mathbf{P}(A) = \frac{|A|}{36}$

Uniform distribution on $[0, 1]$:

- $\Omega = [0, 1]$
- $\mathcal{F} = \mathcal{B}([0, 1])$
- $\mathbf{P} = \lambda$, Lebesgue measure